UNIFORM GROWTH RATE

KASRA RAFI AND JING TAO

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Abstract. In an evolutionary system in which the rules of mutation are local in nature, the number of possible outcomes after $m$ mutations is an exponential function of $m$ but with a rate that depends only on the set of rules and not the size of the original object. We apply this principle to find a uniform upper bound for the growth rate of certain groups including the mapping class group. We also find a uniform upper bound for the growth rate of the number of homotopy classes of triangulations of an oriented surface that can be obtained from a given triangulation using $m$ diagonal flips.

1. Introduction

Let $G$ be a group and $S$ be a generating set for $G$. We denote the word length in $G$ associated to $S$ with $\| \cdot \|_S$. Recall that the growth rate of $G$ (relative to $S$) is defined to be

$$h_G = \lim_{R \to \infty} \frac{\log \# B_R(G)}{R},$$

where $B_R(G) = \{ g \in G \mid \|g\|_S \leq R \}$.

At his 60th birthday conference, Bill Thurston mentioned that the mapping class group has a growth rate that is independent of its genus. Namely, consider the following set of curves on a surface $\Sigma = \Sigma_{g,p}$ of genus $g$ with $p$ punctures:

Let $S$ be the set of Dehn (or half) twists around these curves. This set $S$ generates $\text{MCG}(\Sigma)$, the mapping class group of $\Sigma$ [Lic64, FM12, Art47, Bir74]. (Note that $S$ is a combination of the Lickorish generators of the mapping class group of a closed surface and the standard generators of a braid group.) We will refer to $S$ as the set of extended Lickorish generators for $\text{MCG}(\Sigma)$. Then, the growth rate of
MCG(\(\Sigma\)) equipped with the word metric associated to \(S\) has an upper bound that is independent of the topology of \(\Sigma\). This, Thurston asserted, is true since most pairs of elements in \(S\) commute.

Note that, in fact, the number of elements in \(S\) that do not commute with a given element in \(S\) is uniformly bounded. We show that this is enough to obtain the uniform growth rate in general.

**Theorem A.** Given any \(c_0\) let \(S\) be a generating set for a group \(G\) such that, for every \(s \in S\), the number of elements of \(S\) that do not commute with \(s\) is bounded by \(c_0\). Then \(h_G \leq \log(2c_0 + 2) + 1\).

Since each curve in the extended Lickorish generators intersects at most 3 other curves, we obtain:

**Corollary B.** The growth rate of MCG(\(\Sigma\)) relative to the extended Lickorish generators is bounded by \(\log 8 + 1\).

Uniform growth rate can also be shown regarding groups \(\text{Aut}(F_n)\), \(\text{Out}(F_n)\), \(\text{GL}_n(\mathbb{Z})\), and similar groups if the generating set is chosen such that the number of generators that do not commute with a given generator is uniformly bounded. In fact, these groups have natural generating sets with this property. For example, in the case of \(\text{Aut}(F_n)\), let \(F_n\) be the free group with basis \(\{a_1, \ldots, a_n\}\), and consider the following three types of automorphisms of \(F_n\):

1. **Inversion:** For \(1 \leq i \leq n\), \(I_i(a_i) = \overline{a_i}\) and fixes all other \(a_j\).
2. **Transposition:** For \(1 \leq i \leq n - 1\), \(P_i(a_i) = a_{i+1}\) and \(P_i(a_{i+1}) = a_i\) and fixes all other \(a_j\).
3. **Multiplication:** For \(1 \leq i \leq n - 1\), \(M_i(a_i) = a_ia_{i+1}\) and fixes all other \(a_j\).

The collection of inversions, transpositions, and multiplications generate \(\text{Aut}(F_n)\) and is called the set of local Nielsen generators. For each \(s \in S\), the number of elements that do not commute with \(s\) is at most 7, and we obtain:

**Corollary C.** The growth rate of \(\text{Aut}(F_n)\) relative to local Nielsen generators is bounded by \(\log 16 + 1\).

1.1. **Evolving structures.** Another context in which to apply this philosophy is the setting of evolving structures. We follow the footsteps of the work of Sleator-Tarjan-Thurston [STT92] where they showed that if a graph is allowed to evolve using a set of rules that change the graph locally, then the growth rate of the number of possible outcomes after \(R\) mutations is bounded by a constant depending on the rules of evolution and not the size of the graph. This was used in [STT92] to estimate the diameter of the space of plane triangulations equipped with the diagonal flip metric and in [RT13] to estimate the diameter of the space of cubic graphs equipped with the Whitehead move metric. Similar to their work, one can also consider the evolution of labeled graphs. Generalizing the results in [STT92] slightly, we prove:

**Theorem D.** Let \(G\) be any group and \(\Gamma\) be a \(G\)-labeled trivalent graph (see Section 3 for definition). Let \(B_R(\Gamma)\) be the set of \(G\)-labeled graphs that are obtained from \(\Gamma\) by at most \(R\) splits. Then

\[
\lim_{R \to \infty} \frac{\log \#B_R(\Gamma)}{R} \leq 3 \log 4.
\]
That is, the growth rate of \( B_R(\Gamma) \) is independent of the size and shape of the starting graph \( \Gamma \) and of the group \( G \).

As an application, we can prove a combinatorial version of Corollary B. Namely, let \( T_n(\Sigma) \) be the space of homotopy classes of triangulations of the surface \( \Sigma \) with \( n \) vertices.

**Theorem E.** For \( T \in T_n(\Sigma) \), let \( B_R(T) \) be the set of triangulations in \( T_n(\Sigma) \) that are obtained from \( T \) using \( R \) diagonal flips. Then

\[
\lim_{R \to \infty} \frac{\log \# B_R(T)}{R} \leq 3 \log 4
\]

for every surface \( \Sigma \) and any number of vertices \( n \).

Note that, even though Theorem E is a direct analogue of Corollary B, it does not follow from it. This is because the quotient of \( T_n(\Sigma) \) by \( \text{MCG}(\Sigma) \) has a size that goes to infinity as the number of vertices \( n \) approaches infinity.

### 1.2. Remarks and references

Our Theorem A follows immediately from an upper bound on the growth rate of a right-angled Artin group \( A(\Theta) \) with defining graph \( \Theta \), in terms of the maximum degree of the complementary graph \( \overline{\Theta} \) (Theorem 2.1). Other results relating the growth rate of \( A(\Theta) \) to the shape of \( \Theta \) have been obtained in the past. For instance, it was shown in [Sco07] that the growth series of \( A(\Theta) \) can be computed in terms of the clique polynomial of \( \Theta \). Similar results can be found in [AP14] and [McM14]. However, the degree of \( \overline{\Theta} \) cannot be recovered from the coefficients of the clique polynomial of \( \Theta \), so these results are independent from ours.

Our proof of Theorem 2.1 is related to normal forms for elements of a right-angled Artin group. A normal form for a word representing an element in \( A(\Theta) \) is obtained by shuffling commuting elements and removing inverse pairs of generators of \( A(\Theta) \) whenever possible ([HM99]). By fixing an ordering of \( V(\Theta) \), then every element of \( A(\Theta) \) admits a unique normal form, obtained by additionally shuffling lower-order letters to lower positions whenever possible. In our proof of Theorem 2.1, we construct a canonical representative for a given word, obtained similarly by shuffling lower-order letters to lower positions. However, we do not need to cancel inverse pairs, so the canonical representative of a word may not be in normal form.

### 2. Uniform growth rates

#### 2.1. Preliminaries

Let \( G \) be a finitely generated group. By convention, the inverse of an element \( g \in G \) will be represented by \( \overline{g} \); and for any subset \( S \subseteq G \), let \( \overline{S} = \{ \overline{s} : s \in S \} \). A word in \( S \cup \overline{S} \) is a sequence \( w = [s_1, \ldots, s_R] \), where \( s_i \in S \cup \overline{S}; R \) is the length of \( w \). We allow the empty word whose length is 0. A word \( w = [s_1, \ldots, s_R] \) represents an element \( g \in G \) if \( g = s_1 \cdots s_R \). (The empty word represents the identity element.) By a generating set for \( G \) we will mean a finite set \( S \subseteq G \setminus \{1\} \) such that every element \( g \in G \) is represented by a word in \( S \cup \overline{S} \). The word length \( ||g||_S \) of \( g \) relative to a generating set \( S \) is the length of the shortest word in \( S \cup \overline{S} \) representing \( g \). For any \( R \), \( B_R(G) \) is the set of elements
of $G$ with word length at most $R$. The growth rate (also called the entropy) of $G$ relative to $S$ is
\[ h_G = \lim_{R \to \infty} \frac{\log \#B_R(G)}{R}, \]
where the above limit exists by sub-additivity.

We remark that the growth rate of $G$ depends on the generating set, but positivity of the growth rate does not. The growth rate of $F_n$ relative to a basis is $\log(2n-1)$. If $G$ contains a subgroup isomorphic to $F_2$, then $h_G$ is strictly positive. See [GdlH97] and the references within for more details.

2.2. RAAGs. A graph is a 1-dimensional CW complex. It is simple if there are no self-loops or double edges.

Let $\Theta$ be a finite simple graph. Let $V(\Theta)$ and $E(\Theta)$ be the set of vertices and edges of $\Theta$. An element of $E(\Theta)$ will be denoted by $vw$, where $v$ and $w$ are the vertices of the edge. The complementary graph of $\Theta$ is the graph $\Theta^c$ with $V(\Theta) = V(\Theta^c)$, but two vertices span an edge in $\Theta$ if and only if they do not in $\Theta$.

The right-angled Artin group or RAAG associated to $\Theta$ is the group $A(\Theta)$ with the presentation:
\[ A(\Theta) = \langle s_v \mid v \in V(\Theta) \rangle : [s_v, s_w] = 1 \text{ for } vw \in E(\Theta). \]
The collection $S = \{s_v\}$ will be called the standard generating set of $A(\Theta)$. We will often ignore the distinction between a vertex $v$ and the generator $s_v$.

**Theorem 2.1.** If the valence of every vertex in $\Theta$ is bounded above by a constant $c_0$, then the growth rate of $A(\Theta)$ relative to the standard generating set $S$ is bounded by $\log(2c_0 + 2) + 1$.

From Theorem 2.1 we derive Theorem A as a corollary.

**Corollary 2.2.** Let $S_G$ be a generating set for a group $G$ such that for every $s \in S_G$, the number of elements of $S_G$ that do not commute with $s$ is bounded by $c_0$. Then $h_G \leq \log(2c_0 + 2) + 1$.

**Proof.** Let $\Theta$ be the graph with vertex set $S_G$ and $ss' \in E(\Theta)$ if and only if $[s, s'] = 1$ in $G$. The natural map from $A(\Theta)$ to $G$ taking the standard generating set $S$ to $S_G$ extends to a surjective homomorphism, and the hypothesis on $S_G$ implies the valence of every vertex in $\Theta$ is bounded by $c_0$. All together, we obtain $h_G \leq h_{A(\Theta)} \leq \log(2c_0 + 2) + 1$.

The rest of the section is dedicated to proving Theorem 2.1.

Given a word $w = [s_1, \ldots, s_R]$ in $S \cup S$, the $j$-th letter of $w$ is $w(j) = s_j$. A word $w' = [t_1, \ldots, t_R]$ is a reordering of $w$ if $t_1 \cdots t_R = s_1 \cdots s_R$ and there is a permutation $\sigma$ such that $t_j = s_{\sigma(j)}$. We say the letter $s_k$ in $w$ is ready for position $i, i \leq k$, if $s_k$ commutes with every $s_j$, for $i \leq j \leq k$.

At every vertex $v$ of $\Theta$, label the half-edges at $v$ from 1 to $d_v$, where $d_v \leq c_0$ is the valence of $v$. Let $n$ be the cardinality of $V(\Theta)$. Fix a labeling $L_0 : V(\Theta) \rightarrow \mathbb{N}$ whose image is $\{1, \ldots, n\}$.

Fix $w_0 = [s_1, \ldots, s_R]$. We will inductively construct a sequence $w_1, \ldots, w_R$ of words that reorders $w_0$, in conjunction with a sequence $L_1, \ldots, L_R$ of labeling of $V(\Theta)$. The final word $w_R$ will be called the canonical representative of $g = s_1 \cdots s_R$ induced by $w_0$. ($W_R$ depends on $W_0$.) Along this process, we produce an encoding of the canonical representative by a sequence of integers $\ell_1, \ldots, \ell_R$. 
Suppose for $i \geq 0$, $w_i = [u_1, \ldots, u_i, t_{i+1}, \ldots, t_R]$, a labeling $L_i$ of $V(\overline{G})$, and a sequence $\ell_1, \ldots, \ell_i$ are given. Among $\{t_{i+1}, \ldots, t_R\}$, let $U$ be the subset of letters that are ready for position $i + 1$. Pick $t \in U$ such that $L_i(t)$ is minimal among all elements of $U$. Let $w_{i+1} = [u_1, \ldots, u_i, t, t_{i+1}, \ldots, t_R]$.

and $u_{i+1} = t$. If $t \in S$, then let $\ell_{i+1} = L_i(t)$; if $t \in S$, then let $\ell_{i+1} = -L_i(t)$. The word $w_{i+1}$ is a reordering of $w_i$ and hence of $w_0$ by induction.

We now define the labeling $L_{i+1} : V(\overline{G}) \to \mathbb{N}$. Let $n_i$ be the largest value of $L_i$. Let $w$ be the code of $G$ and hence of $\overline{G}$. We set $\ell_{i+1}(w_k) = n_i + k$ for each $k = 1, \ldots, d$, and $L_{i+1}(v) = L_i(v)$ for all other $v \in V(\overline{G})$.

**Lemma 2.3.** Let $n = \#V(\overline{G})$. Then

$$1 \leq |\ell_1| \leq |\ell_2| \leq \cdots \leq |\ell_R| \leq n + c_0 R.$$  

**Proof.** For each $i \geq 1$, we show $|\ell_i| \leq |\ell_{i+1}|$. Let $w_R = [u_1, \ldots, u_R]$. We have $|\ell_i| = L_i(u_i)$. For $v \in V(\overline{G})$, $L_i(v) = L_{i+1}(v)$ unless $v$ is in the link of $u_i$; in the latter case, $L_{i+1}(v)$ is bigger than the maximal value of $L_i$. If $u_i$ and $u_{i+1}$ do not commute, then $u_{i+1}$ is in the link of $u_i$, therefore $L_{i+1}(u_{i+1})$ exceeds the maximal value of $L_i$, and in particular $L_{i+1}(u_{i+1}) > L_i(u_i)$. If $u_i$ and $u_{i+1}$ commute, then they were both ready for position $i$. In this case, $L_i(u_{i+1}) = L_{i+1}(u_{i+1})$, and $u_i$ was chosen precisely so that $L_i(u_i)$ is minimal among all elements in the set $u_{i+1}, \ldots, u_R$ that were ready for position $i$. We conclude $|\ell_i| \leq |\ell_{i+1}|$.

The largest value of $L_{i+1}$ is at most $c_0$ plus the largest value of $L_i$. Hence the largest value of $L_{i+1}$ is at most $n + c_0 R$. This bounds all $|\ell_i|$. \[ \square \]

Set $C_R = n + c_0 R$. Let $D_R = \{ \pm 1, \pm 2, \ldots, \pm C_R, C_R + 1 \}$ and let

$$W_R = \{ (\ell_1, \ldots, \ell_R) : \ell_i \in D_R \text{ and } |\ell_1| \leq \cdots \leq |\ell_R| \}.$$  

**Proposition 2.4.** There exists an embedding of $B_R(G)$ into $W_R$, hence $\#B_R(G) \leq \#W_R$.

**Proof.** Let $g \in B_R(G)$ have $\|g\|_S = r$. Pick any word $w = [s_1, \ldots, s_r]$ representing $g$ and let $w_r$ be the canonical representative of $g$ induced from $w$. Let $(\ell_1, \ldots, \ell_r)$ be the code of $w_r$. If $r < R$, then extend the sequence to $(\ell_1, \ell_2, \ell_3, \ldots, \ell_R)$ by setting $\ell_{r+i} = C_R + 1$ for all $i = 1, \ldots, R - r$. By Lemma 2.3 $(\ell_1, \ldots, \ell_R) \in W_R$. This gives a map $B_R(G) \to W_R$.

To see this is an embedding, we show how to recover $w_r$ and hence $g$ from the sequence $(\ell_1, \ldots, \ell_R)$. Recall $\overline{G}$ is equipped with a cyclic ordering of the half-edges at every vertex and a labeling $L_0$ of the vertices from 1 to $n$. Let $w_0$ be the empty word. Suppose for $0 \leq i \leq r - 1$, $L_i : V(\overline{G}) \to \mathbb{N}$ and a word $w_i = [u_1, \ldots, u_i]$ are defined. If $\ell_{i+1} = C_R + 1$, then set $u_{i+1} = u_{i+2} = \cdots u_R = 1$. Otherwise, let $v$ be the unique vertex in $\overline{G}$ with label $|\ell_{i+1}| = L_i(v)$. Set $u_{i+1} = v$ if $\ell_{i+1}$ is positive and $u_{i+1} = \overline{v}$ if $\ell_{i+1}$ is negative. Let $(v_1, \ldots, v_d)$ be the vertices in the link of $u_{i+1}$ listed in cyclic order. Let $n_i$ be the largest value of $L_i$. Construct $L_{i+1} : V(\overline{G}) \to \mathbb{N}$ by setting $L_{i+1}(v_k) = n_i + k$ and $L_{i+1}(u) = L_i(u)$ for all other $u \in \overline{G}$. Then $w_r = [u_1, \ldots, u_r]$. \[ \square \]
We now give an upper bound for the growth rate of $\#W_R$, which will complete the proof of Theorem 2.1.

**Lemma 2.5.** \( \lim_{R \to \infty} \frac{\log \#W_R}{R} \leq \log(2c_0 + 2) + 1 \).

**Proof.** Suppose \( p(R) \) and \( q(R) \) are two functions of \( R \) with \( \lim_{R \to \infty} \frac{p(R)}{q(R)} = \frac{1}{\epsilon} \).

Then, using Stirling’s formula, \( \log \left( \frac{p}{q} \right) \) is asymptotic to \( pH(\epsilon) \) as \( R \to \infty \), where

\[ H(\epsilon) = \epsilon \log \frac{1}{\epsilon} + (1 - \epsilon) \log \frac{1}{1 - \epsilon} \]

is the binary entropy function. (See [Mac03, Ch. 1].)

For any \( R \geq 1 \) and \( C \geq R \), by a simple counting argument, the set

\[ W(R,C) = \{(x_1, \ldots, x_R): x_i \in \{1, \ldots, C\}, x_1 \leq \cdots \leq x_R \} \]

has cardinality \( \#W(R,C) = \binom{C + R - 1}{R} \).

Let \( C = C_R + 1 = n + c_0 R + 1 \). We have:

\[ \#W_R \leq 2^R \left( n + R(c_0 + 1) \right) \quad \text{and} \quad \lim_{R \to \infty} \frac{n + R(c_0 + 1)}{R} = c_0 + 1. \]

Therefore,

\[
\lim_{R \to \infty} \frac{\log \#W_R}{R} \leq \lim_{R \to \infty} \frac{\log 2^R \left( n + R(c_0 + 1) \right)}{R} = \lim_{R \to \infty} \frac{R \log 2 + (n + R(c_0 + 1))H \left( \frac{1}{c_0 + 1} \right)}{R} = \log 2 + (c_0 + 1)H \left( \frac{1}{c_0 + 1} \right) = \log 2 + \log(c_0 + 1) + c_0 \log \left( 1 + \frac{1}{c_0} \right) \leq \log(2c_0 + 2) + 1. \]

\( \Box \)

### 3. Evolving structures on \( G \)–labeled graphs

A graph is oriented if each edge is oriented. For any edge \( e \) of an oriented graph, denote by \( i(e) \) and \( t(e) \) the initial and terminal vertex of \( e \). If \( e \) is a loop, then \( i(e) = t(e) \). The orientation of \( e \) induces an orientation on each half-edge of \( e \): the half-edge \( e_l \) containing \( i(e) \) is oriented so that \( i(e_l) = i(e) \) \( (t(e_l) \) is a point in the interior of \( e \), and the half-edge \( e_r \) containing \( t(e) \) is oriented so that \( t(e_r) = t(e) \).

Given a group \( G \), an oriented graph is \( G \)–labeled if each edge is labeled by an element of \( G \). Two \( G \)–labeled graphs are equivalent if one is obtained from the other by reversing the orientation of some of the edges and relabeling those edges with the inverse words. This defines an equivalent relation on the set of \( G \)–labeled graphs. Fix \( n \) and let \( G_n(G) \) be the set of equivalent classes of trivalent \( G \)–labeled graphs of rank \( n \). (Recall the rank of a graph is the rank of its fundamental group.)

We now consider operations that derive from an element in \( G_n(G) \) another element in \( G_n(G) \).
Let $\Gamma \in \mathcal{G}_n(G)$. Let $e$ be an edge with label $g_e$. There are two types of edges in $\Gamma$: loop or non-loop. First assume $e$ is not a loop. Choose a half-edge $\tilde{a}$ (not a half-edge of $e$) incident at $i(e)$, and let $a$ be the edge associated to $\tilde{a}$ with label $g_a$. Disconnect $\tilde{a}$ from $i(e)$ and reattach it to $t(e)$, while changing the label $g_a \rightarrow g_a g_e$ if $t(\tilde{a}) = i(e)$, or $g_a \rightarrow g_e g_a$ if $i(\tilde{a}) = i(e)$. We call this a forward split along $e$. A forward split along $e$ is well defined: if we reverse the orientation of $a$ and invert $g_a$, then the resulting graph is equivalent. Similarly, take a half-edge $\tilde{b}$ incident at $t(e)$. A backward split along $e$ is obtained by disconnecting $\tilde{b}$ from $t(e)$ and reattaching it to $i(e)$, while changing the label $g_b \rightarrow g_b g_e$ if $i(\tilde{b}) = t(e)$, or $g_b \rightarrow g_e g_b$ if $t(\tilde{b}) = i(e)$. This is again well defined. A double split along $e$ (see Figure 1) is the composition of a forward and a backward split along $e$. The resulting graph from a double split is trivalent, unlike from a backward or forward split alone. If we reverse the orientation of $e$ and invert $g_e$, then a forward split along $e$ becomes a backward split along $e$ and vice versa. Therefore, a double split is well defined on the equivalent class of $\Gamma$.

For a loop $e$, let $a$ be the edge connected to $e$ with label $g_a$. A forward split along $e$ changes the label $g_a \rightarrow g_a g_e$ if $t(a) = i(e)$, or $g_a \rightarrow g_e g_a$ if $i(a) = i(e)$. A backward split changes the label $g_a \rightarrow g_a g_e$ if $t(a) = i(e)$, or $g_a \rightarrow g_e g_a$ if $i(a) = t(e)$ (see Figure 1). By a loop split along $e$ we will mean either a forward or a backward split along $e$. This is again well defined on the equivalent class of $\Gamma$.

For any edge $e$ of $\Gamma$, a split along $e$ will mean either a double split or a loop split depending on the type of $e$. We will represent a split along $e$ by $\Gamma \xrightarrow{s} \Gamma'$ and call $e = \text{supp}(s)$ the support of $s$.

Fix $\Gamma_0 \in \mathcal{G}_n(G)$. A derivation $D = [s_1, \ldots, s_R]$ of length $R$ is a sequence of splits

$$\Gamma_0 \xrightarrow{s_1} \Gamma_1 \xrightarrow{s_2} \cdots \xrightarrow{s_R} \Gamma_R.$$ 

Set $\Gamma_i = [s_1, \ldots, s_i](\Gamma_0)$; also write $\Gamma_R = D(\Gamma_0)$. We will say $\Gamma_R$ is derived from $\Gamma_0$ by $D$. Let $B_R(\Gamma_0)$ be the set of all trivalent graphs (up to equivalence) that are derived from $\Gamma_0$ by a derivation of length at most $R$. Our main result is Theorem 1 restated below.
Theorem 3.1. For any \( \Gamma_0 \in G_n(G) \),
\[
\lim_{R \to \infty} \frac{\log \# B_R(\Gamma_0)}{R} \leq 3 \log 4.
\]

The main idea behind the proof of Theorem 3.1 is to give a normal form for a derivation. This was done in [STT92] for unlabeled graphs. It turns out labeled graphs do not pose significant additional difficulties. We are also careful to obtain an explicit upper bound for the growth rate of \( \# B_R(\Gamma) \).

![Figure 2. Configurations of splits.](image-url)

A split \( \Gamma \xrightarrow{s} \Gamma' \) defines a bijection between the edges of \( \Gamma \) and \( \Gamma' \). For any edge \( e \) in \( \Gamma \), let \( s(e) \) be its image in \( \Gamma' \). We say \( \text{supp}(s) \) and its vertices are destroyed by \( s \), and \( s(\text{supp}(s)) \) and its vertices are created by \( s \). All other vertices of \( \Gamma \) survive \( s \). An edge of \( \Gamma \) survives \( s \) if all of its vertices survive \( s \).

Given \( \Gamma \xrightarrow{s} \Gamma' \). If a representative of \( \Gamma \) is chosen, then \( s \) naturally induces a representative for \( \Gamma' \). Fix a representative in the equivalent class of \( \Gamma_0 \). This way, for any derivation \( D = [s_1, \ldots, s_R] \), we can inductively define a representative for each \( \Gamma_i = [s_1, \ldots, s_i](\Gamma_0) \).

We will refer to Figure 2 for the following discussion. For each vertex \( v \) of \( \Gamma_0 \), label the half-edges at \( v \) from 1 to 3 so they can be cyclically ordered. Let \( D = [s_1, \ldots, s_R] \) be a derivation. We will cyclically order the half-edges at each vertex of \( \Gamma_i = [s_1, \ldots, s_i](\Gamma_0) \) and label each \( s_i \) as follows. Let \( X \) be a fixed planar binary tree with four valence-1 vertices. The distinguished middle edge of \( X \) is oriented (see Figure 2). Let \( P \) be the planar graph which is the wedge of an interval and an oriented loop (also see Figure 2). Now suppose for \( i \geq 0 \), the half-edges of \( \Gamma_i \) are labeled. Let \( e \) be the support of \( s_{i+1} \) in \( \Gamma_i \). If \( e \) is not a loop, then the cyclic ordering at \( t(e) \) and \( t(e) \) allows us to identify a contractible neighborhood of \( e \) with \( X \). The four configurations in the left column of Figure 2 represent all possible double splits with support the middle edge of \( X \). Record the label of the configuration that \( s_{i+1} \) identifies with; this is the label of \( s_{i+1} \). Similarly, if \( e \) is a loop, then identify a neighborhood of \( e \) with \( P \). Label \( s_{i+1} \) by 0 if \( s_{i+1} \) is a forward split along \( e \) and label \( s_{i+1} \) by 1 otherwise (see Figure 2). Let \( \ell \in \{0, 1, 2, 3\} \) be the
label of \(s_{i+1}\). Note that we always know if \(e\) is a loop or not so there is no confusion with the duplication of the labels 0 and 1. For each vertex \(v\) of \(\Gamma_{i+1}\), if \(v\) is not created by \(s_{i+1}\), then the half-edges at \(v\) will inherit their labels from \(\Gamma_i\); otherwise, label the half-edges at \(v\) from 1 to 3 according to the right side of configuration \(\ell\) in Figure 2.

Let \(D = [s_1, \ldots, s_R]\). Compute the label of each \(s_i\) from above. Fix \(i \geq 0\) and let \(e\) be any edge in \(\Gamma_i\). For \(k > i + 1\), we will say \(e\) survives \([s_{i+1}, \ldots, s_{k-1}]\) if for all \(j = i + 1, \ldots, k - 1\), the image of \(e\) in \(\Gamma_j\) survives \(s_j\). In particular, \(e\) remains the same type from \(\Gamma_i\) to \(\Gamma_{k-1}\). Let \(e_i\) be the preimage of \(\text{supp}(s_k)\) in \(\Gamma_i\). We say \(s_k\) is ready for \(\Gamma_i\) if \(e_i\) survives \([s_{i+1}, \ldots, s_{k-1}]\). In this case, we can apply \(s_k\) to \(\Gamma_i\) with support \(e_i\) using the label of \(s_k\); this is well defined since \(e_i\) is the same type as \(\text{supp}(s_k)\).

Consider
\[\Gamma_{k-2} \xrightarrow{s_{k-1}} \Gamma_{k-1} \xrightarrow{s_k} \Gamma_k.\]
Suppose \(s_k\) is ready for \(\Gamma_{k-2}\). Apply \(s_k\) to \(\Gamma_{k-2}\) and let \(\Gamma'_{k-1}\) be the resulting graph. Propagate the labels of half-edges from \(\Gamma_{k-2}\) to \(\Gamma'_{k-1}\) as before. Since \(e_{k-2}\) and \(e = \text{supp}(s_{k-1})\) are disjoint in \(\Gamma_{k-2}\), \(e\) survives \(s_k\), so we may apply \(s_{k-1}\) to \(\Gamma'_{k-1}\) with support \(s_k(e)\) using the label of \(s_{k-1}\). Let
\[\Gamma_{k-2} \xrightarrow{s_k} \Gamma_{k-1} \xrightarrow{s_{k-1}} \Gamma'_{k}\]
be the derivation obtained by switching the order of \(s_{k-1}\) and \(s_k\). We claim the following:

**Lemma 3.2.** With the same notation as above. If \(s_k\) is ready for \(\Gamma_{k-2}\), then \(s_{k-1}\) and \(s_k\) commute; that is, \(\Gamma_k = \Gamma'_{k}\).

**Figure 3.** If \(s_k\) is ready for position \(k - 2\), then \(s_{k-1}\) and \(s_k\) commute.

**Proof.** For any split \(s\), we say an edge is affected by \(s\) if its label is changed by \(s\). Any split affects at most two edges. Let \(e\) and \(e'\) be the supports of \(s_{k-1}\) and \(s_k\) in \(\Gamma_{k-1}\) respectively. If \(s_{k-1}\) and \(s_k\) do not affect the same edge in \(\Gamma_{k-2}\), then they clearly commute. So let \(a\) be an edge in \(\Gamma_{k-2}\) affected by both \(s_{k-1}\) and \(s_k\). \(a\) must share a vertex with both \(e\) and \(e'\). Since \(e\) and \(e'\) are disjoint, \(a\) cannot be a loop. The proof that the labels of \(s_{k-1} \circ s_k(a)\) and \(s_k \circ s_{k-1}(a)\) are the same now follows.
from considering different cases. The proofs in all cases are similar. Figure 3 shows the case when neither $e$ and $e'$ are loops and $t(e') = i(a)$ and $t(a) = i(e)$. Since $s_k \circ s_{k-1}$ and $s_{k-1} \circ s_k$ affect the edges labels the same way, $\Gamma'_k = \Gamma_k$. 

Let $D = [s_1, \ldots, s_R]$. For any $i \leq k - 1$, if $s_k$ is ready for $\Gamma_i$, then $s_k$ is ready for $\Gamma_j$, for all $i \leq j \leq k - 1$. By applying Lemma 3.2 $k - i$ times, we see that

$$D' = [s_1, \ldots, s_t, s_k, s_{i+1}, \ldots, s_R]$$

is a well-defined derivation and $D(\Gamma_0) = D'(\Gamma_0)$.

Set $D_0 = D$ and let $\Gamma_R = D(\Gamma_0)$. Inductively, we will construct a sequence $D_1, \ldots, D_R$ of derivations such that $D_j(\Gamma_0) = \Gamma_R$ for all $j = 1, \ldots, R$. The final derivation $D_R$ will be called the canonical derivation of $\Gamma_R$ coming from $D_0$.

Let $M = 2n - 2$ be the number of vertices of $\Gamma_0$. Label each vertex of $\Gamma_0$ by a distinct integer from 1 to $M$. Similarly, label the edges of $\Gamma_0$ from 1 to $N$, where $N = 3n - 3$ is the number of edges of $\Gamma_0$.

Suppose for $i \geq 0$, $D_i = [u_1, \ldots, u_i, t_{i+1}, \ldots, t_R]$ has been constructed. Also, suppose the vertices and edges of $\Gamma'_i$ have been labeled, where $\Gamma'_i = [u_1, \ldots, u_i](\Gamma_0)$. Let $U$ be the subset of $\{t_{i+1}, \ldots, t_R\}$ consisting of splits that are ready for $\Gamma'_i$. Pick $t \in U$ such that $t$ destroys the vertex of $\Gamma'_i$ with the lowest label. Set

$$D_{i+1} = [u_1, \ldots, u_i, t, t_{i+1}, \ldots, t_R].$$

Set $u_{i+1} = t$ and let $\Gamma'_{i+1} = [u_1, \ldots, u_{i+1}](\Gamma_0)$. Let $M_i$ and $N_i$ be the maximal vertex and edge label of $\Gamma'_i$. Let $f$ be the edge in $\Gamma'_{i+1}$ created by $t$. Label $f$ by $N_i + 1$. If $f$ is a loop, then label its vertex by $M_i + 1$. If $f$ is not a loop, then label $i(f)$ by $M_i + 1$ and $t(f)$ by $M_i + 2$. All other vertices and edges of $\Gamma'_{i+1}$ will inherit their label from $\Gamma'_i$. This completes the construction.

Since $\Gamma_0$ has $2n - 2$ vertices and at most two vertices are created in each stage, the maximal vertex label of $\Gamma_R$ is at most $2n - 2 + 2R$. Similarly, the maximal edge label of $\Gamma_R$ is at most $3n - 3 + R$. Let

$$V = \{1, \ldots, 2n - 2 + 2R\}, \quad E = \{1, \ldots, 3n - 3 + R\}.$$

Let $F_R$ be the set of all pairs of functions $(\phi, \psi)$, where $\phi: V \to \{0, 1, 2, 3\}$ and $\psi: E \to \{0, 1, 2, 3\}$.

**Proposition 3.3.** There exists an embedding of $B_R(\Gamma_0)$ into $F_R$, hence $\# B_R(\Gamma_0) \leq \# F_R$.

**Proof.** By definition, any $\Gamma \in B_R(\Gamma_0)$ can be obtained from $\Gamma_0$ by a derivation $D$ of length $r \leq R$. Let $D_r$ be the canonical derivation of $\Gamma$ coming from $D$. We now encode $D_r$ by a pair of maps $\phi: V \to \{0, 1, 2, 3\}$ and $\psi: E \to \{0, 1, 2, 3\}$. Set $D_r$

$$\Gamma_0 \xrightarrow{u_1} \Gamma_1 \xrightarrow{u_2} \cdots \xrightarrow{u_r} \Gamma_r.$$

For $i \in V$, let $j$ be the largest index so that $\Gamma_j$ has a vertex $v$ with label $i$. If $j \geq R$ ($j > R$ means no such label exists), then set $\phi(i) = 0$. If $j < R$, then $u_{j+1}$ must destroy $v$, so the support of $u_{j+1}$ is an edge $e$ in $\Gamma_j$ where $v$ is either $i(e)$ or $t(e)$. Define $\phi(i) \in \{1, 2, 3\}$ to be the label of the half-edge of $e$ containing vertex $v$. If $e$ is a loop, then choose $\phi(i)$ to be the label of any half-edge of $e$. For $i \in E$, let
$k$ be the largest index so that $\Gamma_k$ has an edge $e$ of label $i$. If $k \geq R$, then $\psi(i) = 0$. If $k < R$, then $e$ is the support of $t_{k+1}$. In this case, define $\psi(i) \in \{0, \ldots, 3\}$ to be the label of $t_{k+1}$.

To see this is an embedding, we will give a decoding procedure that will recover from $(\phi, \psi)$ the canonical derivation $D_r = [u_1, \ldots, u_r]$ and hence $\Gamma$.

For each $k \geq 0$, suppose $\Gamma'_k$ has been constructed. In $\Gamma'_k$, let $i$ range from 1 to $2n - 2 + 2k$ in order and let $v$ be the vertex in $\Gamma_k$ with label $i$. If $\phi(i) = 0$, then move on to $i + 1$. Otherwise, $\phi(i)$ determines a unique edge $e$, where $v$ is either the initial or terminal vertex of $e$, such that the half-edge of $e$ at $v$ has label $\phi(i)$. We now explain a matching procedure that can occur in two ways. If $e$ is a loop, then we have a match. If $e$ is not a loop, then let $w$ be the other vertex of $e$ with label $i'$. If $\phi(i')$ is exactly the label of the half-edge of $e$ at $w$, then we have a match. In all other cases, there is no match and we move on to $i + 1$. If there is a match, then let $j \in E$ be the label of $e$. The configuration $\psi(j)$ determines a split supported on $e$ which we call $u'_{k+1}$, and applying $u'_{k+1}$ to $\Gamma_k$ yields $\Gamma'_{k+1}$. Proceed this way until $k = r$ results in a derivation $D'$.

To see that $D' = D_r$. Let $e$ be the support of $u_k$. The encoding procedure ensures that the values of $\phi$ on the labels of $i(e)$ and $t(e)$ determine $e$, and hence a match, and the value of $\psi$ on the label of $e$ agrees with $u_k$. Furthermore, since only the splits that are ready at $\Gamma_{k-1}$ can determine a match, and $u_k$ is the unique one among them that destroys the vertex of $\Gamma_{k-1}$ with the smallest label, the match coming from $u_k$ will always be the first match the decoding procedure finds. This shows $\Gamma_k = \Gamma'_k$ and $t_k = t'_k$ for all $k$. Therefore, $B_R(\Gamma_0)$ embeds in $F_R$. $\square$

Since $\# F_R = 4^{5n-5+3R}$, $\lim_{R \to \infty} \frac{\log \# F_R}{R} = 3 \log 4$. This completes the proof of Theorem 3.1.

### 3.1. Triangulations of a surface

Let $\Sigma = \Sigma_{g,p}$ be an oriented surface of genus $g$ with $p$ punctures. For any $n \geq p$, let $\mathcal{T}_n = \mathcal{T}_n(\Sigma)$ be the set of homotopy classes of triangulations of $\Sigma$ with $n$ vertices (the punctures of $\Sigma$ are always vertices of triangles.) A natural transformation of a triangulation is a diagonal flip. Given $T \in \mathcal{T}_n$. Let $\Delta$ and $\Delta'$ be two triangles in $T$ that share a common edge $E$. View $\Delta \cup \Delta'$ as a quadrilateral with diagonal $E$. Replacing $E$ by the other diagonal in the quadrilateral yields a triangulation $T' \in \mathcal{T}_n$. Call this process a (diagonal) flip about $E$ and denote it by $T \xrightarrow{d} T'$. Let $B_R(T)$ be the set of all triangulations of $\Sigma$ obtained from $T$ by a sequence of at most $R$ diagonal flips.

Fix $T_0 \in \mathcal{T}_n$. Dual to $T_0$ is a trivalent graph $\Gamma_0$ obtained by putting a vertex in the interior of each triangle and connecting two vertices by an edge when two triangles share an edge. Pick a vertex $x_0$ in $\Gamma_0$ and let $G = \pi_1(\Sigma, x_0)$. We will label each edge of $\Gamma_0$ by an element of $G$ as follows. Orient the edges of $\Gamma_0$ arbitrarily. Pick a spanning tree $K_0$ in $\Gamma_0$ and label each edge of $K_0$ by 1. Each edge $e$ in the complement of $K_0$ represents an element of $G$: connect the end points of $e$ to $x_0$ along $K_0$ and orient the resulting closed curve so that it matches the orientation of $e$ in $\Gamma_0$. Now, label $e$ by the element that this closed curve represents in $G$. This makes a $G$-labeled graph $\Gamma_0 \in \mathcal{G}_m(G)$, where $m = 2g + n - 1$ is the rank of $\Gamma_0$.

By a pair $(\Gamma, f)$ we will mean a $G$-labeled graph $\Gamma \in \mathcal{G}_m(G)$ together with an embedding $f : \Gamma \to \Sigma$. We say a pair $(\Gamma, f)$ is well-labeled if for any closed path $p$ in $\Gamma$, the product of labels of edges along $p$ is in the conjugacy class in $G$ represented
by $f(p)$. By construction, $(\Gamma_0, i)$, where $\Gamma_0$ is the dual graph to $T_0$ and $i$ is the inclusion map, is well-labeled.

**Proposition 3.4.** There exists an embedding of $B_R(T_0)$ into $B_R(\Gamma_0)$, hence $\#B_R(T_0) \leq \#B_R(\Gamma_0)$.

**Proof.** Assume $T$ and a well-labeled dual graph $(\Gamma, i)$ are given. Consider a flip $T \xrightarrow{d} T'$ about an edge $E$ in $T$ and let $e$ be the edge in $\Gamma$ dual to $E$. Identify the quadrilateral containing $E$ and a contractible neighborhood of $e$ dual to the quadrilateral with the left-hand side of Figure 4. We define a split move $\Gamma \xrightarrow{s} \Gamma'$ supported on $e$, where $(\Gamma', i)$ is also embedded in $\Sigma$, as indicated by Figure 4. We refer to $s$ as the split associated to the flip $d$. Note that a closed path $p$ in $\Gamma$ can naturally be mapped to a homotopic closed path $p'$ in $\Gamma'$ and the products of labels along edges of $p$ and $p'$ are the same. That is, the pair $(\Gamma', i)$ is still well-labeled.

We now define a map from $B_R(T_0)$ to $B_R(\Gamma_0)$. For any $T \in B_R(T_0)$, choose an arbitrary sequence of flips $T_0 \xrightarrow{d_1} T_1 \xrightarrow{d_2} \cdots \xrightarrow{d_R} T_R = T$ and let $\Gamma_0 \xrightarrow{s_1} \Gamma_1 \xrightarrow{s_2} \cdots \xrightarrow{s_R} \Gamma_R$ be the associated sequence of dual splits as constructed above. The map from $B_R(T_0)$ to $B_R(\Gamma_0)$ is defined by sending $T_R$ to $(\Gamma_R, i)$ and then to $\Gamma_R$.

We show that this map is injective. In fact, for triangulations $T$ and $T'$ and dual labeled graphs $(\Gamma, i)$ and $(\Gamma', i)$ that are well-labeled, we show that if $\Gamma$ and $\Gamma'$ are equivalent $G$-labeled graphs, then there exists a homeomorphism of $\Sigma$ homotopic to the identity taking $T$ to $T'$.

Since $\Gamma$ and $\Gamma'$ are equivalent, there is a graph isomorphism $\phi: \Gamma \to \Gamma'$ such that the label of any edge $e \in \Gamma$ matches the label of $\phi(e) \in \Gamma'$. Since $\Gamma$ and $\Gamma'$ are dual graphs to the triangulations $T$ and $T'$ respectively, we can build a homeomorphism $f: \Sigma \to \Sigma$ mapping a triangle of $T$ associated to a vertex $v \in \Gamma$ to the triangle of $T'$ associated to the vertex $\phi(v)$. To show that $\phi$ is homotopic to identity, it is sufficient to show that every closed path $q$ in $\Sigma$ is homotopic to $f(q)$.

First perturb $q$ so it misses the vertices of $T$. Then $q$ can be pushed to a closed path $p$ in $\Gamma$. Since $q$ is homotopic to $p$, we have $f(q)$ is homotopic to $p' = f(p)$. But the product of labels along the closed paths $p$ and $p'$ are identical, which means $p$ and $p'$ represent the same conjugacy class in $G$ and hence are homotopic. This finishes the proof.

Theorem E from the introduction now follows from Theorem 3.1 and Proposition 3.4.
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Department of Mathematics, University of Toronto, Toronto, Canada
E-mail address: rafi@math.toronto.edu

Department of Mathematics, University of Oklahoma, Norman, Oklahoma 73019-0315
E-mail address: jing@math.ou.edu