

ARE LINES MUCH BIGGER THAN LINE SEGMENTS?

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ABSTRACT. We pose the following conjecture:

(\star) *If A is the union of line segments in \mathbb{R}^n , and B is the union of the corresponding full lines, then the Hausdorff dimensions of A and B agree.*

We prove that this conjecture would imply that every Besicovitch set (compact set that contains line segments in every direction) in \mathbb{R}^n has Hausdorff dimension at least $n - 1$ and (upper) Minkowski dimension n . We also prove that conjecture (\star) holds if the Hausdorff dimension of B is at most 2, so in particular it holds in the plane.

1. INTRODUCTION

Let A be the union of line segments in \mathbb{R}^n , and let B be the union of the corresponding full lines. Can B be much bigger than A ? According to Lebesgue measure *yes*: Nikodym's classical construction [16] is a subset F of the unit square $(0, 1)^2$ with Lebesgue measure 1 such that for each $x \in F$ there is a line l_x through x intersecting F in the single point x . Thus if we take inside $(0, 1)^2$ an open subsegment of each l_x with endpoint x , then the union of these open line segments must have Lebesgue measure zero (since it is contained in $(0, 1)^2 \setminus F$) but by extending each open line segment only with the endpoints the new union has Lebesgue measure 1 (since it contains F). In higher dimension the line segments can be even pairwise disjoint: Larman [12] constructed a compact set F made up of a disjoint union of closed line segments in \mathbb{R}^3 such that the union of the closed segments have positive Lebesgue measure but the union of the corresponding open segments has Lebesgue measure zero.

In this paper we study the question if extending the line segments to full lines can increase the Hausdorff dimension. We pose the following:

Line Segment Extension Conjecture 1.1. *If A is the union of line segments in \mathbb{R}^n , and B is the union of the corresponding full lines, then the Hausdorff dimensions of A and B agree.*

This conjecture turns out to be closely related to the so-called Kakeya Conjecture. Recall that a compact subset of \mathbb{R}^n is called a *Besicovitch set* if it contains unit line segments in every direction. Besicovitch [1] constructed in 1919 such sets

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of Lebesgue measure zero. The Kakeya Conjecture asserts that every Besicovitch set in \mathbb{R}^n has Hausdorff (or at least upper Minkowski) dimension n . Davies [2] proved this in the plane in 1971.

Although the Kakeya Conjecture is very important in many areas of mathematics, especially in harmonic analysis, because of deep connections with several important conjectures of different areas (see the survey papers [17] and [19] about the Kakeya Conjecture and these connections), all the known partial results in dimension at least 3 are much weaker than the conjecture.

Wolff [18] proved in 1995 that the Hausdorff dimension of a Besicovitch set in \mathbb{R}^n is at least $\frac{n+2}{2}$, which is still the best estimate for the Hausdorff dimension for $n = 3, 4$. The current best estimate for $n \geq 5$ is due to Katz and Tao [9] (2002) and is of the form $An + B$ where $A = 2 - \sqrt{2} = 0.585\dots$ and $B = 4\sqrt{2} - 5 = 0.6\dots$. For (upper) Minkowski dimension slightly better estimates exist for $n = 3$ (Katz-Laba-Tao [10], 2000), for $n = 4$ (Laba-Tao [11], 2001) and for $n \geq 24$ (Katz and Tao [9], 2002).

Clearly, the Line Segment Extension Conjecture would imply that in the Kakeya problem it does not matter if we consider line segments or full lines. But this would not be very interesting, (this is probably what everybody expects) since the constructions for Besicovitch sets of zero Lebesgue measure also work for full lines. It might be much more surprising that the new conjecture would have not only the above consequence but in the following sense it would imply the Kakeya Conjecture itself.

Theorem 1.2. (i) *The Line Segment Extension Conjecture for n would imply that every Besicovitch set in \mathbb{R}^n has Hausdorff dimension at least $n - 1$.*
(ii) *If the Line Segment Extension Conjecture holds for every $n \geq 2$, then every Besicovitch set in \mathbb{R}^n has packing and upper Minkowski dimension n .*

After seeing these strong consequences we show some evidence in support of the new conjecture.

Theorem 1.3. (i) *The Line Segment Extension Conjecture holds in the plane.*
(ii) *More generally, the Line Segment Extension Conjecture holds if the Hausdorff dimension of the union of the line segments is less than 2 or the Hausdorff dimension of the union of the lines is at most 2.*

The above theorems are proved in Section 2. In Section 3 we present some examples that show that certain variants of the Line Segment Extension Conjecture are not valid.

2. PROOF OF THE RESULTS

Notation 2.1. Let \mathcal{S} be a collection of line segments. By extending each line segment to a line we get a collection of lines that we denote by $\mathcal{L}(\mathcal{S})$.

For any $a, b \in \mathbb{R}^{n-1}$ let $l(a, b)$ denote the line $(x_2, \dots, x_{n-1}) = x_1 a + b$.

Let \mathcal{L} be a collection of lines in \mathbb{R}^n such that none of them are orthogonal to the x_1 axis. Then let

$$\mathcal{C}(\mathcal{L}) = l^{-1}(\mathcal{L}) = \{(a, b) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} : l(a, b) \in \mathcal{L}\}.$$

Note that l is continuous, so if \mathcal{L} is Borel, then so is $\mathcal{C}(\mathcal{L}) = l^{-1}(\mathcal{L})$.

Let \dim_H , \dim_P , \dim_M and $\overline{\dim}_M$ denote the Hausdorff dimension, the packing dimension, the Minkowski dimension and the upper Minkowski dimension, respectively; see the definitions in [6] or [15].

Proof of Theorem 1.2. First we present a well-known short argument (see e.g. in [3]) that shows that if for every $n \geq 2$ any Besicovitch set in \mathbb{R}^n has Hausdorff dimension at least $n - 1$, then every Besicovitch set in \mathbb{R}^n has packing and upper Minkowski dimension n , and so (i) implies (ii). We will need the simple observation that the Cartesian product of Besicovitch sets is a Besicovitch set and the well-known facts (see [15]) that for any Borel sets A and B we have $\dim_H(A) \leq \dim_P(A) \leq \overline{\dim}_M(A)$ and $\dim_P(A \times B) \leq \dim_P(A) + \dim_P(B)$. Suppose that B is a Besicovitch set in \mathbb{R}^n with $\dim_P(B) < n$. Then for large enough k we have $k \dim_P(B) < kn - 1$. Then $B^k = B \times \dots \times B$ is a Besicovitch set in \mathbb{R}^{kn} with $\dim_H(B^k) \leq \dim_P(B^k) \leq k \dim_P(B) < kn - 1$, which would contradict our assumption.

Therefore it is enough to prove (i). The claim of (i) follows from the observation that if we consider \mathbb{R}^n inside the n -dimensional projective space, then lines in a given direction correspond to lines through a given point of the “hyperplane at infinity”, so after applying a projective map that takes the hyperplane at infinity to a one codimensional plane of \mathbb{R}^n , then, by extending the line segments of the image of the Besicovitch set, we get a set that contains that hyperplane. To make the argument more accessible we present the same argument more formally, without referring to projective geometry.

For $x \in \mathbb{R} \setminus \{0\}, y \in \mathbb{R}^{n-1}$, let $P(x, y) = (\frac{1}{x}, \frac{y}{x})$. This is a locally Lipschitz map, so it preserves Hausdorff dimension.

We claim that P maps the punched line $l(a, b) \setminus \{(0, b)\}$ to the punched line $l(b, a) \setminus \{(0, a)\}$. Indeed,

$$\begin{aligned} P(l(a, b) \setminus \{(0, b)\}) &= P(\{(t, at + b) : t \in \mathbb{R} \setminus \{0\}\}) \\ &= \left\{ \left(\frac{1}{t}, \frac{at + b}{t} \right) : t \in \mathbb{R} \setminus \{0\} \right\} \\ &= \{(u, a + ub) : u \in \mathbb{R} \setminus \{0\}\} = l(b, a) \setminus \{(0, a)\}. \end{aligned}$$

If B is a Besicovitch set in \mathbb{R}^n , then for every $a \in \mathbb{R}^{n-1}$ the set B contains a subsegment $s(a)$ of a line $l(a, b)$ for some $b \in \mathbb{R}^{n-1}$ and we can clearly guarantee that $s(a) \subset \mathbb{R}^n \setminus (\{0\} \times \mathbb{R}^{n-1})$. Thus, by the claim of the above paragraph $P(s(a))$ is a line segment and its line extension contains $(0, a)$. Therefore if $S = \{s(a) : a \in \mathbb{R}^{n-1}\}$ and $S' = \{P(s(a)) : a \in \mathbb{R}^{n-1}\}$, then $\cup \mathcal{L}(S') \supset \{0\} \times \mathbb{R}^{n-1}$, so $\dim_H \cup \mathcal{L}(S') \geq n - 1$. Since $B \supset \cup S$ and P preserves Hausdorff dimension, we have $\dim_H B \geq \dim_H(\cup S) = \dim_H(\cup S')$. Therefore applying the Line Segment Extending Conjecture to S' would give $\dim_H B \geq n - 1$. □

Although in some arguments one might need measurability assumptions, the following observation shows that we do not need to assume measurability. The author learned this from Márton Elekes [4], and this proof is also essentially due to him.

- Lemma 2.2.** (i) *For any collection \mathcal{S} of closed line segments in \mathbb{R}^n there exists a collection $\mathcal{S}' \supset \mathcal{S}$ of closed line segments with $\dim_H(\cup \mathcal{S}') = \dim_H(\cup \mathcal{S})$ such that $\mathcal{L}(\mathcal{S}')$ is Borel.*
- (ii) *If \mathcal{L} is a Borel collection of lines, then $\cup \mathcal{L}$ is analytic.*

Proof. (i) We can suppose that for some fixed $\delta > 0$ and bounded open set $B \subset \mathbb{R}^n$ each $s \in \mathcal{S}$ is contained in B and has length at least δ since we can write \mathcal{S} as a countable union $\mathcal{S} = \cup_j \mathcal{S}^j$ of such subcollections and if $\mathcal{S}^{j'}$ is good for \mathcal{S}^j , then $\cup_j \mathcal{S}^{j'}$ is good for \mathcal{S} .

Let $A = \cup \mathcal{S}$. Then $A \subset B$. Since for any s any set is contained in a G_δ set of the same s -dimensional Hausdorff (outer) measure (see [8, 471D (b)]), we can take a G_δ set $A' \supset A$ with $A' \subset B$ and $\dim_{\mathbb{H}}(A') = \dim_{\mathbb{H}}(A)$. Write A' in the form $A' = \bigcap_{k=1}^\infty G_k$, where each G_k is open and $B \supset G_1 \supset G_2 \supset \dots$. Let \mathcal{S}_k be the collection of those closed line segments inside G_k that have length at least δ . Let $\mathcal{S}' = \bigcap_{k=1}^\infty \mathcal{S}_k$.

Then $\mathcal{S}' \supset \mathcal{S}$ since each \mathcal{S}_k contains \mathcal{S} . Since for any k , $\cup \mathcal{S}' \subset \cup \mathcal{S}_k \subset G_k$ by construction, we have $\cup \mathcal{S}' \subset \bigcap_k G_k = A'$, so $\dim_{\mathbb{H}} \cup \mathcal{S} \leq \dim_{\mathbb{H}} \cup \mathcal{S}' \leq \dim_{\mathbb{H}} A' = \dim_{\mathbb{H}} \cup \mathcal{S}$, hence $\dim_{\mathbb{H}} \cup \mathcal{S} = \dim_{\mathbb{H}} \cup \mathcal{S}'$.

We claim that $\mathcal{L}(\mathcal{S}') = \bigcap_{k=1}^\infty \mathcal{L}(\mathcal{S}_k)$. Indeed, if $l \in \mathcal{L}(\mathcal{S}')$, then l is the extension of a line segment $s \in \bigcap_k \mathcal{S}_k$, so $l \in \mathcal{L}(\mathcal{S}_k)$ for each k , hence $l \in \bigcap_{k=1}^\infty \mathcal{L}(\mathcal{S}_k)$. Conversely, if $l \in \bigcap_{k=1}^\infty \mathcal{L}(\mathcal{S}_k)$, then for each k the set $l \cap G_k$ contains a closed line segment of length at least δ . Since B is bounded, $B \supset G_1 \supset G_2 \supset \dots$, this implies that there exists a closed line segment $s \subset l$ of length at least δ that is contained in every G_k , so $s \in \mathcal{S}'$ and $l \in \mathcal{L}(\mathcal{S}')$.

Since G_k is open, $\mathcal{L}(\mathcal{S}_k)$ is also open, so $\mathcal{L}(\mathcal{S}') = \bigcap_k \mathcal{L}(\mathcal{S}_k)$ is Borel.

(ii) For $i = 1, \dots, n$ let \mathcal{L}_i contain those lines of \mathcal{L} that are not orthogonal to the i -th axis. Then $\mathcal{L} = \bigcup_{i=1}^n \mathcal{L}_i$, so it is enough to show that each $\cup \mathcal{L}_i$ is analytic. Since each \mathcal{L}_i is Borel, this means that we can assume that $\mathcal{L} = \mathcal{L}_i$ for some i and without loss of generality we can clearly suppose that $i = 1$. Then $\mathcal{C}(\mathcal{L})$ is defined and $\cup \mathcal{L} = F(\mathbb{R} \times \mathcal{C}(\mathcal{L}))$, where $F(t, (a, b)) = (t, ta + b)$, so $\cup \mathcal{L}$ is the continuous image of a Borel set, and so it is analytic. □

Lemma 2.3. *If \mathcal{L} is a Borel collection of nonvertical lines of the plane, then the intersection of $\cup \mathcal{L}$ with almost every vertical line has the same Hausdorff dimension.*

Proof. Let v_t denote the vertical line $x = t$ and let $B = \mathcal{C}(\mathcal{L})$. One can easily check (or see [15, proof of 18.11]) that $(\cup \mathcal{L}) \cap v_t = \{t\} \times \pi_t(B)$, where $\pi_t(x, y) = tx + y$, so $(\cup \mathcal{L}) \cap v_t$ is similar to the projection of B to the line $x = ty$. Then Marstrand's projection theorem [13] gives the claim of the lemma. □

Now we are ready to prove Theorem 1.3. First we prove its first part and then we show how that implies the more general second part.

Theorem 2.4. *Let \mathcal{S} be a collection of line segments in \mathbb{R}^2 and $\mathcal{L}(\mathcal{S})$ be the collection of lines we get by extending each line segment of \mathcal{S} . Then $\dim_{\mathbb{H}}(\cup \mathcal{S}) = \dim_{\mathbb{H}}(\cup \mathcal{L}(\mathcal{S}))$.*

Proof. We can suppose that each $s \in \mathcal{S}$ intersects two fixed segments e and f that are opposite sides of a rectangle since we can decompose \mathcal{S} as a countable union of subcollections with this property. If the result can be applied to each subcollection, then it follows for the union \mathcal{S} . By Lemma 2.2 we can also suppose that $\mathcal{L}(\mathcal{S})$ is Borel and $\cup \mathcal{L}(\mathcal{S})$ is analytic.

Fix an arbitrary u such that $u < \dim_{\mathbb{H}}(\cup \mathcal{L}(\mathcal{S}))$ and the proof will be completed by showing that $\dim_{\mathbb{H}}(\cup \mathcal{S}) \geq u$. If $u = 1$, then this is clear, so we can suppose that $u > 1$. Then we can apply Marstrand's slicing theorem [13] (to $\cup \mathcal{L}(\mathcal{S})$), which states that if $u > 1$ and an analytic subset A of the plane has positive u -dimensional

Hausdorff measure, then in almost every direction, positively many lines meet A in a set of Hausdorff dimension at least $u - 1$. Therefore we get that for almost every unit vector w there exists a set $T \subset \mathbb{R}$ of positive Lebesgue measure such that for any $t \in T$ we have $\dim_{\mathbb{H}}((\cup \mathcal{L}(\mathcal{S})) \cap l_{w,t}) \geq u - 1$, where $l_{w,t}$ is the line $\{a \in \mathbb{R}^2 : a \cdot w = t\}$.

Choose distinct parallel lines l_0 and l_1 so that both of them separate e and f and they are orthogonal to such a nonexceptional unit vector w . Then every segment of \mathcal{S} intersects both l_0 and l_1 . Without loss of generality we can suppose that $w = (1, 0)$, $l_0 = v_0$ and $l_1 = v_1$, where v_t denotes the vertical line $x = t$. Thus by the choice of w and T we have

$$\dim_{\mathbb{H}}((\cup \mathcal{L}(\mathcal{S})) \cap v_t) \geq u - 1, \quad (\forall t \in T).$$

Since T has positive Lebesgue measure, by Lemma 2.3 we get that

$$\dim_{\mathbb{H}}((\cup \mathcal{L}(\mathcal{S})) \cap v_t) \geq u - 1 \quad \text{for almost every } t \in \mathbb{R}.$$

Since every segment of \mathcal{S} intersects both $l_0 = v_0$ and $l_1 = v_1$ we have

$$(\cup \mathcal{L}(\mathcal{S})) \cap v_t = (\cup \mathcal{S}) \cap v_t, \quad (\forall t \in [0, 1]).$$

Thus we obtained that

$$\dim_{\mathbb{H}}((\cup \mathcal{S}) \cap v_t) \geq u - 1 \quad \text{for almost every } t \in [0, 1],$$

which implies (see e.g. [15, Theorem 7.7]) $\dim_{\mathbb{H}}(\cup \mathcal{S}) \geq u$, which completes the proof. □

Corollary 2.5. *Let \mathcal{S} be a collection of line segments in \mathbb{R}^n and $\mathcal{L}(\mathcal{S})$ be the collection of lines we get by extending each line segment of \mathcal{S} . Suppose also that $\dim_{\mathbb{H}}(\cup \mathcal{S}) < 2$ or $\dim_{\mathbb{H}}(\cup \mathcal{L}(\mathcal{S})) \leq 2$. Then $\dim_{\mathbb{H}}(\cup \mathcal{S}) = \dim_{\mathbb{H}}(\cup \mathcal{L}(\mathcal{S}))$.*

Proof. Similarly as in the proof of Theorem 2.4, by decomposing \mathcal{S} , we get that we can suppose that every segment of \mathcal{S} has roughly the same direction and, by Lemma 2.2, we can suppose that $\cup \mathcal{L}(\mathcal{S})$ is analytic.

Let p_W denote the orthogonal projection to the subspace W . By the Marstrand-Mattila projection theorem [14] for almost every 2-dimensional subspace $W \subset \mathbb{R}^n$ we have

$$\dim_{\mathbb{H}} p_W (\cup \mathcal{L}(\mathcal{S})) = \min(2, \dim_{\mathbb{H}} \cup \mathcal{L}(\mathcal{S})).$$

Since the segments of \mathcal{S} have roughly the same direction, we can fix a plane W with the above property so that the projection of every segment of \mathcal{S} into W is a nondegenerated segment. Thus

$$p_W(\cup \mathcal{S}) = \cup \{p_W(s) : s \in \mathcal{S}\} \quad \text{and} \quad p_W(\cup \mathcal{L}(\mathcal{S})) = \cup \mathcal{L}(\{p_W(s) : s \in \mathcal{S}\}).$$

Therefore, applying Theorem 2.4 in the plane W to the projected segments we get that

$$\dim_{\mathbb{H}} (\cup \{p_W(s) : s \in \mathcal{S}\}) = \dim_{\mathbb{H}} (\cup \mathcal{L}(\{p_W(s) : s \in \mathcal{S}\})).$$

Combining the displayed equalities of the above paragraph with obvious inequalities we get that

$$\dim_{\mathbb{H}} \cup \mathcal{L}(\mathcal{S}) \geq \dim_{\mathbb{H}} \cup \mathcal{S} \geq \dim_{\mathbb{H}} p_W(\cup \mathcal{S}) = \min(2, \dim_{\mathbb{H}} \cup \mathcal{L}(\mathcal{S})).$$

Since $\dim_{\mathbb{H}}(\cup \mathcal{S}) < 2$ or $\dim_{\mathbb{H}}(\cup \mathcal{L}(\mathcal{S})) \leq 2$, we get that $\dim_{\mathbb{H}}(\cup \mathcal{S}) = \dim_{\mathbb{H}}(\cup \mathcal{L}(\mathcal{S}))$. □

3. CONCLUDING REMARKS

The following example shows that the Minkowski dimension version of the Line Segment Extension Conjecture would be false even in the plane.

Example 3.1. Let A be the union of the line segments with endpoints $((2m - 1)2^{-n}, 0)$ and $((2m - 1)2^{-n}, 2^{-n})$, where $n = 1, 2, \dots$ and $m = 1, \dots, 2^{m-1}$, and let B be the set we obtain by extending the line segments of A to unit line segments in the square $[0, 1] \times [0, 1]$.

Then B is dense in $[0, 1] \times [0, 1]$, so it has Minkowski dimension 2. But the Minkowski dimension of A is 1, which can be calculated directly or by noticing that A is an inhomogeneous self-similar set (see the definition in [7]) and then the formula of [7, Corollary 2.2] can be applied.

We do not know if such an example exists with the extra requirement that the lengths of the line segments must have a positive lower bound. We do not know if for packing dimension the Line Segment Extension Conjecture would hold at least in the plane.

The next example shows that linearity must be used. The Line Segment Extension Conjecture would be false for general curves instead of line segments.

Example 3.2. We construct an embedding of the infinite rooted quadrarary tree into \mathbb{R}^3 . Consider the points of the form $((2k - 1)2^{-n}, (2l - 1)2^{-n}, 2^{-n})$, where $n = 0, 1, 2, \dots$ and $k, l = 1, \dots, 2^{n-1}$ and join each point $((2k - 1)2^{-n}, (2l - 1)2^{-n}, 2^{-n})$ with a line segment to the four points of the form $((4k - 2 \pm 1)2^{-n-1}, (4l - 2 \pm 1)2^{-n-1}, 2^{-n-1})$.

Then the infinite branches of this tree are rectifiable curves. The union of these curves is a countable union of line segments, so it has Hausdorff dimension 1. But if we extend each of these curves with its limit point, then we get all points of the square $[0, 1] \times [0, 1] \times \{0\}$, so this way we get a set of Hausdorff dimension 2.

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