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WEAK FORM OF EQUIDISTRIBUTION THEOREM FOR HARMONIC MEASURES OF FOLIATIONS BY HYPERBOLIC SURFACES

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Abstract. We show that the equidistribution theorem of C. Bonatti and X. Gómez-Mont for a special kind of foliations by hyperbolic surfaces does not hold in general, and we seek a weaker form valid for general foliations by hyperbolic surfaces.

1. Introduction

Let M be a smooth closed manifold, and let $\mathcal{F}$ be a smooth foliation by hyperbolic surfaces, i.e. a 2-dimensional foliation equipped with a smooth leafwise metric g_P of constant curvature -1. Let v_P be the leafwise Poincaré volume form, and for a point $z \in M$ and $\rho > 0$, let $B_\rho(z)$ be the leafwise ρ-disk centered at z. When $B_\rho(z)$ is an embedded disk in M, let $\beta_\rho(z)$ be the probability measure of M supported on $B_\rho(z)$ defined by

$$\beta_\rho(z) = \frac{1}{\int_{B_\rho(z)} v_P}\, v_P|_{B_\rho(z)}.$$

When $B_\rho(z)$ is not embedded, define $\beta_\rho(z)$ using the universal cover of the leaf.

In [BGM], Christian Bonatti and Xavier Gómez-Mont have shown the following theorem.

Theorem 1.1. *Let Σ be a closed oriented hyperbolic surface, and let $\Phi : \pi_1(\Sigma) \to PSL(2, \mathbb{C})$ be a nonelementary representation. Endow leaves of the associated foliated $\mathbb{P}^1$ bundle $(N, \mathcal{G})$ with a hyperbolic metric lifted from Σ. Then there exists a probability measure μ on N such that for any sequences $z_n \in N$ and $\rho_n \to \infty$, $\beta_{\rho_n}(z_n)$ converges weakly to μ.*

The measure μ turns out to be the unique harmonic measure of the foliation $\mathcal{G}$ in the sense of [G]. See Section 4 for more detail. Thus one may ask the following question.

Question 1.2. For $(M, \mathcal{F})$ as above, if $\beta_{\rho_n}(z_n)$ converges weakly to a measure μ as $\rho_n \to \infty$, is it true that μ is a harmonic measure of $\mathcal{F}$?

In Section 2, we shall answer this question in the negative, and in Section 3, we propose a measure $\mu_{\rho,\rho'}(z)$ modified for the positive answer. In Section 4, we raise

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a further question and give a new example of foliations for which the conclusion of the theorem of Bonatti and Gómez-Mont holds.

2. A counterexample

Let $Solv_3$ be the 3-dimensional unimodular solvable nonnilpotent Lie group. The multiplication of $Solv_3 = \{(x, q, t)\}$ is given by

$$(x, q, t)(x', q', t') = (e^t x' + x,\ e^{-t} q' + q,\ t + t').$$

It has a semidirect product structure:

$$1 \to \mathbb{R}^2 \to Solv_3 \to \mathbb{R} \to 1.$$

Any lattice Γ of $Solv_3$ is a semidirect product

$$1 \to \mathbb{Z}^2 \to \Gamma \to \mathbb{Z} \to 1$$

such that $\Gamma \cap \mathbb{R}^2 = \mathbb{Z}^2$. The multiplication is given by

$$(n, m, \ell)(n', m', \ell') = ((n', m')A^\ell + (n, m),\ \ell + \ell')$$

for some hyperbolic matrix $A \in SL(2, \mathbb{Z})$. The quotient manifold $M = \Gamma \setminus Solv_3$ is a T^2 bundle over S^1 with monodromy ${}^tA$:

$$T^2 = \mathbb{Z}^2 \setminus \mathbb{R}^2 \to M = \Gamma \setminus Solv_3 \to S^1 = \mathbb{Z} \setminus \mathbb{R}.$$

Denote by $G = \{q = 0\}$ the subgroup of $Solv_3$, isomorphic to the 2-dimensional solvable nonabelian Lie group, and let $\tilde{\mathcal{F}}$ be the orbit foliation of the right G-action. Notice that the leaf passing through (x_0, q_0, t_0) is just $L_{q_0} = \{q = q_0\}$. The left action of the lattice Γ commutes with the right G-action, and therefore $\tilde{\mathcal{F}}$ descends to a foliation $\mathcal{F}$ on M. Now

$$g = e^{-2t} dx^2 + e^{2t} dq^2 + dt^2$$

is a left invariant metric on $Solv_3$. The restriction of g to each leaf L_q of $\tilde{\mathcal{F}}$ is written as

$$g_P = e^{-2t} dx^2 + dt^2.$$

If we change the variable by $y = e^t$, then we get

$$g_P = (dx^2 + dy^2)/y^2,$$

the Poincaré metric on the half plane $\mathbb{H}$. That is, we have an identification

$$Solv_3 \ni (x, q, t) \leftrightarrow (x + e^t i, q) \in \mathbb{H} \times \mathbb{R},$$

where the right G-action leaves each leaf $L_q = \mathbb{H} \times \{q\}$ invariant. The action of the one-parameter subgroup $\{Y^t = (0, 0, t)\}$ of G on each leaf $L_q \cong \mathbb{H}$ is given by

$$Y^t(x + yi) = x + e^t yi,$$

and the one-parameter subgroup $\{S^s = (s, 0, 0)\}$ is given by

$$S^s(x + yi) = ys + x + yi.$$

They satisfy

$$Y^t \circ S^s = S^{se^{-t}} \circ Y^t.$$

See Figure 1. On the other hand, the left $Solv_3$-action (in particular Γ-action) leaves the product structure invariant, and the action on the first factor $\mathbb{H}$ is given by

$$(x, q, t) \cdot z = e^t z + x.$$

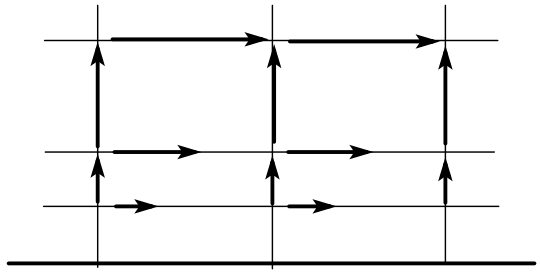

FIGURE 1. Y^t moves points upwards along the geodesics by length t. S^s moves points horizontally along the horocycles by length s. Y^t contracts S^s.

It is not only g_P-isometric but also leaves the point ∞ on $\partial\mathbb{H}$ invariant. That is, each leaf of the foliation $\mathcal{F}$ of the quotient manifold M admits a pointed hyperbolic structure.

The flow $\{S^s\}$ leaves the coordinate y, whence the old coordinate t, invariant. Thus it leaves fibers of the fibration $T^2 \to M \to S^1$ invariant and is a linear flow on it parallel to an eigenvector of the matrix ${}^tA$.

Now $m = dx \wedge dq \wedge dt$ is a biinvariant Haar measure of $Solv_3$. If we denote by $v_P = y^{-2}dx \wedge dy$ the leafwise Poincaré volume form of $\tilde{\mathcal{F}}$, then

$$m = -y\, v_P \wedge dq.$$

The measure m yields a probability measure on M, also denoted by m. By a criterion in [G], m is a harmonic measure of $\mathcal{F}$, since the function y is a harmonic function on $\mathbb{H}$. Moreover by a general theorem of Bertrand Deroin and Victor Kleptsyn [DK], it is the unique harmonic measure. The rest of this section is devoted to the proof of the following theorem.

Theorem 2.1. *There exist $z_n \in M$ such that $\beta_n(z_n)$ converges to $\mu \neq m$.*

We consider an infinite cyclic covering $\hat{M}$ of M and the lift $\hat{\mathcal{F}}$ of the foliation $\mathcal{F}$. Precisely,

$$\hat{M} = \mathbb{Z}^2 \setminus Solv_3 = T^2 \times \mathbb{R},$$

where $\mathbb{Z}^2$ is the normal subgroup of the lattice Γ. Let us denote by $\mathcal{P}(\hat{M})$ the space of the Radon probability measures of $\hat{M}$, endowed with the pointwise convergence topology on the space $C_0(\hat{M})$ of continuous functions on $\hat{M}$ with compact support.

Every leaf of $\hat{\mathcal{F}}$ is pointedly isometric to $\mathbb{H}$. Choose one leaf and identify it with $\mathbb{H}$. For $\rho > 0$, let $z_\rho = e^\rho i \in \mathbb{H} \subset \hat{M}$. Notice that the hyperbolic distance of z_ρ to the horocycle $\{y = 1\}$ is ρ. We shall show that the probability measure $\beta_\rho(z_\rho) \in \mathcal{P}(\hat{M})$ converges to a measure $\hat{\mu} \in \mathcal{P}(\hat{M})$ as $\rho \to \infty$. The boundary $\partial B_\rho(z_\rho)$ of the disk $B_\rho(z_\rho)$ is tangent to $\{y = 1\}$ and satisfies the equation:

$$x^2 + (y - \frac{R+1}{2})^2 = \frac{1}{4}(R-1)^2, \quad \text{where} \quad R = e^{2\rho}.$$

See Figure 2. Putting $y = e^t$, we get

$$x = \pm\sqrt{R(e^t - 1) + e^t - e^{2t}}.$$

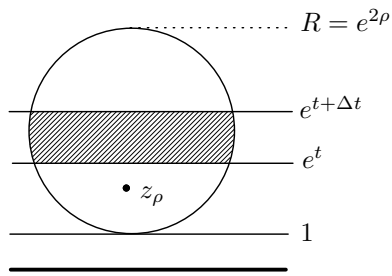

FIGURE 2

Now since $v_P = y^{-2}dx \wedge dy = e^{-t}dx \wedge dt$, the area $A(R, t, \Delta t)$ of the set $B_\rho(z_\rho) \cap (T^2 \times [t, t + \Delta t])$ is given by

$$A(R, t, \Delta t) = 2 \int_t^{t+\Delta t} \sqrt{R(e^{-t} - e^{-2t}) + e^{-t} - 1}\, dt.$$

On the other hand, the area $A(R)$ of $B_\rho(z_\rho)$ is given by

$$A(R) = \pi(e^\rho + e^{-\rho} - 2) = \pi(R^{1/2} + R^{-1/2} - 2).$$

Now we have

$$\beta_\rho(z_\rho)(T^2 \times [t, t + \Delta t]) = \frac{A(R, t, \Delta t)}{A(R)} = \frac{2}{\pi} \int_t^{t+\Delta t} \frac{\sqrt{R(e^{-t} - e^{-2t}) + e^{-t} - 1}}{R^{1/2} + R^{-1/2} - 2} dt.$$

Therefore the limit measure $\hat{\mu}$ as $\rho \to \infty$ should satisfy

$$\hat{\mu}(T^2 \times [t, t + \Delta t]) = \frac{2}{\pi} \int_t^{t+\Delta t} \sqrt{e^{-t} - e^{-2t}} dt.$$

On the other hand, the portion of the measure $\beta_\rho(z_\rho)$ supported on $T^2 \times [t, t + \Delta t]$ $(t > 0)$ becomes more and more invariant by the flow S^s as $\rho \to \infty$, while S^s is a linear flow of irrational slope on $T^2 \times \{t\}$ and $dx \wedge dq$ is the unique measure invariant by S^s. Therefore one concludes that

$$\beta_\rho(z_\rho) \to \hat{\mu} = \hat{\Phi}\, dx \wedge dq \wedge dt \text{ as } \rho \to \infty,$$

where

$$\hat{\Phi}(t) = \begin{cases} \frac{2}{\pi}\sqrt{e^{-t} - e^{-2t}} & \text{if } t \geq 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

The actual proof needs the evaluation on a function from $C_0(\hat{M})$, which is routine and omitted. But $\beta_\rho(z_\rho) \to \hat{\mu}$ does not guarantee that $\hat{\mu}$ is a probability measure, since the constant 1 does not belong to $C_0(\hat{M})$ and some part of $\beta_\rho(z_\rho)$ may escape to ∞. This, however, is assured by the following concrete computation:

$$\int_0^\infty \sqrt{e^{-t} - e^{-2t}} dt = \pi/2.$$

Also this implies a stronger fact that $\beta_\rho(z_\rho) \to \hat{\mu}$ pointwise on any bounded continuous function. The function $\hat{\Phi}$ takes the maximum value at $t = \log 2$. See Figure 3. Returning to the compact manifold M, the previous observation shows that the limit measure μ of $\beta_\rho(z_\rho)$ is the projected image of $\hat{\mu}$ and is written as $\mu = (\Phi \circ p)\, m$,

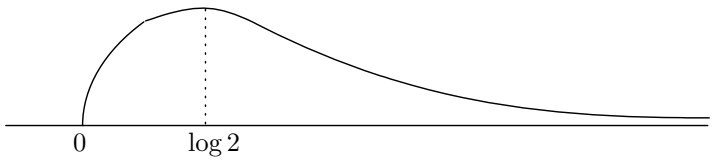

FIGURE 3. The function $\hat{\Phi}$.

where $p : M \to S^1$ is the bundle projection and Φ is a continuous function on S^1 given by

$$\Phi(t) = \sum_{k \in \mathbb{Z}} \hat{\Phi}(t+k).$$

But Φ is not a constant function since $\Phi(0) < \Phi(\log 2)$, showing that $\mu \neq m$, as is required.

The author is greatful to Hiroki Kodama for showing him this simple proof.

3. Weak form of equidistribution

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ be the disk model of the Poincaré plane. For $0 < R < 1$, denote $\mathbb{D}(R) = \{|z| < R\}$ and let ρ be the Poincaré distance from 0 to the circle $\partial\mathbb{D}(R)$, i.e.

$$\rho = \frac{1}{2} \log \frac{1+R}{1-R}.$$

For $R < R' < 1$, let ρ' be the Poincaré distance between $\partial\mathbb{D}(R)$ and $\partial\mathbb{D}(R')$. Define a function $\psi_{\rho,\rho'} : \mathbb{D} \to [0, \infty)$ by

$$\psi_{\rho,\rho'}(z) = \begin{cases} 1 & \text{if } |z| \leq R, \\ \frac{R'-|z|}{R'-R} & \text{if } R \leq |z| \leq R', \\ 0 & \text{if } R' \leq |z| < 1. \end{cases}$$

The function $\psi_{\rho,\rho'}$ is determined by ρ and ρ'. See Figure 4. Define a probability

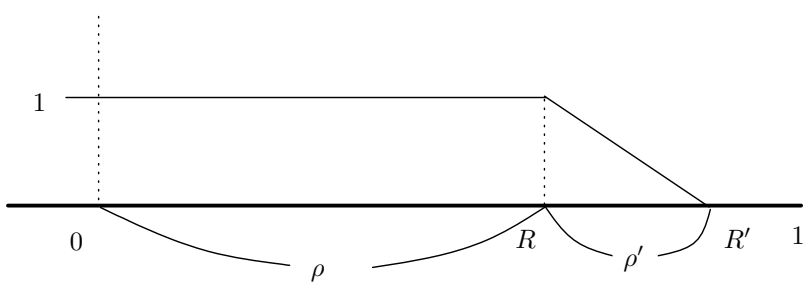

FIGURE 4. The functions $\psi_{\rho,\rho'}$.

measure $\mu_{\rho,\rho'}$ on $\mathbb{D}$ by

$$\mu_{\rho,\rho'} = \frac{1}{\int_{\mathbb{D}} \psi_{\rho,\rho'} v_P} \, \psi_{\rho,\rho'} \, v_P,$$

where v_P denotes the Poincaré volume form.

Let $(M, \mathcal{F})$ be as in Section 1. For any $x \in M$, let L_x be the leaf through x with the universal cover $\tilde{L}_x$ identified with $\mathbb{D}$. Define a map $j_x : \mathbb{D} \to M$ as the composite

$$j_x : \mathbb{D} \cong \tilde{L}_x \to L_x \subset M$$

such that $j_x(0) = x$. Define $\mu_{\rho,\rho'}(x) \in \mathcal{P}(M)$ by $\mu_{\rho,\rho'}(x) = (j_x)_* \mu_{\rho,\rho'}$. The main result of this section is the following.

Theorem 3.1. *If $\mu_{\rho_n,\rho'_n}(x_n)$ converges for some sequences $x_n \in M$, $\rho_n \to \infty$ and $\rho'_n \to \infty$, then the limit is a harmonic measure for $\mathcal{F}$.*

To show this, we approximate $\psi_{\rho,\rho'}$ by another function $\varphi_{\rho,\rho'}$ which is a combination of harmonic functions. Let $A = 1/\log \frac{R'}{R}$. Define a function $\varphi_{\rho,\rho'} : \mathbb{D} \to [0, \infty)$ by

$$\varphi_{\rho,\rho'}(z) = \begin{cases} 1 & \text{if } |z| \le R, \\ A \log \frac{R'}{|z|} & \text{if } R \le |z| \le R', \\ 0 & \text{if } R' \le |z| < 1. \end{cases}$$

Define a probability measure $\nu_{\rho,\rho'}$ on $\mathbb{D}$ by

$$\nu_{\rho,\rho'} = \frac{1}{\int_{\mathbb{D}} \varphi_{\rho,\rho'} v_P} \, \varphi_{\rho,\rho'} \, v_P,$$

and define $\nu_{\rho,\rho'}(x) = (j_x)_* \nu_{\rho,\rho'} \in \mathcal{P}(M)$ just as before. Theorem 3.1 reduces to the following two propositions. Denote by $\|\cdot\|$ the norm of $\mathcal{P}(M) \subset C(M)'$ dual to the sup norm $\|\cdot\|_\infty$ of the Banach space $C(M)$ of the continuous functions of M.

Proposition 3.2. *We have $\|\mu_{\rho_n,\rho'_n}(x_n) - \nu_{\rho_n,\rho'_n}(x_n)\| \to 0$ as $\rho_n, \rho'_n \to \infty$.*

Proposition 3.3. *If $\nu_{\rho_n,\rho'_n}(x_n)$ converges for some sequences $x_n \in M$ and $\rho_n, \rho'_n \to \infty$, then the limit is a harmonic measure for $\mathcal{F}$.*

We shall first show Proposition 3.3. For a C^2 function $f : M \to \mathbb{R}$, we denote by $\Delta_P f$ the leafwise Laplacian with respect to the leafwise Poincaré metric. What we have to prove is that

$$\int_M \Delta_P f \; \nu_{\rho_n,\rho'_n}(x_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Since j_{x_n} is a local isometry onto the leaf, we have $\Delta_P f \circ j_{x_n} = \Delta_P (f \circ j_{x_n})$. Rewriting $f \circ j_{x_n}$ as f, this follows from the following proposition about $\mathbb{D}$.

Proposition 3.4. *For any nonzero bounded C^2 function $f : \mathbb{D} \to \mathbb{R}$, we have*

$$\frac{\int_{\mathbb{D}} \varphi_{\rho,\rho'} \Delta_P f \, v_P \; \cdot \; \|f\|_\infty{}^{-1}}{\int_{\mathbb{D}} \varphi_{\rho,\rho'} v_P} \to 0$$

as $\rho, \rho' \to 0$ uniformly on f

Estimate of the numerator. First notice that $\Delta_P f \, v_P = \Delta_E f \, v_E$ where E stands for Euclidian. (Both are equal to $dJ^* df$, where J is the almost complex structure.) We need the following Green-Riesz formula. See [D], Chap. I, p.30.

Theorem 3.5. *Let Ω be a smoothly bounded compact domain in $\mathbb{R}^n$, and let $\vec{n}_E$ be the outward unit normal vector at $\partial\Omega$. Denote by σ_E the Euclidian area measure on $\partial\Omega$. Then for any C^2 function φ and f defined on $\mathbb{R}^n$, we have*

$$\int_\Omega (\varphi \Delta_E f - f \Delta_E \varphi) v_E = \int_{\partial\Omega} (\varphi \frac{\partial f}{\partial \vec{n}_E} - f \frac{\partial \varphi}{\partial \vec{n}_E}) \sigma_E.$$

Let us apply this formula to $\varphi_{\rho,\rho'}$, f in Proposition 3.4 and the domains $\mathbb{D}(R)$, $\mathbb{D}(R') \setminus \mathbb{D}(R)$. Remark that

$$\Delta_E \log \frac{R'}{|z|} = 0 \quad \text{and} \quad \frac{\partial}{\partial \vec{n}_E}(\log \frac{R'}{|z|}) = -\frac{1}{|z|}.$$

Computation shows that

$$\int_{\mathbb{D}} \varphi_{\rho,\rho'} \Delta_P f \, v_P = A(\frac{1}{R'} \int_{\partial \mathbb{D}(R')} f \sigma_E - \frac{1}{R} \int_{\partial \mathbb{D}(R)} f \sigma_E).$$

This implies that

$$|\int_{\mathbb{D}} \varphi_{\rho,\rho'} \Delta_P f \, v_P| \; \|f\|_\infty^{-1} \leq 4\pi A \; .$$

Estimate of the denominator. We use the following notation.

Notation 3.6. For $0 < R < R' < 1$ and positive valued functions $F(R, R')$ and $G(R, R')$, we write $F \sim G$ if $F/G \to 1$ as $R \to 1$.

Lemma 3.7. (1) *If $F_1 \sim G_1$ and $F_2 \sim G_2$, then $F_1 + F_2 \sim G_1 + G_2$.*

(2) *If $F \sim G$, then we have*

$$\int_R^{R'} F(r, R') dr \sim \int_R^{R'} G(r.R') dr.$$

(3) *We have* $\log(R'/R) \sim R' - R$.

Now since

$$v_P = \frac{4dx \wedge dy}{(1 - |z|^2)^2} = \frac{4rdr \wedge d\theta}{(1 - r^2)^2}$$

in the polar coordinates, we have

$$\frac{1}{8\pi} \int_{\mathbb{D}} \varphi_{\rho,\rho'} v_P = \int_0^R \frac{rdr}{(1 - r^2)^2} + A \int_R^{R'} \log \frac{R'}{r} \; \frac{rdr}{(1 - r^2)^2},$$

where

$$\text{the first term } = \frac{R^2}{2(1 - R)(1 + R)} \sim \frac{1}{4(1 - R)} = \frac{A \log(R'/R)}{4(1 - R)} \sim \frac{A(R' - R)}{4(1 - R)},$$

and by Lemma 3.7 (2)

$$\text{the second term } \sim A \int_R^{R'} \frac{(R' - r)dr}{4(1 - r)^2} = -\frac{A(R' - R)}{4(1 - R)} + \frac{A}{4} \log \frac{1 - R}{1 - R'}.$$

Since both terms are positive, we get from Lemma 3.7 (1),

$$\frac{1}{8\pi} \int_{\mathbb{D}} \varphi_{\rho,\rho'} v_P \sim \frac{A}{4} \log \frac{1 - R}{1 - R'} \sim \frac{A\rho'}{2},$$

where the last $\sim$ holds when ρ' is bounded from below and follows from the formula

$$\rho' = \frac{1}{2}(\log \frac{1 + R'}{1 - R'} - \log \frac{1 + R}{1 - R}).$$

It follows from the two estimates that

$$\lim_{R \to 1} \frac{\int_{\mathbb{D}} \varphi_{\rho,\rho'} \Delta_P f \cdot \|f\|_\infty^{-1}}{\int_{\mathbb{D}} \varphi_{\rho,\rho'} v_P} \leq \lim_{R \to 1} \frac{1}{\rho'} = 0,$$

and the convergence is uniform on f. This shows Propositions 3.4 and 3.3.

Finally Proposition 3.2 follows from the estimate

$$\int_{\mathbb{D}} \psi_{\rho,\rho'} v_P \sim \int_{\mathbb{D}} \varphi_{\rho,\rho'} v_P,$$

since $\psi_{\rho,\rho'} \geq \varphi_{\rho,\rho'}$. We have already shown that

$$\int_{\mathbb{D}} \varphi_{\rho,\rho'} v_P \sim 4\pi A\rho'.$$

Analogous (and easier) computation using $A(R' - R) \sim 1$ shows that

$$\int_{\mathbb{D}} \psi_{\rho,\rho'} v_P \sim 4\pi A\rho'.$$

4. Further question and example

It seems that the counterexample in Section 2 is rather special. There might be more foliations which satisfy the conclusion of the theorem of Bonatti and Gómez-Mont. To consider this problem, let us recall their proof, which consists of two steps. In the first step, they consider general $(M, \mathcal{F}, g_P)$ as in the beginning of Section 1. Let $p : \hat{M} \to M$ be the unit tangent bundle of the foliation $\mathcal{F}$. The space $\hat{M}$ admits a leafwise geodesic flow $\{g^t\}$ and the leafwise stable horocycle flow $\{h^s\}$, which satisfy

$$g^t \circ h^s \circ g^{-t} = h^{se^{-t}}. \tag{4.1}$$

Therefore the two flows form a locally free action of the Lie group B, the 2-dimensional nonabelian Lie group. Given a leafwise submersed ρ-disk $B_\rho(z)$ of $\mathcal{F}$ (see Section 1), they considered the lift $\sigma : B_\rho(z) \setminus \{z\} \to \hat{M}$ by the radial unit vector fields, and showed that the limit $\lim_{\rho_n \to \infty} \sigma_* \beta_{\rho_n}(z_n)$ is h^s-invariant, if it exists. This part is true for any $(M, \mathcal{F}, g_P)$.

On the other hand, Yuri Bakhtin and Matilde Martínez [BM] showed that the map $p_* : \mathcal{P}(\hat{M}) \to \mathcal{P}(M)$ between the space of the probability measures gives a bijection from the subset of the B-invariant measures on $\hat{M}$ to the subset of the harmonic measures on M.

Now assume that the horocycle flow $\{h^s\}$ is uniquely ergodic. Then the unique invariant measure $\hat{\mu}$ is also g^t-invariant by (4.1), and thus $p_*\hat{\mu}$ is a unique harmonic measure of $(M, \mathcal{F}, g)$. In the second step, Bonatti and Goméz-Mont showed the unique ergodicity of the horocycle flow $\{h^s\}$ for foliations in Theorem 1.1. It is plausible to expect that there are more foliations with this property. We shall raise one example.

Example 4.1. Let G be an arbitrary connected unimodular Lie group, and let $\Gamma \subset PSL(2, \mathbb{R}) \times G$ be a cocompact lattice such that $p_2(\Gamma)$ is dense in G, where p_2 is the projection onto the second factor. Then the manifold $M = \Gamma \setminus (\mathbb{H} \times G)$ admits a horizontal foliation $\mathcal{F} = \Gamma \setminus \{\mathbb{H} \times \{g\}\}$ by hyperbolic surfaces.

The unit tangent bundle $\hat{M}$ of the above foliation $\mathcal{F}$ is identified with $\Gamma \setminus (PSL(2, \mathbb{R}) \times G)$. According to Marina Ratner [R], any ergodic probability measure $\hat{\mu}$ invariant by the leafwise stable horocycle flow is *algebraic* in the following sense. For any x in the support of $\hat{\mu}$, there is a closed subgroup $H \subset PSL(2, \mathbb{R}) \times G$ such that the closure of the horocycle orbit of x is $x \cdot H$ and that $\hat{\mu} = x_* m$, where m is the normalized Haar measure of $(g^{-1}\Gamma g \cap H) \setminus H$ and $x = \Gamma g$.

On the other hand, it is shown by Fernando Alcalde Cuesta and Françoise Dal'bo [ACD] that the leafwise stable horocycle flow is minimal. Therefore $\hat{\mu}$ is the Haar measure of $\Gamma\backslash(PSL(2,\mathbb{R})\times G)$ and is unique. In conclusion the foliation in Example 4.1 is equidistributed, i.e. satisfies the conclusion of Theorem 1.1.

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