

AN UNUSUAL CONTINUED FRACTION

DZMITRY BADZIAHIN AND JEFFREY SHALLIT

(Communicated by Matthew A. Papanikolas)

ABSTRACT. We consider the real number σ with continued fraction expansion $[a_0, a_1, a_2, \dots] = [1, 2, 1, 4, 1, 2, 1, 8, 1, 2, 1, 4, 1, 2, 1, 16, \dots]$, where a_i is the largest power of 2 dividing $i + 1$. We show that the irrationality measure of σ^2 is at least $8/3$. We also show that certain partial quotients of σ^2 grow doubly exponentially, thus confirming a conjecture of Hanna and Wilson.

1. INTRODUCTION

By a *continued fraction* we mean an expression of the form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_n}}}$$

or

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_n + \cdots}}}$$

where a_1, a_2, \dots are positive integers and a_0 is an integer. To save space, as usual, we write $[a_0, a_1, \dots, a_n]$ for the first expression and $[a_0, a_1, \dots]$ for the second. For properties of continued fractions, see, for example, [9, 14].

It has been known since Euler and Lagrange that a real number has an ultimately periodic continued fraction expansion if and only if it is a quadratic irrational. But the expansions of some other “interesting” numbers are also known explicitly; for example [25],

$$\begin{aligned} e &= [2, (1, 2n, 1)_{n=1}^\infty] = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \dots], \\ e^2 &= [7, (3n - 1, 1, 1, 3n, 12n + 6)_{n=1}^\infty] = [7, 2, 1, 1, 3, 18, 5, 1, 1, 6, 30, \dots], \\ \tan 1 &= [(1, 2n - 1)_{n=1}^\infty] = [1, 1, 1, 3, 1, 5, 1, 7, \dots]. \end{aligned}$$

These three are examples of “Hurwitz continued fractions”, where there is a “quasiperiod” of terms that grow linearly (see, for example, [15, 19] and [20, pp. 110–123]). By contrast, no simple pattern is known for the expansions of e^3 or e^4 .

Recently there has been some interest in understanding the Diophantine properties of numbers whose continued fraction expansion is generated by a simple

Received by the editors May 4, 2015 and, in revised form, May 26, 2015.

2010 *Mathematics Subject Classification*. Primary 11J70, 11J82; Secondary 11Y65, 11A55.

The research of the first author was supported by EPSRC Grant EP/L005204/1.

computational model, such as a finite automaton. One famous example is the Thue-Morse sequence on the symbols $\{a, b\}$, where $\bar{a} = b$ and $\bar{b} = a$, is given by

$$\mathbf{t} = t_0 t_1 t_2 \cdots = abbabaab \cdots$$

and is defined by

$$t_n = \begin{cases} a, & \text{if } n = 0; \\ t_{n/2}, & \text{if } n \text{ even}; \\ \overline{t_{n-1}}, & \text{if } n \text{ odd.} \end{cases}$$

Queffélec [21] proved that if a, b are distinct positive integers, then the real number $[\mathbf{t}] = [t_0, t_1, t_2, \dots]$ is transcendental. Later, a simpler proof was found by Adamczewski and Bugeaud [3]. Queffélec [22] also proved the transcendence of a much wider class of automatic continued fractions. More recently, in a series of papers, several authors explored the transcendence properties of automatic, morphic, and Sturmian continued fractions [1, 2, 5, 6, 10, 11].

All automatic sequences (and the more general class of morphic sequences) are necessarily bounded. A more general class, allowing unbounded terms, is the k -regular sequences of integers, for integer $k \geq 2$. These are sequences $(a_n)_{n \geq 0}$ where the k -kernel, defined by

$$\{(a_{k^e n + i})_{n \geq 0} : e \geq 0, 0 \leq i < k^e\},$$

is contained in a finitely generated module [7, 8]. We state the following conjecture:

Conjecture 1. *Every continued fraction where the terms form a k -regular sequence of positive integers is transcendental or quadratic.*

In this paper we study a particular example of a k -regular sequence:

$$\mathbf{s} = s_0 s_1 s_2 \cdots = (1, 2, 1, 4, 1, 2, 1, 8, \dots),$$

where $s_i = 2^{\nu_2(i+1)}$ and $\nu_p(x)$ is the p -adic valuation of x (the exponent of the largest power of p dividing x). To see that \mathbf{s} is 2-regular, notice that every sequence in the 2-kernel is a linear combination of \mathbf{s} itself and the constant sequence $(1, 1, 1, \dots)$.

The corresponding real number σ has continued fraction expansion

$$\begin{aligned} \sigma = [\mathbf{s}] &= [s_0, s_1, s_2, \dots] = [1, 2, 1, 4, 1, 2, 1, 8, \dots] \\ &= 1.35387112842988237438889 \dots \end{aligned}$$

The sequence \mathbf{s} is sometimes called the “ruler sequence”, and is sequence A006519 in Sloane’s *Encyclopedia of Integer Sequences* [24]. The decimal expansion of σ is sequence A100338.

Although σ has slowly growing partial quotients (indeed, $s_i \leq i + 1$ for all i), empirical calculation for $\sigma^2 = 1.832967032396003054427219544210417324 \cdots$ demonstrates the appearance of some exceptionally large partial quotients. For example, here are the first few terms:

$$\begin{aligned} \sigma^2 &= [1, 1, 4, 1, 74, 1, 8457, 1, 186282390, 1, 1, 1, 2, 1, 430917181166219, \\ &11, 37, 1, 4, 2, 41151315877490090952542206046, 11, 5, 3, 12, 2, 34, 2, 9, 8, 1, 1, 2, 7, \\ &13991468824374967392702752173757116934238293984253807017, \dots]. \end{aligned}$$

The terms of this continued fraction form sequence A100864 in [24], and were apparently first noticed by Paul D. Hanna and Robert G. Wilson in November 2004. The very large terms form sequence A100865 in [24]. In this note, we explain

the appearance of these extremely large partial quotients. The techniques have some similarity with those of Maillet (see [18] and [9, Section 2.14]).

Throughout the paper we use the following conventions. Given a real irrational number x with partial quotients

$$x = [a_0, a_1, a_2, \dots],$$

we define the sequence of convergents by

$$\begin{aligned} p_{-2} = 0, & & p_{-1} = 1, & & p_n = a_n p_{n-1} + p_{n-2}, & (n \geq 0), \\ q_{-2} = 1, & & q_{-1} = 0, & & q_n = a_n q_{n-1} + q_{n-2}, & (n \geq 0), \end{aligned}$$

and then

$$[a_0, a_1, \dots, a_n] = \frac{p_n}{q_n}.$$

The basic idea of this paper is to use the following classical estimate:

$$(1) \quad \frac{1}{(a_{n+1} + 2)q_n^2} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1}q_n^2}.$$

Therefore, in order to show that some partial quotients of x are huge, it is sufficient to find convergents p_n/q_n of x such that $|x - p_n/q_n|$ is much smaller than q_n^{-2} . We quantify this idea in Section 3.

Furthermore, we use the Hurwitz-Kolden-Frame representation of continued fractions [13, 16, 17] via 2×2 matrices, as follows:

$$(2) \quad M(a_0, \dots, a_n) := \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix}.$$

By taking determinants we immediately deduce the classical identity

$$(3) \quad p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$$

for $n \geq 0$.

Given a finite sequence $z = (a_0, \dots, a_n)$ we let z^R denote the reversed sequence (a_n, \dots, a_0) . A sequence is a *palindrome* if $z = z^R$. By taking the transpose of equation (2) it easily follows that

$$(4) \quad M(a_n, \dots, a_0) := \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{bmatrix}.$$

Hence if

$$[a_0, a_1, \dots, a_n] = p_n/q_n,$$

then

$$[a_n, \dots, a_1, a_0] = p_n/p_{n-1}.$$

We now briefly mention ultimately periodic continued fractions. By an expression of the form $[x, \overline{w}]$, where x and w are finite strings, we mean the continued fraction $[x, w, w, w, \dots]$, where the overbar or “vinculum” denotes the repeating portion. Thus, for example,

$$\sqrt{7} = [2, \overline{1, 1, 1, 4}] = [2, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, \dots].$$

We now recall a classical result (see, e.g., [20, Satz 3.9, p. 79] or [9, Ex. 2.50, p. 45]).

Lemma 2. *Let a_0 be a positive integer and let w denote a finite palindrome of positive integers. Then*

$$[a_0, \overline{w, 2a_0}] = \sqrt{\frac{p}{q}},$$

where

$$M(w) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

$p = a_0^2\alpha + 2a_0\beta + \delta$, and $q = \alpha$.

2. THREE SEQUENCES

We now define three related sequences for $n \geq 2$:

$$\begin{aligned} u(n) &= (s_1, s_2, \dots, s_{2^n-3}), \\ v(n) &= (s_1, s_2, \dots, s_{2^n-2}) = (u(n), 1), \\ w(n) &= (s_1, s_2, \dots, s_{2^n-3}, 2) = (u(n), 2). \end{aligned}$$

Recall that $s_n = 2^{v_2(n+1)}$ is defined in the introduction. The following table gives the first few values of these quantities:

n	$u(n)$	$v(n)$	$w(n)$
2	2	21	22
3	21412	214121	214122
4	2141218121412	21412181214121	21412181214122

The following proposition, which is easily proved by induction, gives the relationship between these sequences, for $n \geq 2$:

Proposition 3.

- (a) $u(n + 1) = (v(n), 2^n, v(n)^R)$;
- (b) $u(n)$ is a palindrome;
- (c) $v(n + 1) = (v(n), 2^n, 1, v(n))$.

Furthermore, we can define the sequence of associated matrices with $u(n)$ and $v(n)$:

$$\begin{aligned} M(u(n)) &:= \begin{bmatrix} c_n & e_n \\ d_n & f_n \end{bmatrix}, \\ M(v(n)) &:= \begin{bmatrix} w_n & y_n \\ x_n & z_n \end{bmatrix}. \end{aligned}$$

The first few values of these arrays are given in the following table. As $d_n = e_n = z_n$ and $c_n = y_n$ for $n \geq 2$, we omit the duplicate values.

n	c_n	d_n	f_n	w_n	x_n
2	2	1	0	3	1
3	48	17	6	65	23
4	40040	14169	5014	54209	19183
5	51358907616	18174434593	6431407678	69533342209	24605842271

If we now define

$$\sigma_n = [1, \overline{w(n)}],$$

then Lemma 2 with $a_0 = 1$ and $w = u(n)$ gives

$$\sigma_n = \sqrt{\frac{c_n + 2e_n + f_n}{c_n}}.$$

Write $\sigma = [s_0, s_1, \dots]$ and $[s_0, s_1, \dots, s_n] = \frac{p_n}{q_n}$. Furthermore, define $\hat{\sigma}_n = [1, u(n)]$. Notice that σ , σ_n , and $\hat{\sigma}_n$ all agree on the first $2^n - 2$ partial quotients. We have

$$|\sigma - \hat{\sigma}_n| < \frac{1}{q_{2^n-3}q_{2^n-2}},$$

by a classical theorem on continued fractions (e.g., [14, Theorem 171]), and furthermore, since $s_{2^n-3} = 2$, $s_{2^n-2} = 1$, we have, for $n \geq 3$, that

$$\sigma < \sigma_n < \hat{\sigma}_n.$$

Hence

$$|\sigma - \sigma_n| < \frac{1}{q_{2^n-3}q_{2^n-2}}.$$

Now by considering

$$M(1)M(u(n))M(1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_n & e_n \\ d_n & f_n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2c_n + d_n & e_n + f_n \\ c_n + d_n & c_n \end{bmatrix},$$

we see that $q_{2^n-3} = c_n$ and $q_{2^n-2} = c_n + d_n$. For simplicity write $g_n = c_n + 2e_n + f_n$. Then

$$|\sigma - \sigma_n| = \left| \sigma - \sqrt{\frac{g_n}{c_n}} \right| < \frac{1}{c_n^2},$$

and so

$$\left| \sigma^2 - \frac{g_n}{c_n} \right| = \left| \sigma - \sqrt{\frac{g_n}{c_n}} \right| \cdot \left| \sigma + \sqrt{\frac{g_n}{c_n}} \right| < \frac{3}{c_n^2}.$$

So we have already found good approximations of σ^2 by rational numbers. In the next section we will show that g_n and c_n have a large common factor, which will improve the quality of the approximation.

3. IRRATIONALITY MEASURE OF σ^2

From Proposition 3(a), we get that the matrix

$$\begin{bmatrix} c_{n+1} & e_{n+1} \\ d_{n+1} & f_{n+1} \end{bmatrix}$$

associated with $u(n + 1)$ is equal to the matrix associated with $(v(n), 2^n, v(n)^R)$, which is

$$\begin{aligned} & \begin{bmatrix} w_n & y_n \\ x_n & z_n \end{bmatrix} \begin{bmatrix} 2^n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_n & x_n \\ y_n & z_n \end{bmatrix} \\ &= \begin{bmatrix} 2^n w_n^2 + 2w_n y_n & 2^n w_n x_n + x_n y_n + w_n z_n \\ 2^n w_n x_n + x_n y_n + w_n z_n & 2^n x_n^2 + 2x_n z_n \end{bmatrix}. \end{aligned}$$

Notice that

$$(5) \quad c_{n+1} = (2^n w_n + 2y_n)w_n.$$

On the other hand, we have

$$\begin{aligned}
 g_{n+1} &= c_{n+1} + 2d_{n+1} + f_{n+1} \\
 &= c_{n+1} + 2(2^n w_n x_n + x_n y_n + w_n z_n) + 2^n x_n^2 + 2x_n z_n \\
 &= c_{n+1} + 2(2^n w_n x_n + x_n y_n + w_n z_n) + 2^n x_n^2 \\
 &\quad + 2x_n z_n + 2(x_n y_n - w_n z_n + 1) \\
 &= c_{n+1} + (2^n w_n + 2y_n)2x_n + 2^n x_n^2 + 2x_n z_n + 2 \\
 &= (2^n w_n + 2y_n)(2x_n + w_n) + 2^n x_n^2 + 2x_n z_n + 2,
 \end{aligned}$$

where we have used equation (3). By Euclidean division, we get

$$\gcd(g_{n+1}, 2^n w_n + 2y_n) = \gcd(2^n w_n + 2y_n, 2^n x_n^2 + 2x_n z_n + 2).$$

Next, we interpret Proposition 3(c) in terms of matrices. We get that the matrix

$$\begin{bmatrix} w_{n+1} & y_{n+1} \\ x_{n+1} & z_{n+1} \end{bmatrix}$$

associated with $v(n + 1)$ is equal to the matrix associated with $(v(n), 2^n, 1, v(n))$, which is

$$\begin{aligned}
 &(6) \\
 &\begin{bmatrix} w_n & y_n \\ x_n & z_n \end{bmatrix} \begin{bmatrix} 2^n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_n & y_n \\ x_n & z_n \end{bmatrix} \\
 &= \begin{bmatrix} (2^n+1)w_n^2 + 2^n w_n x_n + y_n(w_n+x_n) & (2^n+1)w_n y_n + 2^n w_n z_n + y_n(y_n+z_n) \\ (2^n+1)x_n w_n + 2^n x_n^2 + z_n(w_n+x_n) & (2^n+1)x_n y_n + 2^n x_n z_n + z_n(y_n+z_n) \end{bmatrix}.
 \end{aligned}$$

Letting $r_n := 2^n(w_n + x_n) + w_n + y_n + z_n$, we see that

$$\begin{aligned}
 2^{n+1}w_{n+1} + 2y_{n+1} &= 2^{n+1}((2^n + 1)w_n^2 + 2^n w_n x_n + y_n(w_n + x_n)) \\
 &\quad + 2((2^n + 1)w_n y_n + 2^n w_n z_n + y_n(y_n + z_n)) \\
 &= 2(2^n w_n + y_n)(2^n(w_n + x_n) + w_n + y_n + z_n) \\
 (7) \qquad \qquad \qquad &= 2(2^n w_n + y_n)r_n.
 \end{aligned}$$

Now

$$\begin{aligned}
 x_{n+1} &= (2^n + 1)x_n w_n + 2^n x_n^2 + z_n(w_n + x_n) \\
 &= x_n(2^n(w_n + x_n) + w_n + y_n + z_n) + w_n z_n - x_n y_n \\
 &= x_n r_n + 1
 \end{aligned}$$

and

$$\begin{aligned}
 z_{n+1} &= (2^n + 1)x_n y_n + 2^n x_n z_n + z_n(y_n + z_n) \\
 &= z_n(2^n(w_n + x_n) + w_n + y_n + z_n) + (2^n + 1)(x_n y_n - w_n z_n) \\
 &= z_n r_n - 2^n - 1.
 \end{aligned}$$

It now follows, from some tedious algebra, that

$$\begin{aligned}
 (8) \qquad \qquad \qquad &\frac{2^n x_{n+1}^2 + x_{n+1} z_{n+1} + 1}{r_n} = (2^n + 1)w_n x_n z_n + 2^n(2^n + 1)w_n x_n^2 + z_n + (2^n - 1)x_n \\
 &\quad + 2^n x_n^2 y_n + 2^{n+1}x_n^2 z_n + x_n y_n z_n + 2^{2n}x_n^3 + x_n z_n^2.
 \end{aligned}$$

From equation (5) and reindexing, we get

$$\begin{aligned} c_{n+2} &= w_{n+1}(2^{n+1}w_{n+1} + 2y_{n+1}) \\ &= 2w_{n+1}(2^n w_n + y_n)r_n, \end{aligned}$$

where we used equation (7). Also, from the argument above about gcd's and equation (8), we see that $2r_n \mid g_{n+2}$. Hence for $n \geq 2$ we have

$$\frac{g_{n+2}}{c_{n+2}} = \frac{P_{n+2}}{Q_{n+2}}$$

for integers $P_{n+2} := \frac{g_{n+2}}{2r_n}$ and $Q_{n+2} := w_{n+1}(2^n w_n + y_n)$. It remains to see that P_{n+2}/Q_{n+2} are particularly good rational approximations to σ^2 .

Since w_n/x_n and y_n/z_n denote successive convergents to a continued fraction, we clearly have $w_n \geq x_n$, $w_n \geq y_n$, and $w_n \geq z_n$. It follows that

$$\begin{aligned} Q_{n+2} &= w_{n+1}(2^n w_n + y_n) \\ &= ((2^n + 1)w_n^2 + 2^n w_n x_n + y_n(w_n + x_n))(2^n w_n + y_n) \\ &\leq (2^{n+1} + 3)w_n^2 \cdot (2^n + 1)w_n \\ &= (2^{n+1} + 3)(2^n + 1)w_n^3. \end{aligned}$$

On the other hand,

$$\begin{aligned} c_{n+2} &= 2Q_{n+2}r_n \\ &> 2(2^n + 1)w_n^2 \cdot 2^n w_n \cdot (2^n + 1)w_n = 2^{n+1}(2^n + 1)^2 w_n^4 \\ &\geq Q_{n+2}^{4/3} \frac{2^{n+1}(2^n + 1)^2}{((2^{n+1} + 3)(2^n + 1))^{4/3}} \\ &> Q_{n+2}^{4/3}. \end{aligned}$$

This gives

$$\left| \sigma^2 - \frac{P_{n+2}}{Q_{n+2}} \right| < Q_{n+2}^{-8/3}$$

for all integers $n \geq 2$.

The result we have just shown can be nicely formulated in terms of the irrationality measure. Recall that the *irrationality measure* of a real number x is defined to be the infimum, over all real μ , for which the inequality

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^\mu}$$

is satisfied by at most finitely many integer pairs (p, q) .

Theorem 4. *The irrationality measure of σ^2 is at least 8/3.*

It would be very interesting to find the precise value of the irrationality measure of σ_2 . The numerical experiments conducted by the second author suggest that the bound 8/3 in Theorem 4 is sharp; however, we do not currently know how to prove it.

Note that the classical Khintchine Theorem (e.g., [12, Chapter VII, Theorem I]) states that for almost all real numbers (in terms of Lebesgue measure), the irrationality exponent equals two. Hence Theorem 4 says that the number σ^2 belongs to a very tiny set of zero Lebesgue measure.

Furthermore, the famous Roth Theorem [23] states that the irrationality exponent of every irrational algebraic number is two. Therefore, we conclude that

σ^2 (and hence σ) are transcendental numbers. However, this result is not new. By [4, Theorem 2.1], if a continued fraction of a real number x starts with arbitrarily long palindromes, then x is transcendental. Clearly σ satisfies this property.

We now provide a lower bound for some very large partial quotients of σ^2 . For each $n \geq 2$, we certainly have

$$\left| \sigma - \frac{P_{n+2}}{Q_{n+2}} \right| < Q_{n+2}^{-8/3} < \frac{1}{2Q_{n+2}^2}.$$

In particular this implies that the rational number P_{n+2}/Q_{n+2} is a convergent of σ^2 .

Notice that P_{n+2} and Q_{n+2} are not necessarily relatively prime. Let $\tilde{P}_{n+2}/\tilde{Q}_{n+2}$ denote the reduced fraction of P_{n+2}/Q_{n+2} . If $\tilde{P}_{n+2}/\tilde{Q}_{n+2}$ is the m -th convergent of σ^2 , then define A_{n+2} to be the $(m + 1)$ -th partial quotient of σ^2 . Then the estimate (1) implies that

$$\frac{1}{(A_{n+2} + 2)\tilde{Q}_{n+2}^2} < \left| \sigma^2 - \frac{\tilde{P}_{n+2}}{\tilde{Q}_{n+2}} \right| < \frac{3}{c_{n+2}^2} \leq \frac{3}{4r_n^2\tilde{Q}_{n+2}^2}.$$

Hence $A_{n+2} \geq 4r_n^2 - 2$.

From the formula for r_n and the inequalities $w_n \geq x_n, w_n \geq y_n, w_n \geq z_n$, one can easily derive

$$(2^n + 1)w_n \leq r_n \leq (2^{n+1} + 3)w_n.$$

This, together with the formula (6) for w_{n+1} , gives the estimate

$$r_{n+1} \geq (2^{n+1} + 1)w_{n+1} \geq (2^{n+1} + 1)(2^n + 1)w_n^2 > r_n^2 + 1.$$

Therefore, the sequence $4r_n^2 - 2$, and in turn A_{n+2} , grows doubly exponentially. This phenomenon explains the observation of Hanna and Wilson for the sequence A100864 in [24].

The first few values of the sequences we have been discussing are given below:

n	σ_n^2	$\tilde{\sigma}_n$	g_n	r_n	P_n	Q_n	A_n
3	$\frac{11}{6}$	$\frac{65}{48}$	88	834	11	6	74
4	$\frac{834}{455}$	$\frac{54209}{40040}$	73392	1282690	1668	910	8457
5	$\frac{7054795}{3848839}$	$\frac{69533342209}{5135807616}$	94139184480	3151520587778	56438360	30790712	186282390

4. ADDITIONAL REMARKS

The same idea can be used to bound the irrationality exponent of an infinite collection of numbers $\sigma = [s_0, s_1, s_2, \dots]$. Indeed, there is nothing particularly special about the terms 2^n appearing in Proposition 3. One can check that the same result holds if the strings $u(n)$ and $v(n)$ satisfy the following modified properties from Proposition 3 for infinitely many numbers $n \geq 2$:

- (a') $u(n + 1) = (v(n), k_n, v(n)^R)$;
- (b') $u(n + 2)$ is a palindrome;
- (c') $v(n + 2) = (v(n + 1), 2k_n, 1, v(n + 1))$.

In particular one can easily check these properties for a string $\mathbf{s} = s_0s_1s_2\cdots$ such that $s_i = f(\nu_2(i+1))$ where $f : \mathbb{N} \rightarrow \mathbb{N}$ is a function satisfying the following conditions:

- (1) $f(0) = 1$;
- (2) $f(n+1) = 2f(n)$ for infinitely many positive integers n .

ACKNOWLEDGMENTS

The authors are grateful to Yann Bugeaud and the referee for their helpful comments.

REFERENCES

- [1] Boris Adamczewski and Jean-Paul Allouche, *Reversals and palindromes in continued fractions*, Theoret. Comput. Sci. **380** (2007), no. 3, 220–237, DOI 10.1016/j.tcs.2007.03.017. MR2330994 (2008d:11074)
- [2] Boris Adamczewski and Yann Bugeaud, *On the complexity of algebraic numbers. II. Continued fractions*, Acta Math. **195** (2005), 1–20, DOI 10.1007/BF02588048. MR2233683 (2008b:11080)
- [3] Boris Adamczewski and Yann Bugeaud, *A short proof of the transcendence of Thue-Morse continued fractions*, Amer. Math. Monthly **114** (2007), no. 6, 536–540. MR2321257 (2008d:11072)
- [4] Boris Adamczewski and Yann Bugeaud, *Palindromic continued fractions* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **57** (2007), no. 5, 1557–1574. MR2364142 (2008m:40009)
- [5] Boris Adamczewski and Yann Bugeaud, *Transcendence measures for continued fractions involving repetitive or symmetric patterns*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 4, 883–914, DOI 10.4171/JEMS/218. MR2654083 (2011g:11138)
- [6] J.-P. Allouche, J. L. Davison, M. Queffélec, and L. Q. Zamboni, *Transcendence of Sturmian or morphic continued fractions*, J. Number Theory **91** (2001), no. 1, 39–66, DOI 10.1006/jnth.2001.2669. MR1869317 (2002k:11117)
- [7] Jean-Paul Allouche and Jeffrey Shallit, *The ring of k -regular sequences*, Theoret. Comput. Sci. **98** (1992), no. 2, 163–197, DOI 10.1016/0304-3975(92)90001-V. MR1166363 (94c:11021)
- [8] Jean-Paul Allouche and Jeffrey Shallit, *The ring of k -regular sequences. II*, Theoret. Comput. Sci. **307** (2003), no. 1, 3–29, DOI 10.1016/S0304-3975(03)00090-2. Words. MR2014728 (2004m:68172)
- [9] J. Borwein, A. van der Poorten, J. Shallit, and W. Zudilin, *Neverending Fractions: An Introduction to Continued Fractions*. Cambridge University Press, 2014.
- [10] Yann Bugeaud, *Continued fractions with low complexity: Transcendence measures and quadratic approximation*, Compos. Math. **148** (2012), no. 3, 718–750, DOI 10.1112/S0010437X11007524. MR2925396
- [11] Yann Bugeaud, *Automatic continued fractions are transcendental or quadratic* (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) **46** (2013), no. 6, 1005–1022. MR3134686
- [12] J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 45, Cambridge University Press, New York, 1957. MR0087708 (19,396h)
- [13] J. S. Frame, *Classroom Notes: Continued Fractions and Matrices*, Amer. Math. Monthly **56** (1949), no. 2, 98–103, DOI 10.2307/2306169. MR1527170
- [14] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, 5th edition, 1985.
- [15] A. Hurwitz, *Über die Kettenbrüche, deren Teilnenner arithmetische Reihen bilden*. Vierteljahrsschrift d. Naturforsch. Gesellschaft in Zürich, Jahrg. 41 (1896), In Mathematische Werke, Band II, Birkhäuser, Basel, 1963, pp. 276–302.
- [16] Adolf Hurwitz, *Lectures on number theory*, Universitext, Springer-Verlag, New York, 1986. Translated from the German and with a preface by William C. Schulz; Translation edited and with a preface by Nikolaos Kritikos. MR816942 (88f:11002)

- [17] Kjell Kolden, *Continued fractions and linear substitutions*, Arch. Math. Naturvid. **50** (1949), no. 6, 141–196. MR0032033 (11,244a)
- [18] E. Maillet, *Introduction à la Théorie des Nombres Transcendants et des Propriétés Arithmétiques des Fonctions*. Gauthier-Villars, 1906.
- [19] K. R. Matthews and R. F. C. Walters, *Some properties of the continued fraction expansion of $(m/n)e^{1/q}$* , Proc. Cambridge Philos. Soc. **67** (1970), 67–74. MR0252889 (40 #6104)
- [20] Oskar Perron, *Die Lehre von den Kettenbrüchen. Bd I. Elementare Kettenbrüche* (German), B. G. Teubner Verlagsgesellschaft, Stuttgart, 1954. 3te Aufl. MR0064172 (16,239e)
- [21] M. Queffélec, *Transcendance des fractions continues de Thue-Morse* (French, with French summary), J. Number Theory **73** (1998), no. 2, 201–211, DOI 10.1006/jnth.1998.2305. MR1658023 (99j:11081)
- [22] Martine Queffélec, *Irrational numbers with automaton-generated continued fraction expansion*, Dynamical systems (Luminy-Marseille, 1998), World Sci. Publ., River Edge, NJ, 2000, pp. 190–198. MR1796159 (2001h:11008)
- [23] K. F. Roth, *Rational approximations to algebraic numbers*, Mathematika **2** (1955), 1–20; corrigendum, 168. MR0072182 (17,242d)
- [24] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, Available at <http://oeis.org>.
- [25] R. F. C. Walters, *Alternative derivation of some regular continued fractions*, J. Austral. Math. Soc. **8** (1968), 205–212. MR0226245 (37 #1835)

DEPARTMENT OF MATHEMATICAL SCIENCES, DURHAM UNIVERSITY, LOWER MOUNTJOY, STOCKTON ROAD, DURHAM, DH1 3LE, UNITED KINGDOM

E-mail address: dzmitry.badziahin@durham.ac.uk

SCHOOL OF COMPUTER SCIENCE, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO N2L 3G1, CANADA

E-mail address: shallit@cs.uwaterloo.ca