OPTIMISTIC LIMIT OF THE COLORED JONES POLYNOMIAL 
AND THE EXISTENCE OF A SOLUTION

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Abstract. For the potential function of a link diagram induced by the optimistic limit of the colored Jones polynomial, we show the existence of a solution of the hyperbolicity equations by directly constructing it. This construction is based on the shadow-coloring of the conjugation quandle induced by a boundary-parabolic representation $\rho : \pi_1(L) \to \text{PSL}(2, \mathbb{C})$. This gives us a very simple and combinatorial method to calculate the complex volume of $\rho$.

1. Introduction

The optimistic limit of the Kashaev invariant appeared in [10] when the volume conjecture was first introduced. It can be considered as an informal way to predict the actual limit of the Kashaev invariant using a potential function, and it has been widely considered the actual limit by general physicists. Since the appearance, much work has been done to provide a mathematically rigorous definition in [8], [14], [4] and [2].

Let $L$ be a link. The author, with several collaborators, defined a potential function combinatorially from the link diagram in [4] and showed that the evaluation of the function at a saddle point becomes the complex volume of a certain representation. Furthermore, it was shown in [2] that, if we modify the potential function slightly using the information of a given boundary-parabolic representation $\rho : \pi_1(L) \to \text{PSL}(2, \mathbb{C})$, then the set of hyperbolicity equations always has the solution which induces $\rho$ up to conjugation. This solution was directly constructed from the shadow-coloring of $\mathcal{P}$ induced by $\rho$, where $\mathcal{P}$ is the conjugation quandle consisting of parabolic elements of $\text{PSL}(2, \mathbb{C})$.

On the other hand, the colored Jones polynomial was shown to be a generalization of the Kashaev invariant in [12], and the optimistic limit of the colored Jones polynomial was also developed in [13], [6], [7] and [4]. In particular, following the idea of [4], another potential function $W(w_1, \ldots, w_n)$ from the optimistic limit of

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$\rho$ Boundary-parabolic representation means the images of the meridians and the longitudes of the cusp tori are all parabolic elements of $\text{PSL}(2, \mathbb{C})$.

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the colored Jones polynomial was defined in [1] combinatorially from the link diagram. At first, we fix the link diagram $D$. Then we assign variables $w_1, \ldots, w_n$ to regions of the diagram and define a potential function of a crossing $j$ as in Figure 1.

\[
W_j := -\text{Li}_2\left(\frac{w_c}{w_b}\right) - \text{Li}_2\left(\frac{w_c}{w_d}\right) + \text{Li}_2\left(\frac{w_a w_c}{w_b w_d}\right) + \text{Li}_2\left(\frac{w_d}{w_a}\right) + \frac{\pi^2}{6} + \log \frac{w_b}{w_a} \log \frac{w_d}{w_a}
\]

(a) Positive crossing

\[
W_j := \text{Li}_2\left(\frac{w_c}{w_b}\right) + \text{Li}_2\left(\frac{w_c}{w_d}\right) - \text{Li}_2\left(\frac{w_a w_c}{w_b w_d}\right) - \text{Li}_2\left(\frac{w_b}{w_a}\right) - \frac{\pi^2}{6} - \log \frac{w_b}{w_a} \log \frac{w_d}{w_a}
\]

(b) Negative crossing

**Figure 1. Potential function of the crossing $j$**

Then the potential function of $D$ is defined by

\[
W(w_1, \ldots, w_n) := \sum_{j : \text{crossings}} W_j,
\]

and we modify it to

\[
W_0(w_1, \ldots, w_n) := W(w_1, \ldots, w_n) - \sum_{k=1}^{n} \left( w_k \frac{\partial W}{\partial w_k} \right) \log w_k.
\]

Also, from the potential function $W(w_1, \ldots, w_n)$, we define a set of equations

\[
\mathcal{I} := \left\{ \exp\left( w_k \frac{\partial W}{\partial w_k} \right) = 1 \bigg| k = 1, \ldots, n \right\}.
\]

Then, from Proposition 1.1 of [1], $\mathcal{I}$ becomes the set of hyperbolicity equations of the five-term triangulation of $S^3 \setminus (L \cup \{\text{two points}\})$ in Figure 7. Here, hyperbolicity equations are the equations that determine the complete hyperbolic structure of the triangulation, and which consist of gluing equations of edges and the completeness condition. According to Yoshida’s construction in Section 4.5 of [1], a solution $w = (w_1, \ldots, w_n)$ of $\mathcal{I}$ determines the boundary-parabolic representation

\[
\rho_w : \pi_1(S^3 \setminus (L \cup \{\text{two points}\})) = \pi_1(S^3 \setminus L) \rightarrow \text{PSL}(2,\mathbb{C}).
\]

\footnote{We always assume the diagram does not contain a trivial knot component which has only over-crossings or under-crossings or no crossing. If this happens, then we change the diagram of the trivial component slightly. For example, applying the Reidemeister second move to make different types of crossings or the Reidemeister first move to add a kink is good enough. This assumption is necessary to guarantee that the five-term triangulation becomes a topological triangulation of $S^3 \setminus (L \cup \{\text{two points}\})$.}
Theorem 1.2 of [1] shows that, for the solution \( w \) of \( \mathcal{I} \),
\[
W_0(w) \equiv i(\text{vol}(\rho_w) + i\text{cs}(\rho_w)) \pmod{\pi^2},
\]
where \( \text{vol}(\rho_w) \) and \( \text{cs}(\rho_w) \) are the volume and the Chern-Simons invariant of the representation \( \rho_w \) defined in [15], respectively. We call \( \text{vol}(\rho_w) + i\text{cs}(\rho_w) \) the complex volume of \( \rho_w \).

Although the potential function in [1] determines the complex volume very nicely, there are two major problems.

1. When \( \mathcal{I} \) has no solution, we cannot do anything with the potential function \( W \).
2. We do not know whether the set \( \{\rho_w \mid w \text{ is a solution of } \mathcal{I}\} \) contains all possible boundary-parabolic representations \( \rho : \pi_1(L) \to \text{PSL}(2, \mathbb{C}) \).

In the case of the optimistic limit of the Kashaev invariant, we solved these problems in [2] by using the shadow-coloring of the conjugation quandle \( \mathcal{P} \) defined in [9]. The purpose of this article is to solve the above problems by constructing a solution of \( \mathcal{I} \) using the same method.

**Theorem 1.1.** For any boundary-parabolic representation \( \rho : \pi_1(L) \to \text{PSL}(2, \mathbb{C}) \) and any link diagram \( D \) of \( L \), there exists the solution \( w^{(0)} \) of \( \mathcal{I} \) satisfying \( \rho_{w^{(0)}} = \rho \), up to conjugation.

The exact formula of \( w^{(0)} \) is in (2.7), which is amazingly simple. Using this solution, we define the colored Jones version of the optimistic limit of \( \rho \) by \( W_0(w^{(0)}) \).

Then, from (1.1), the optimistic limit is always the complex volume of \( \rho \). The author believes calculating this optimistic limit is the most convenient method to obtain the complex volume of a given boundary-parabolic representation because everything is combinatorially obtained from the link diagram.

Note that the potential function and the triangulation of the Kashaev version in [4] were slightly modified in [2] according to the information of the representation \( \rho \) so as to guarantee the existence of a solution. However, we do not need any modification of the colored Jones version in [1], which is a major advantage of this article. Actually, if a link diagram contains Figure 2 or a kink, then the unmodified Kashaev version does not have solutions. (See [1] for the proof. The modification needs extra information other than the link diagram.) Due to the existence of a solution of the colored Jones version for any diagram and any \( \rho \), several combinatorial applications are possible. See [3] and [5] for those applications.

**Figure 2.** Example that the Kashaev version does not have a solution
2. CONSTRUCTION OF THE SOLUTION

2.1. Reviews on shadow-coloring. This section is a summary of definitions and properties we need. For complete descriptions, see [9], especially Section 5.

Let $\mathcal{P}$ be the set of parabolic elements of $\text{PSL}(2, \mathbb{C}) = \text{Isom}^+(\mathbb{H}^3)$. We identify $\mathbb{C}^2 \setminus \{0\}/\pm$ with $\mathcal{P}$ by

$$
\left( \begin{array}{c}
\alpha \\
\beta 
\end{array} \right) \mapsto \left( \begin{array}{cc}
1 + \alpha\beta & -\alpha^2 \\
\beta^2 & 1 - \alpha\beta 
\end{array} \right),
$$

and define operation $\ast$ by

$$
\left( \begin{array}{c}
\alpha \\
\beta 
\end{array} \right) \ast \left( \begin{array}{c}
\gamma \\
\delta 
\end{array} \right) = \left( \begin{array}{cc}
1 + \gamma\delta & -\gamma^2 \\
\delta^2 & 1 - \gamma\delta 
\end{array} \right) \left( \begin{array}{c}
\alpha \\
\beta 
\end{array} \right) \in \mathcal{P},
$$

where this operation is actually induced by the conjugation as follows:

$$
\left( \begin{array}{c}
\alpha \\
\beta 
\end{array} \right) \ast \left( \begin{array}{c}
\gamma \\
\delta 
\end{array} \right) \in \mathcal{P} \leftrightarrow \left( \begin{array}{c}
\gamma \\
\delta 
\end{array} \right) \left( \begin{array}{c}
\alpha \\
\beta 
\end{array} \right) \left( \begin{array}{c}
\gamma \\
\delta 
\end{array} \right)^{-1} \in \text{PSL}(2, \mathbb{C}).
$$

The inverse operation $\ast^{-1}$ is expressed by

$$
\left( \begin{array}{c}
\alpha \\
\beta 
\end{array} \right) \ast^{-1} \left( \begin{array}{c}
\gamma \\
\delta 
\end{array} \right) = \left( \begin{array}{cc}
1 - \gamma\delta & \gamma^2 \\
-\delta^2 & 1 + \gamma\delta 
\end{array} \right) \left( \begin{array}{c}
\alpha \\
\beta 
\end{array} \right) \in \mathcal{P},
$$

and $(\mathcal{P}, \ast)$ becomes a conjugation quandle. Here, quandle means, for any $a, b, c \in \mathcal{P}$, the map $\ast b : a \mapsto a \ast b$ is bijective and

$$
a \ast a = a, \ (a \ast b) \ast c = (a \ast c) \ast (b \ast c)
$$

hold.

We define the Hopf map $h : \mathcal{P} \rightarrow \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ by

$$
\left( \begin{array}{c}
\alpha \\
\beta 
\end{array} \right) \mapsto \frac{\alpha}{\beta}.
$$

For an oriented link diagram $D$ of $L$ and the boundary-parabolic representation $\rho$, we assign arc-colors $a_1, \ldots, a_n \in \mathcal{P}$ to arcs of $D$ so that each $a_k$ is the image of the meridian around the arc under the representation $\rho$. Note that, in Figure 3, we have

$$
a_m = a_l \ast a_k.
$$

![Figure 3. Arc-coloring](image)

We also assign region-colors $s_1, \ldots, s_m \in \mathcal{P}$ to regions of $D$ satisfying the rule in Figure 4. Note that, if an arc-coloring is given, then a choice of one region-color determines all the other region-colors.
Lemma 2.1. Consider the arc-coloring induced by the boundary-parabolic representation \( \rho : \pi_1(L) \to \text{PSL}(2, \mathbb{C}) \). Then, for any triple \((a_k, s, s \ast a_k)\) of an arc-color \(a_k\) and its surrounding region-colors \(s\), \(s \ast a_k\) as in Figure 4, there exists a region-coloring satisfying 
\[
h(a_k) \neq h(s) \neq h(s \ast a_k) \neq h(a_k).
\]

Proof. Although this lemma was already proved in Proposition 2 of [9] and Lemma 2.4 of [2], we write down the proof again for the reader’s convenience.

For the given arc-colors \(a_1, \ldots, a_n\), we choose region-colors \(s_1, \ldots, s_m\) so that
\[
\{h(s_1), \ldots, h(s_m)\} \cap \{h(a_1), \ldots, h(a_n)\} = \emptyset.
\]
This is always possible because the number of \(h(s_1)\) satisfying \(h(s_1) \in \{h(a_1), \ldots, h(a_n)\}\) is finite, and \(h(s_2), \ldots, h(s_m)\) are uniquely determined by \(h(s_1)\). Therefore, the number of \(h(s_1)\) satisfying
\[
\{h(s_1), \ldots, h(s_m)\} \cap \{h(a_1), \ldots, h(a_n)\} \neq \emptyset
\]
is finite, but we have infinitely many choices for the value \(h(s_1) \in \mathbb{CP}^1\).

Now consider Figure 4 and assume \(h(s \ast a_k) = h(s)\). Then we obtain
\[
h(s \ast a_k) = \hat{a}_k(h(s)) = h(s),
\]
where \(\hat{a}_k : \mathbb{CP}^1 \to \mathbb{CP}^1\) is the M"obius transformation
\[
\hat{a}_k(z) = \frac{(1 + \alpha_k \beta_k)z - \alpha_k^2}{\beta_k z + (1 - \alpha_k \beta_k)}
\]
of \(a_k = \left( \begin{array}{c} \alpha_k \\ \beta_k \end{array} \right) \). Then (2.3) implies \(h(s)\) is the fixed point of \(\hat{a}_k\), which means \(h(a_k) = h(s)\) which contradicts (2.2). \(\square\)

We remark that Lemma 2.1 holds for any choice of region-colors with only finitely many exceptions. Therefore, if we want to find a region-color explicitly, we choose \(h(s_1) \notin \{h(a_1), \ldots, h(a_n)\}\) and then decide \(h(s_2), \ldots, h(s_m)\) using
\[
h(s_1 \ast a) = \hat{a}(h(s_1)), \ h(s_1 \ast^{-1} a) = \hat{a}^{-1}(h(s_1)).
\]
If this choice does not satisfy Lemma 2.1 then we change \(h(s_1)\) and do the same process again. This process is very simple and it ends in finite steps. If proper \(h(s_1)\) is chosen, then we can easily extend it to \(s_1 \in \mathcal{P}\) and decide proper region-coloring \(\{s_1, \ldots, s_m\}\).

The arc-coloring induced by \(\rho\) together with the region-coloring satisfying Lemma 2.1 is called the shadow-coloring induced by \(\rho\). We choose \(p \in \mathcal{P}\) so that
\[
h(p) \notin \{h(a_1), \ldots, h(a_n), h(s_1), \ldots, h(s_m)\}.
\]
The geometric shape of the five-term triangulation will be determined by the shadow-coloring induced by $\rho$ and $p$ in the following section.

From now on, we fix the representatives of shadow-colors in $\mathbb{C}^2\setminus\{0\}$. As mentioned in [2], the representatives of some arc-colors may satisfy (2.1) up to sign, in other words, $a_m = \pm (a_l \ast a_k)$ in Figure 3. However, the representatives of the region-colors are uniquely determined due to the fact that $s \ast (\pm a) = s \ast a$ for any region-color $s$ and any arc-color $a$.

For $a = \left( \frac{\alpha_1}{\alpha_2} \right)$ and $b = \left( \frac{\beta_1}{\beta_2} \right)$ in $\mathbb{C}^2\setminus\{0\}$, we define the determinant $\det(a, b)$ by

$$\det(a, b) := \det \left( \frac{\alpha_1}{\alpha_2}, \frac{\beta_1}{\beta_2} \right) = \alpha_1 \beta_2 - \beta_1 \alpha_2.$$ 

Then the determinant satisfies $\det(a \ast c, b \ast c) = \det(a, b)$ for any $a, b, c \in \mathbb{C}^2\setminus\{0\}$. Furthermore, for $v_0, \ldots, v_3 \in \mathbb{C}^2\setminus\{0\}$, the cross-ratio can be expressed using the determinant by

$$[h(v_0), h(v_1), h(v_2), h(v_3)] = \frac{\det(v_0, v_3) \det(v_1, v_2)}{\det(v_0, v_2) \det(v_1, v_3)}.$$ 

(For the proof of these, see Lemma 2.9 of [2].)

2.2. Geometric shape of the five-term triangulation. The five-term triangulation is obtained by placing octahedra on each crossing and subdividing them into five tetrahedra. (See Section 3 of [1] for an exact description.)

Consider the crossing in Figure 5 with the shadow-coloring induced by $\rho$, and let $w_a, \ldots, w_d$ be the variables assigned to regions of $D$.

![Figure 5. Crossing with shadow-coloring and region-variables](image)

We place tetrahedra at each crossing of $D$ and assign coordinates as in Figure 6 so as to make them hyperbolic ideal ones. As a matter of fact, Figure 6 is the same as Figure 11 of [2] without orientation, which was used only for a degenerate crossing. Interestingly, this subdivision is good enough for our purpose whether the crossing is degenerate or not.

---

A crossing is called degenerate when $h(a_k) = h(a_l)$ holds for the two arcs of the crossing with arc-colors $a_k$ and $a_l$. 

---

3}
Lemma 2.2. All the tetrahedra in Figure [6] are non-degenerate.

Proof. The proof is trivial because the shadow-coloring we are considering satisfies Lemma 2.1 and all endpoints of edges are adjacent, as \( h(a_k), h(s), h(s \ast a_k) \) in Figure [4] or one of them is \( h(p) \).

According to Yoshida’s construction in Section 4.5 of [11], the shape of the triangulation according to the coordinates in Figure [5] determines a boundary-parabolic representation \( \rho' : \pi_1(L) \rightarrow PSL(2, \mathbb{C}) \). However, \( \rho' \) equals \( \rho \) up to conjugation because, due to the Poincaré polyhedron theorem, \( \pi_1(L) \) is generated by face-pairing maps. In Figure [6] the face-pairing maps are the isomorphisms induced by Möbius transformations of \( a_1, \ldots, a_n \in PSL(2, \mathbb{C}) \). Therefore, two representations \( \rho \) and \( \rho' \) send generators to the same elements \( a_1, \ldots, a_n \), which means \( \rho = \rho' \) up to conjugate.

To make the five-term triangulation, we glue the face \( \{ h(a_k), h(s), h(s \ast a_k) \} \) to \( \{ h(a_k), h(s \ast a_k), h((s \ast a_l) \ast a_k) \} \) by sending the tetrahedron \( \{ h(a_k), h(s), h(s \ast a_l), h(p) \} \) by the isomorphism induced by \( a_k \). After applying 2-3 moves along the edge \( \{ h(p \ast a_k), h(p) \} \), we obtain Figure [7]. Here we assigned the vertex-orientation according to Figure 9 of [1].

Proposition 2.3. All the tetrahedra in Figure [7] are non-degenerate.

Proof. All the edges of the tetrahedra already appeared in Lemma 2.2 except \( \{ h(p \ast a_k), h(p) \} \), so it is enough to show that \( h(p \ast a_k) \neq h(a_k) \).
Assume $h(p \ast a_k) = h(a_k)$. Then

\[(2.6) \quad h(p \ast a_k) = \hat{a}_k(h(p)) = h(p),\]

where $\hat{a}_k : \mathbb{CP}^1 \to \mathbb{CP}^1$ is the Möbius transformation of $a_k$ defined in (2.4). Then (2.6) implies $h(p)$ is the fixed point of $\hat{a}_k$, which means $h(a_k) = h(p)$. This contradicts the definition (2.5) of $p$. \qed

2.3. Formula of the solution $\mathbf{w}^{(0)}$. Consider the crossing in Figure 5 and the tetrahedra in Figure 7. For the positive crossing, we assign shape parameters to the edges as follows:

- $\frac{w_d}{w_a}$ to $(h(a_k), h(s \ast a_k))$ of $(h(p \ast a_k), h(p), h(a_k), h(s \ast a_k))$,
- $\frac{w_b}{w_c}$ to $(h(a_k), h((s \ast a_l) \ast a_k))$ of $-(h(p \ast a_k), h(p), h(a_k), h((s \ast a_l) \ast a_k))$,
- $\frac{w_a}{w_d}$ to $(h(p), h(a_l \ast a_k))$ of $(h(p), h(a_l \ast a_k), h(s \ast a_k), h((s \ast a_l) \ast a_k))$ and
- $\frac{w_d}{w_c}$ to $(h(p), h(a_l))$ of $-(h(p), h(a_l), h(s), h(s \ast a_l))$, respectively.
On the other hand, for the negative crossing, we assign shape parameters to the edges as follows:

- \( \frac{w_a}{w_b} \) to \( (h(a_k), h((s \ast a_l) \ast a_k)) \) of \( -(h(p), h(p \ast a_k), h(a_k), h((s \ast a_l) \ast a_k)) \),
- \( \frac{w_c}{w_d} \) to \( (h(a_k), h(s \ast a_k)) \) of \( (h(p), h(p \ast a_k), h(a_k), h(s \ast a_k)) \),
- \( \frac{w_b}{w_d} \) to \( (h(p), h(a_l)) \) of \( -(h(p), h(a_l), h(s \ast a_l), h(s)) \), respectively.

According to Proposition 1.1 of [1], \( \mathcal{I} \) becomes the set of hyperbolicity equations of the five-term triangulation with the above shape parameters.

For a region of \( D \) with region-color \( s_k \) and region-variable \( w_k \), we define

\[
 w_k^{(0)} := \det(p, s_k).
\]

Then, by the definition of \( p \), we know \( w_k^{(0)} \neq 0 \). Furthermore, direct calculation shows \( w^{(0)} = (w_1^{(0)}, \ldots, w_n^{(0)}) \) is a solution of \( \mathcal{I} \). Specifically, for the first two cases of the positive crossing, the shape parameters assigned to edges \( (h(a_k), h(s \ast a_k)) \) and \( (h(a_k), h((s \ast a_l) \ast a_k)) \) are the cross-ratios

\[
 [h(p \ast a_k), h(p), h(a_k), h(s \ast a_k)] = \frac{\det(p \ast a_k, s \ast a_k) \det(p, a_k)}{\det(p \ast a_k, a_k) \det(p, s \ast a_k)} = \frac{w_d^{(0)}}{w_a^{(0)}},
\]

\[
 [h(p \ast a_k), h(p), h(a_k), h((s \ast a_l) \ast a_k)]^{-1} = \frac{\det(p \ast a_k, a_k) \det(p, (s \ast a_l) \ast a_k) \det(p, a_k)}{\det(p \ast a_k, (s \ast a_l) \ast a_k) \det(p, a_k)} = \frac{w_b^{(0)}}{w_c^{(0)}},
\]

respectively, and all the other cases can be verified in the same way. The proof of Theorem 1.1 follows from the above and the discussion below Lemma 2.2.

3. Examples

We consider the same examples in Section 4 of [2].

3.1. Figure-eight knot \( 4_1 \). For the figure-eight knot diagram in Figure 8, let the representatives of the shadow-coloring be

\[
 a_1 = \left( \begin{array}{c} 0 \\ t \end{array} \right), \quad a_2 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad a_3 = \left( \begin{array}{c} -t \\ 1 + t \end{array} \right), \quad a_4 = \left( \begin{array}{c} -t \\ t \end{array} \right),
\]

\[
 s_1 = \left( \begin{array}{c} 1 \\ 1 \end{array} \right), \quad s_2 = \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad s_3 = \left( \begin{array}{c} -t - 1 \\ t + 2 \end{array} \right), \quad s_4 = \left( \begin{array}{c} -2t - 1 \\ 2t + 3 \end{array} \right),
\]

\[
 s_5 = \left( \begin{array}{c} -2t - 1 \\ t + 4 \end{array} \right), \quad s_6 = \left( \begin{array}{c} 1 \\ t + 2 \end{array} \right), \quad p = \left( \begin{array}{c} 2 \\ 1 \end{array} \right),
\]

where \( t \) is a solution of \( t^2 + t + 1 = 0 \), and let \( \rho : \pi_1(4_1) \to \operatorname{PSL}(2, \mathbb{C}) \) be the boundary-parabolic representation determined by \( a_1, \ldots, a_4 \).
The potential function $W(w_1, \ldots, w_6)$ of Figure 8 is

$$W = \begin{cases} 
\text{Li}_2\left(\frac{w_1}{w_2}\right) + \text{Li}_2\left(\frac{w_1}{w_4}\right) - \text{Li}_2\left(\frac{w_1 w_3}{w_2 w_4}\right) - \text{Li}_2\left(\frac{w_2}{w_3}\right) - \text{Li}_2\left(\frac{w_4}{w_3}\right) + \frac{\pi^2}{6} - \log \frac{w_2}{w_3} \log \frac{w_4}{w_3} \\
\text{Li}_2\left(\frac{w_3}{w_2}\right) + \text{Li}_2\left(\frac{w_3}{w_6}\right) - \text{Li}_2\left(\frac{w_1 w_3}{w_2 w_6}\right) - \text{Li}_2\left(\frac{w_2}{w_1}\right) - \text{Li}_2\left(\frac{w_6}{w_1}\right) + \frac{\pi^2}{6} - \log \frac{w_2}{w_1} \log \frac{w_6}{w_1} \\
- \text{Li}_2\left(\frac{w_4}{w_3}\right) - \text{Li}_2\left(\frac{w_4}{w_5}\right) + \text{Li}_2\left(\frac{w_4 w_6}{w_3 w_5}\right) + \text{Li}_2\left(\frac{w_3}{w_6}\right) + \text{Li}_2\left(\frac{w_5}{w_6}\right) - \frac{\pi^2}{6} + \log \frac{w_3}{w_6} \log \frac{w_5}{w_6} \\
- \text{Li}_2\left(\frac{w_6}{w_1}\right) - \text{Li}_2\left(\frac{w_6}{w_5}\right) + \text{Li}_2\left(\frac{w_4 w_6}{w_1 w_5}\right) + \text{Li}_2\left(\frac{w_1}{w_4}\right) + \text{Li}_2\left(\frac{w_2}{w_4}\right) - \frac{\pi^2}{6} + \log \frac{w_1}{w_4} \log \frac{w_5}{w_4} 
\end{cases}.$$ 

Applying (2.7), we obtain

$$w^{(0)}_1 = \det(p, s_1) = 1, \quad w^{(0)}_2 = \det(p, s_2) = 2, \quad w^{(0)}_3 = \det(p, s_3) = 3t + 5, \quad w^{(0)}_4 = \det(p, s_4) = 6t + 7, \quad w^{(0)}_5 = \det(p, s_5) = 4t + 9, \quad w^{(0)}_6 = \det(p, s_6) = 2t + 3,$$

and $(w^{(0)}_1, \ldots, w^{(0)}_6)$ becomes a solution of $\mathcal{I} = \{\exp(w_k \frac{\partial W}{\partial w_k}) = 1 \mid k = 1, \ldots, 6\}$. Applying (1.1), we obtain

$$W_0(w^{(0)}_1, \ldots, w^{(0)}_6) \equiv i(\text{vol}(p) + i \text{cs}(p)) \pmod{\pi^2},$$

and numerical calculation verifies it by

$$W_0(w^{(0)}_1, \ldots, w^{(0)}_6) = \begin{cases} 
i(2.0299 \cdots + 0 i) = i(\text{vol}(4_1) + i \text{cs}(4_1)) & \text{if } t = \frac{-1 - \sqrt{3} i}{2}, \\
i(-2.0299 \cdots + 0 i) = i(-\text{vol}(4_1) + i \text{cs}(4_1)) & \text{if } t = \frac{-1 + \sqrt{3} i}{2}. 
\end{cases}$$
3.2. Trefoil knot $3_1$. For the trefoil knot diagram in Figure 9, let the representatives of the shadow-coloring be

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a_3 = a_4 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$s_1 = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad s_3 = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}, \quad s_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$s_5 = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}, \quad s_6 = \begin{pmatrix} -2 & 3 \\ 1 & 1 \end{pmatrix}, \quad p = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

and let $\rho : \pi_1(3_1) \to \text{PSL}(2, \mathbb{C})$ be the boundary-parabolic representation determined by $a_1, a_2, a_3, a_4$.

The potential function $W(w_1, \ldots, w_6)$ of Figure 9 is

$$W = \left\{-\text{Li}_2\left(\frac{w_4}{w_1}\right) - \text{Li}_2\left(\frac{w_3}{w_5}\right) + \text{Li}_2\left(\frac{w_2 w_3}{w_1 w_5}\right) + \text{Li}_2\left(\frac{w_2}{w_5}\right) + \text{Li}_2\left(\frac{w_4}{w_2}\right) - \frac{\pi^2}{6} + \log \frac{w_1}{w_2} \log \frac{w_6}{w_5}\right\}$$

$$+ \left\{-\text{Li}_2\left(\frac{w_2}{w_1}\right) - \text{Li}_2\left(\frac{w_2}{w_5}\right) + \text{Li}_2\left(\frac{w_2 w_4}{w_1 w_5}\right) + \text{Li}_2\left(\frac{w_1}{w_4}\right) + \text{Li}_2\left(\frac{w_5}{w_4}\right) - \frac{\pi^2}{6} + \log \frac{w_1}{w_4} \log \frac{w_6}{w_5}\right\}$$

$$+ \left\{-\text{Li}_2\left(\frac{w_4}{w_1}\right) - \text{Li}_2\left(\frac{w_4}{w_5}\right) + \text{Li}_2\left(\frac{w_3 w_4}{w_1 w_5}\right) + \text{Li}_2\left(\frac{w_1}{w_3}\right) + \text{Li}_2\left(\frac{w_5}{w_3}\right) - \frac{\pi^2}{6} + \log \frac{w_1}{w_3} \log \frac{w_6}{w_3}\right\}$$

$$+ \left\{\text{Li}_2\left(\frac{w_1}{w_4}\right) + \text{Li}_2\left(\frac{w_1}{w_6}\right) - \text{Li}_2\left(\frac{w_1^2}{w_4 w_6}\right) - \text{Li}_2\left(\frac{w_1}{w_5}\right) - \text{Li}_2\left(\frac{w_4}{w_1}\right) + \frac{\pi^2}{6} - \log \frac{w_4}{w_1} \log \frac{w_6}{w_1}\right\}.$$
and numerical calculation verifies it by
\[ W_0(w^{(0)}_1, \ldots, w^{(0)}_6) = i(0 + 1.6449 \ldots i), \]
where \( \text{vol}(3_1) = 0 \) holds trivially and \( 1.6449 \ldots = \frac{\pi^2}{6} \) holds numerically.

REFERENCES


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