QUANTUM DIMENSIONS AND FUSION RULES
FOR PARAFERMION VERTEX OPERATOR ALGEBRAS

CHONGYING DONG AND QING WANG

(Communicated by Kailash Misra)

ABSTRACT. The quantum dimensions and the fusion rules for the parafermion vertex operator algebra associated to the irreducible highest weight module for the affine Kac-Moody algebra $A_1^{(1)}$ of level $k$ are determined.

1. Introduction

Parafermion vertex operator algebra is the commutant of the Heisenberg vertex operator algebra in the affine vertex operator algebra. It comes from a special kind of coset construction [23]. Precisely speaking, let $L_{\widehat{g}}(k,0)$ be the level $k$ integrable highest weight module with weight zero for affine Kac-Moody algebra $\widehat{g}$ associated to a finite-dimensional simple Lie algebra $g$. Then $L_{\widehat{g}}(k,0)$ contains the Heisenberg vertex operator subalgebra generated by the Cartan subalgebra $h$ of $g$. The commutant $K(g,k)$ of the Heisenberg vertex operator subalgebra in $L_{\widehat{g}}(k,0)$ is the parafermion vertex operator algebra. The structure and representation theory of parafermion vertex operator algebras has been widely studied since 2009 (see [14], [15], [17], [18], [4]). In particular, [17] and [18] show that the role of parafermion vertex operator algebra $K(sl_2,k)$ in the parafermion vertex algebra $K(g,k)$ is similar to the role of 3-dimensional simple Lie algebra $sl_2$ played in Kac-Moody Lie algebras. We denote $K(sl_2,k)$ by $K_0$ in this paper. Moreover, it was proved that $K_0$ coincides with a certain W-algebra in [14] and [15]. Later in [4], the $C_2$-cofiniteness of parafermion vertex operator algebra $K(g,k)$ has been established by proving the $C_2$-cofiniteness of parafermion vertex operator algebra $K_0$, and irreducible modules for parafermion vertex operator algebra $K_0$ were also classified therein. Recently, the rationality of $K_0$ was established [5] by identifying the parafermion vertex operator algebra $K_0$ with certain W-algebra [3]. Also see [16] for the rationality of parafermion vertex operator algebra $K(g,k)$ for any $g$ and classification of irreducible modules.

The notion of quantum dimensions of modules for vertex operator algebras was introduced in [8]. It was proved therein for rational and $C_2$-cofinite vertex operator algebras that quantum dimensions do exist. In this paper, we first determine the
quantum dimensions for the parafermion vertex operator algebra $K_0$. Then by using the important formula obtained in [8] which shows that quantum dimensions are multiplicative under tensor product, we give the fusion rules for the parafermion vertex operator algebra $K_0$. The quantum dimensions of irreducible modules and the fusion rules for any $K(g, k)$ were determined recently in [2].

The paper is organized as follows. In Section 2, we recall some results about parafermion vertex operator algebra $K_0$. In Section 3, after reviewing the notion and properties of quantum dimensions of modules for vertex operator algebras, we give the quantum dimensions of parafermion vertex operator algebra $K_0$. In the final section, we obtained the fusion rules of parafermion vertex operator algebra $K_0$ by using the results of quantum dimensions of parafermion vertex operator algebra $K_0$.

2. Preliminary

In this section, we recall from [14], [15] and [4] some basic results on the parafermion vertex operator algebra associated to the irreducible highest weight module of the affine Kac-Moody algebra $A_1^{(1)}$ of level $k$ with $k \geq 2$ being an integer. First we recall the notion of the parafermion vertex operator algebra.

We are working in the setting of [14]. Let $\{h, e, f\}$ be a standard Chevalley basis of $sl_2$ with brackets $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Let $\hat{sl}_2 = sl_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be the affine Lie algebra associated to $sl_2$. Let $k \geq 2$ be an integer and let

$$V(k, 0) = V_{\hat{sl}_2}(k, 0) = \text{Ind}_{sl_2 \otimes \mathbb{C}[t] \oplus \mathbb{C}K}^{sl_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K}$$

be an induced $\hat{sl}_2$-module such that on $1 = 1$, $sl_2 \otimes \mathbb{C}[t]$ acts as 0 and $K$ acts as $k$. We denote by $a(n)$ the operator on $V(k, 0)$ corresponding to the action of $a \otimes t^n$. Then

$$[a(m), b(n)] = [a, b](m + n) + m\langle a, b \rangle \delta_{m+n, 0} k$$

for $a, b \in sl_2$ and $m, n \in \mathbb{Z}$. It is well known [22] that there is a vertex operator algebra structure on $V(k, 0)$ and it has a unique maximal ideal $\mathcal{J}$, which is generated by a weight $k + 1$ vector $e(-1)^{k+1}$. The quotient algebra $L(k, 0) = V(k, 0)/\mathcal{J}$ is the simple vertex operator algebra associated to an affine Lie algebra $\hat{sl}_2$ of type $A_1^{(1)}$ with level $k$. The subspace $V_0^h(k, 0)$ of $V(k, 0)$ spanned by $h(-i_1) \cdots h(-i_p)1$ for $i_1 \geq \cdots \geq i_p \geq 1$ and $p \geq 0$ is a vertex operator subalgebra of $V(k, 0)$ associated to the Heisenberg algebra. The parafermion vertex operator algebra $K_0$ is defined as the commutant of $V_0^h(k, 0)$ in $L(k, 0)$, that is,

$$K_0 = \{v \in L(k, 0) \mid h(m)v = 0 \text{ for } m \geq 0 \}.$$

It was proved that $K_0$ is a simple vertex operator algebra and the irreducible $K_0$-modules $M_i^{i,j}$ for $0 \leq i \leq k, 0 \leq j \leq k - 1$ were constructed in [14]. Note that $K_0 = M_0^{0,0}$. It was also proved that $M_i^{i,j} \cong M_{k-i,k-i-j}$ as $K_0$-module in [14] Theorem 4.4] and moreover, Theorem 8.2 in [4] showed that the $k(k+1)$ irreducible $K_0$-modules $M_i^{i,j}$ for $0 \leq i \leq k, 0 \leq j \leq i - 1$ constructed in [14] form a complete set of isomorphism classes of irreducible $K_0$-module. Moreover, $K_0$ is $C_2$-cofinite [4] and rational [16].

Recall from [14] that $L = \mathbb{Z} \alpha_1 + \cdots + \mathbb{Z} \alpha_k$ with $\langle \alpha_p, \alpha_q \rangle = 2 \delta_{pq}$. $V_L = M(1) \otimes \mathbb{C}[L]$ is the lattice vertex operator algebra associated with the lattice $L$. Let $\gamma = \alpha_1 + \cdots + \alpha_k$. Thus $\langle \gamma, \gamma \rangle = 2k$. It is well known that the vertex operator algebra
associated with a positive definite even lattice is rational \[7\]. And any irreducible module for the lattice vertex operator algebra \(V_{\mathbb{Z}^2}\) is isomorphic to one of \(V_{\mathbb{Z}^2+n}\), \(0 \leq n \leq 2k - 1\ \[7\]. Let \(L(k, i)\) for \(0 \leq i \leq k\) be the irreducible modules for the rational vertex operator algebra \(L(k, 0)\). The following result was due to \[14\].

**Lemma 2.1.** \(L(k, i) = \bigoplus_{j=0}^{k-1} V_{\mathbb{Z}^2+(i-2j) \gamma/2k} \otimes M^{i,j}\) as \(V_{\mathbb{Z}^2}\)-modules.

3. **Quantum dimensions for irreducible \(K_0\)-modules**

In this section, we first recall the notion and some basic facts about quantum dimension from \[8\]. Then we determine the quantum dimensions of the irreducible \(K_0\)-modules.

Let \((V, Y, 1, \omega)\) be a vertex operator algebra (see \[21, \ 27\]). We define weak module, module and admissible module following \[10, \ 11\]. Let \(W\{z\}\) denote the space of \(W\)-valued formal series in arbitrary complex powers of \(z\) for a vector space \(W\).

**Definition 3.1.** A **weak \(V\)-module** \(M\) is a vector space with a linear map \(Y_M : V \rightarrow (\text{End}M)\{z\}\)

\[
v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \text{End}M),
\]

which satisfies the following conditions for \(u, v \in V, w \in M\):

\[
u_n w = 0 \quad \text{for} \quad n \gg 0,
\]

\[
Y_M(1, z) = \text{Id}_M,
\]

\[
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(u, z_1) Y_M(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y_M(v, z_2) Y_M(u, z_1)
\]

\[
= \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(u, z_0) v, z_2),
\]

where \(\delta(z) = \sum_{n \in \mathbb{Z}} z^n\).

Recall that the canonical central element \(C\) of the Virasoro algebra acts on \(V\) as a scalar \(c \in \mathbb{C}\), called the central charge of \(V\). We remark that the component operators of \(Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}\) still satisfy the Virasoro algebra relation on \(M\) \[10\] with the same central charge \(c\).

**Definition 3.2.** A **\(V\)-module** is a weak \(V\)-module \(M\) which carries a \(\mathbb{C}\)-grading \(M = \bigoplus_{\lambda \in \mathbb{C}} M_{\lambda}\), where \(M_{\lambda} = \{w \in M|L(0)w = \lambda w\}\). Moreover, we require that \(\dim M_{\lambda}\) is finite and for fixed \(\lambda\), \(M_{\lambda+n} = 0\) for all small enough integers \(n\).

**Definition 3.3.** An **admissible \(V\)-module** \(M = \bigoplus_{n \in \mathbb{Z}^+} M(n)\) is a \(\mathbb{Z}^+\)-graded weak module such that \(u_m M(n) \subset M(wt u - m - 1 + n)\) for homogeneous \(u \in V\) and \(m, n \in \mathbb{Z}\).

**Definition 3.4.** (1) A vertex operator algebra \(V\) is called **rational** if the admissible module category is semisimple.

(2) \(V\) is called \(C_2\)-cofinite if \(\dim V/C_2(V) < \infty\) where \(C_2(V)\) is a subspace of \(V\) spanned by \(u_{-2}v\) for \(u, v \in V\).

(3) \(V\) is of CFT type if \(V = \bigoplus_{n \geq 0} V_n\) and \(\dim V_0 = 1\).

The following lemma about rational vertex operator algebras is well known \[12\].
Lemma 3.5. If $V$ is rational and $M$ is an irreducible admissible $V$-module, then
(1) $M$ is a $V$-module and there exists a $\lambda \in \mathbb{C}$ such that $M = \bigoplus_{n \in \mathbb{Z}^+} M_{\lambda+n}$ where $M_{\lambda} \neq 0$. And $\lambda$ is called the conformal weight of $M$.
(2) There are only finitely many irreducible admissible $V$-modules up to isomorphism.

Let $M = \bigoplus_{\lambda \in \mathbb{C}} M_{\lambda}$ be a $V$-module. Set $M' = \bigoplus_{\lambda \in \mathbb{C}} M_{\lambda}^*$, the restricted dual of $M$, where $M_{\lambda}^* = \text{Hom}_{\mathbb{C}}(M_{\lambda}, \mathbb{C})$. It was proved in [20] that $M'$ is naturally a $V$-module where the vertex operator $Y_{M'}(v, z)$ is defined for $v \in V$ via

$$\langle Y_{M'}(v, z)f, u \rangle = \langle f, Y_M(e^{zL(1)}(-z^{-2})L(0)v, z^{-1})u \rangle,$$

where $\langle f, w \rangle = f(w)$ is the natural pairing $M' \times M \to \mathbb{C}$. The $V$-module $M'$ is called the contragredient module of $M$. A $V$-module $M$ is called self-dual if $M$ and $M'$ are isomorphic $V$-modules. The following result was proved in [26].

Lemma 3.6. Let $V$ be a simple vertex operator algebra such that $L(1)V(1) \neq V(0)$. Then $V$ is self-dual.

Remark 3.7. Note that the weight one subspace of $K_0$ is zero. By using lemma 3.6, parafermion vertex operator algebra $K_0$ is obviously self-dual.

Now we recall from [20] the notions of intertwining operators and fusion rules.

Definition 3.8. Let $(V, Y)$ be a vertex operator algebra and let $(W^1, Y^1)$, $(W^2, Y^2)$ and $(W^3, Y^3)$ be $V$-modules. An intertwining operator of type

$$\begin{pmatrix} W^3 \\ W^1 W^2 \end{pmatrix}$$

is a linear map

$$I(\cdot, z) : W^1 \to \text{Hom}(W^2, W^3) \{z\}$$

$$w^1 \to I(w^1, z) = \sum_{n \in \mathbb{C}} w^1_n z^{-n-1},$$

satisfying for $v \in V, w^i \in W^i$ with $i = 1, 2$ and $\lambda \in \mathbb{C}$:
(1) $w^1_{\lambda+n} w^2 = 0$ for $n \in \mathbb{Z}$ sufficiently large.
(2) $I(L(-1)w^1, z) = \frac{d}{dz} I(w^1, z)$.
(3) The Jacobi identity holds:

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y^3(v, z_1) I(w^1, z_2) - z_0^{-1} \delta \left( \frac{-z_2 + z_1}{z_0} \right) I(w^1, z_2) Y^2(v, z_1)$$

$$= z_2^{-1} \left( \frac{z_1 - z_0}{z_2} \right) I(Y^1(w^1, z_0) v, z_2).$$

The space of all intertwining operators of type $\begin{pmatrix} W^3 \\ W^1 W^2 \end{pmatrix}$ is denoted by

$$I_V \left( \begin{pmatrix} W^3 \\ W^1 W^2 \end{pmatrix} \right).$$

Let $N_{W^1, W^2}^{W^3} = \dim I_V \left( \begin{pmatrix} W^3 \\ W^1 W^2 \end{pmatrix} \right)$. These integers $N_{W^1, W^2}^{W^3}$ are usually called the fusion rules.
Definition 3.9. Let $V$ be a vertex operator algebra, and $W^1$, $W^2$ be two $V$-modules. A module $(W, I)$, where $I \in I_V \left( \begin{array}{c} W^1 \\ W^2 \end{array} \right)$, is called a tensor product (or fusion product) of $W^1$ and $W^2$ if for any $V$-module $M$ and $Y \in I_V \left( \begin{array}{c} M \\ W^1 \\ W^2 \end{array} \right)$, there is a unique $V$-module homomorphism $f : W \to M$, such that $Y = f \circ I$. As usual, we denote $(W, I)$ by $W^1 \boxtimes_V W^2$.

Remark 3.10. It is well known that if $V$ is rational, then for any two irreducible $V$-modules $W^1$ and $W^2$, the fusion product $W^1 \boxtimes_V W^2$ exists and $W^1 \boxtimes_V W^2 = \sum W N_{W^1, W^2} W$, where $W$ runs over the set of equivalence classes of irreducible $V$-modules.

Now we recall some notions about quantum dimensions.

Definition 3.11. Let $M = \bigoplus_{n \in \mathbb{Z}^+} M_{\lambda+n}$ be a $V$-module, the formal character of $M$ is defined as $\text{ch}_q M = \text{tr}_M q^{L(0) - c/24} = q^{\lambda - c/24} \sum_{n \in \mathbb{Z}^+} (\dim M_{\lambda+n}) q^n$, where $c$ is the central charge of the vertex operator algebra $V$ and $\lambda$ is the conformal weight of $M$.

We now assume that the vertex operator algebra $V$ is rational and $C_2$-cofinite. It is proved [29] that $\text{ch}_q M$ converges to a holomorphic function in the domain $|q| < 1$. We denote the holomorphic function $\text{ch}_q M$ by $Z_M(\tau)$. Here and below, $\tau$ is in the upper half plane $\mathbb{H}$ and $q = e^{2\pi i \tau}$.

Let $M^0, \cdots, M^d$ be the inequivalent irreducible $V$-modules with corresponding conformal weights $\lambda_i$ and $M^0 \cong V$. It is proved in [29] that for any $i$

$$Z_{M^i} \left( \frac{-1}{\tau} \right) = \sum_{j=0}^d S_{i,j} Z_{M^j}(\tau).$$

The complex matrix $S = (S_{i,j})_{i,j=0}^d$ is called the $S$-matrix.

The following definition of quantum dimension was introduced in [8].

Definition 3.12. Let $V$ be a vertex operator algebra and $M$ a $V$-module such that $Z_V(\tau)$ and $Z_M(\tau)$ exist. The quantum dimension of $M$ over $V$ is defined as $\text{qdim}_V M = \lim_{y \to 0} \frac{Z_M(iy)}{Z_V(iy)}$, where $y$ is real and positive.

The following result was obtained in [8 Lemma 4.2].

Lemma 3.13. Let $V$ be a simple, rational and $C_2$-cofinite vertex operator algebra of CFT type with $V \cong V'$. Let $M^i$ for $0 \leq i \leq d$ be the inequivalent irreducible $V$-modules with corresponding conformal weights $\lambda_i$ and $M^0 \cong V$. Assume $\lambda_0 = 0$ and $\lambda_i > 0 \ \forall i \neq 0$. Then $\text{qdim}_V M^i = \frac{S_{i,0}}{S_{0,0}}$. 
We now also assume that $V$ is of CFT type with $V \cong V'$, the conformal weights $\lambda_i$ of $M^i$ are positive for all $i > 0$. From Remark 3.7 and statements in Section 2, the parafermion vertex operator algebra $K_0$ satisfies all the assumptions.

The following result shows that the quantum dimensions are multiplicative under tensor product $\otimes$.

**Proposition 3.14.** Let $V$ and $M_i$ for $0 \leq i \leq d$ be as in Lemma 3.13. Then

$$qdim_V( M^i \otimes M^j ) = qdim_V M^i \cdot qdim_V M^j$$

for $i, j = 0, \cdots, d$.

Before giving the main result of this section, we recall the following character formula of irreducible $K_0$-modules $M^{i,j}$ which is given in [6][24]:

$$\text{ch} M^{i,j} = \eta(\tau)c_{i-2j}^j(\tau),$$

where $\eta(\tau) = q^{-1/24} \prod_{n\geq 1}(1-q^n)$ is the Dedekind $\eta$-function and $c_{i-2j}^j(\tau)$ are the string functions [25]. Note that $k, l$ and $m$ in [24] are $k, i$ and $i - 2j$, respectively in our notation.

**Theorem 3.15.** The quantum dimensions for all irreducible $K_0$-modules $M^{m,n}$ are

$$qdim_{K_0} M^{m,n} = \frac{\sin \frac{\pi(m+1)}{k+2}}{\sin \frac{\pi}{k+2}}$$

for $0 \leq m \leq k, 0 \leq n \leq k - 1$, where $M^{m,n}$ are the irreducible modules of $K_0$ constructed in [14].

**Proof.** Let $M^{m,n}(\tau)$ denote the character of $M^{m,n}$ for $0 \leq m \leq k, 0 \leq n \leq k - 1$. The $S$-modular transformation of characters has the following form [24], [25]:

$$M^{m,n}(-\frac{1}{\tau}) = \sum_{m',n'} S_{m,n}^{m',n'} M^{m',n'}(\tau),$$

where $S_{m,n}^{m',n'} = (k(k+2))^{-\frac{1}{2}} \exp \frac{i\pi(m-2n)(m'-2n')}{k} \sin \frac{\pi(m+1)(m'+1)}{k+2}$. From [14] and [4], we see that $K_0$ has $k(k+2)$ irreducible modules $M^{m,n}$ with the conformal weights

$$\lambda_{m,n} = \frac{1}{2k(k+2)}(k(m-2n) - (m-2n)^2 + 2kn(m-n+1))$$

for $0 \leq m \leq k, 0 \leq n \leq m - 1$. It is easy to check that $\lambda_{k,0} = 0$ and $\lambda_{m,n} > 0$ for $(m,n) \neq (k,0)$. Thus by using Lemma 3.13, we have

$$qdim_{K_0} M^{m,n} = \frac{S_{m,n}^{0,0}}{S_{0,0}^{0,0}} = \frac{\sin \frac{\pi(m+1)}{k+2}}{\sin \frac{\pi}{k+2}}.$$

\[ \square \]

4. Fusion rule for irreducible $K_0$-modules

In this section, we give the fusion rules for irreducible $K_0$-modules. First we fix some notation. Let $W^1, W^2, W^3$ be irreducible $K_0$-modules. In the following, we use $I\left(\begin{array}{c}
W^3 \\
W^1 W^2
\end{array}\right)$ to denote the space $I_{K_0}\left(\begin{array}{c}
W^3 \\
W^1 W^2
\end{array}\right)$ of all intertwining
operators of type $\left( \begin{array}{c} W^3 \\ W^1 W^2 \end{array} \right)$, and use $W^1 \otimes W^2$ to denote the fusion product $W^1 \otimes_{K_0} W^2$ for simplicity.

We recall the fusion rules for the affine vertex operator algebra of type $A_1^{(1)}$ [28] for later use.

**Lemma 4.1.**

$$L(k, i) \otimes_{L(k, 0)} L(k, j) = \sum_l L(k, l),$$

where $|i - j| \leq l \leq i + j$, $i + j + l \in 2\mathbb{Z}$, $i + j + l \leq 2k$.

**Theorem 4.2.** The fusion rule for the irreducible modules of parafermion vertex operator algebra $K_0$ is as follows:

$$(4.1) \quad M^{i,i'} \otimes M^{j,j'} = \sum_l M^{l,\frac{1}{2}(2i' - i + 2j' - j + i)},$$

where $\bar{a}$ means the residue of the integer $a$ modulo $k$, $0 \leq i, j \leq k, 0 \leq i', j' \leq k - 1$, $|i - j| \leq l \leq i + j$, $i + j + l \in 2\mathbb{Z}$, $i + j + l \leq 2k$. Moreover, with fixed $i, i', j, j'$, $M^{\frac{1}{2}(2i' - i + 2j' - j + l)}$ for $|i - j| \leq l \leq i + j$, $i + j + l \in 2\mathbb{Z}$, $i + j + l \leq 2k$ are inequivalent irreducible modules.

**Proof.** We take $V = L(k, 0), U = V_{Z_\gamma} \otimes K_0$ in Proposition 2.9 of [1], from Lemma 2.1 we see that

$$\dim I_V \left( \begin{array}{c} L(k, l) \\ L(k, i) L(k, j) \end{array} \right) \leq \dim U \left( V_{Z_\gamma + (\frac{i - 2j'}{2k})\gamma} \otimes M^{i,i'} M^{j,j'} \right)$$

for $0 \leq i, j, l \leq k, 0 \leq i', j' \leq k - 1$. Note that $L(k, l) = \bigoplus_{l' = 0}^{k - 1} V_{Z_\gamma + (\frac{2l'}{2k} - 2l')\gamma} \otimes M^{l,l'}$, thus we have

$$\dim I_U \left( V_{Z_\gamma + (\frac{i - 2j'}{2k})\gamma} \otimes M^{i,i'} M^{j,j'} \right) = \sum_{l' = 0}^{k - 1} D((i', j'), (j, j'), (l, l')),$$

where

$$D((i', j'), (j, j'), (l, l')) = \dim I_U \left( V_{Z_\gamma + (\frac{i - 2j'}{2k})\gamma} \otimes M^{i,i'} M^{j,j'} \right).$$

Using Theorem 2.10 of [1], we have

$$D((i, i'), (j, j'), (l, l')) = \dim I_{V_{Z_\gamma}} \left( V_{Z_\gamma + (\frac{i - 2j'}{2k})\gamma} V_{Z_\gamma + (\frac{i - 2j'}{2k})\gamma} \right) \cdot \dim I_{K_0} \left( M^{i,i'} M^{j,j'} \right).$$

Recall from [9] that the fusion rule for lattice vertex operator algebras is given by

$$V_{Z_\gamma + \lambda} \otimes_{V_{Z_\gamma}} V_{Z_\gamma + \mu} = V_{Z_\gamma + \lambda + \mu}.$$
for \(\lambda, \mu \in (\mathbb{Z}\gamma)\), where \((\mathbb{Z}\gamma)\) is the dual lattice of \(\mathbb{Z}\gamma\). Using this together with the fusion rule for the affine vertex operator algebra given in Lemma 4.1, we see that if \(l' \neq \frac{l-i+2l'-j+2j'}{2}\), then

\[
D((i,i'),(j,j'),(l,l')) = 0.
\]

Thus for any \(l\) satisfying \(|i-j| \leq l \leq i+j\), \(i+j+l \in 2\mathbb{Z}\), \(i+j+l \leq 2k\), we have

\[
\dim I_{K_0}\left(\frac{M^l,\frac{1}{2}(2l'-i+2l'-j+l)}{M^i,i' M^j,j'}\right) \geq 1.
\]

In the following, we use the quantum dimension of the irreducible modules over \(K_0\) to prove that

\[
\dim I_{K_0}\left(\frac{M^l,\frac{1}{2}(2l'-i+2l'-j+l)}{M^i,i' M^j,j'}\right) = 1.
\]

From Proposition 3.14, we know

\[
\text{qdim}_{K_0}\left(M^{i,i'} \mathbin{\boxtimes} M^{j,j'}\right) = \text{qdim}_{K_0} M^{i,i'} \cdot \text{qdim}_{K_0} M^{j,j'}.
\]

We also have the fusion product

\[
M^{i,i'} \mathbin{\boxtimes} M^{j,j'} = \sum_{0 \leq l \leq k, 0 \leq l' \leq l-1} N_{(i,i'),(j,j')}^{(l,l')} M^{l,l'}.
\]

Theorem 3.15 shows that

\[
\text{qdim}_{K_0} M^{i,i'} = \frac{\sin \frac{\pi i+j+2}{k+2} \cdot \sin \frac{\pi l}{k+2}}{\sin \frac{\pi}{k+2} \cdot \sin \frac{\pi i+j}{k+2}} = \sum_{l} \sin \frac{\pi l}{k+2} \cdot \sum_{l} \sin \frac{\pi i+j}{k+2} - \sin \frac{\pi i+j}{k+2} - \sin \frac{\pi l}{k+2} - \sin \frac{\pi l}{k+2}.
\]

for \(0 \leq i, j \leq k, 0 \leq i', j' \leq k-1, |i-j| \leq l \leq i+j\), \(i+j+l \in 2\mathbb{Z}\), \(i+j+l \leq 2k\). This identity is equivalent to the following identity:

\[
\cos \frac{\pi (i+j+2)}{k+2} \cos \frac{\pi (j-i)}{k+2} = \sum_{l} \cos \frac{\pi l}{k+2} - \cos \frac{\pi l}{k+2}.
\]

We note that if \(i+j < k\), then \(l_{\text{min}} = i-j, l_{\text{max}} = i+j\), thus (4.2) holds. If \(i+j > k\), \(l_{\text{min}} = i-j, l_{\text{max}} = i+j - 2n\) for some \(n\) satisfying that \(i+j-2n+i+j = 2k\), that is, \(l_{\text{max}} = 2k - i-j,\) thus (4.2) also holds.

Now we prove that the modules \(M^{l,\frac{1}{2}(2l'-i+2l'-j+l)}\) appearing in the sum (4.1) are not isomorphic to each other. If \(M^{l,\frac{1}{2}(2l'-i+2l'-j+l)} \cong M^{l_1,\frac{1}{2}(2l_1'-i+2l_1'-j+l)}\) for some \(l, l_1\) satisfying \(|i-j| \leq l \leq i+j, i+j+l \in 2\mathbb{Z}, i+j+l \leq 2k, i+j+l \leq 2k\), and \(l_1 \leq 2k\), we know that \(M^{i,j} \cong M^{k-i,k-i+j}\) as \(K_0\)-module for \(0 \leq i \leq k, 0 \leq j \leq k-1\) and the \(\frac{k(k+1)}{2}\) irreducible \(K_0\)-modules \(M^{i,j}\) for \(0 \leq i \leq k, 0 \leq j \leq i-1\) exhaust all the isomorphism classes of irreducible \(K_0\)-modules, it follows that \(l_1 = k-l\) and

\[
\frac{k-l+\frac{1}{2}(2l'-i+2j'-j+k-l)}{2} = \frac{1}{2}(2l'-i+2j'-j+k-l),
\]

which is impossible by a direct calculation. This proves the assertion.
References


Department of Mathematics, University of California, Santa Cruz, California 95064
School of Mathematical Sciences, Xiamen University, Xiamen 361005, People’s Republic of China