BOUNDS ON THE NORMAL HILBERT COEFFICIENTS

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ABSTRACT. In this paper we consider extremal and almost extremal bounds on
the normal Hilbert coefficients of $\mathfrak{m}$-primary ideals of an analytically unramified
Cohen-Macaulay ring R of dimension $d > 0$ and infinite residue field. In these
circumstances we show that the associated graded ring of the normal filtration
of the ideal is either Cohen-Macaulay or almost Cohen-Macaulay.

1. INTRODUCTION

The examination of the asymptotic properties of $\mathfrak{m}$-primary ideals of a Cohen-Macaulay local ring $(R, \mathfrak{m})$ of dimension d and infinite residue field has evolved into a challenging area of research, touching most aspects of commutative algebra, including its interaction with computational algebra and algebraic geometry. It takes expression in two graded algebras attached to I: the *Rees algebra* $\mathcal{R} = \mathcal{R}(I)$ and the *associated graded ring* $\mathcal{G} = \mathcal{G}(I)$; namely,

$$\mathcal{R} = \bigoplus_{k=0}^{\infty} I^k t^k \subset R[t] \qquad \text{and} \qquad \mathcal{G} = \mathcal{R}/I\,\mathcal{R} = \bigoplus_{k=0}^{\infty} I^k/I^{k+1},$$

where $R[t]$ is the polynomial ring in the variable t over R. These two graded objects are collectively referred to as *blowup algebras* of I as they play a crucial role in the process of blowing up the variety $\mathrm{Spec}(R)$ along the subvariety $V(I)$.

A successful approach in the study of the ring-theoretic properties of the blowup algebras uses a minimal reduction of the ideal. This notion was first introduced and exploited by Northcott and Rees more than half a century ago for its effectiveness in studying multiplicities in local rings [11]: an ideal J is a reduction of I if the inclusion of Rees algebras $\mathcal{R}(J) \hookrightarrow \mathcal{R}(I)$ is module finite. Since I is also an $\mathfrak{m}$-primary ideal, another pathway to studying blowup algebras – and more precisely $\mathcal{G}$ – is to make use of information encoded in the Hilbert-Samuel function of I, that is, the function that measures the growth of the length of R/I^n, denoted $\lambda(R/I^n)$, for all $n \geq 1$. For $n \gg 0$, it is known that $\lambda(R/I^n)$ is a polynomial in n of degree d, whose normalized coefficients $e_i = e_i(I)$ are called the Hilbert coefficients of I. The general philosophy, pioneered by Sally in a sequence of remarkable papers

(see [18–22]), is that an 'extremal' behavior of bounds involving the e_i's yields good depth properties of the associated graded ring of I. These results are somewhat unexpected since the Hilbert coefficients encode asymptotic information on the Hilbert-Samuel function of I. The literature is very rich with results, especially relating e_0 through e_3 to other data of our ideal. We refer to the monograph by Rossi and Valla [17] for a collective overview.

Over the years, various filtrations other than the I-adic one have proved to be of far-reaching applications in commutative algebra: the integral closure filtration, the Ratliff-Rush filtration, the tight closure filtration, and the symbolic power filtration, just to name a few. It is important to observe that the theory and results that are valid in the case of the I-adic filtration cannot be trivially extended to these other types of filtrations. These in fact are not, in general, good or stable filtrations. In other words, the Rees algebra associated to these filtrations may not be generated in degree one or may even fail to be Noetherian.

The focus of our paper is on the significance of the asymptotic properties encoded in the Hilbert function of the integral closure filtration of an $\mathfrak{m}$-primary ideal I, namely $\{\overline{I^n}\}$. In addition to the local Cohen-Macaulay property of our ambient ring R we also require R to have a canonical module and to be analytically unramified, that is, its $\mathfrak{m}$-adic completion $\widehat{R}$ is reduced. This latter assumption guarantees that the normalization $\overline{\mathcal{R}}$ of the Rees algebra $\mathcal{R}$ of I in $R[t]$ is Noetherian (see [15]). Hence we have that $\lambda(R/\overline{I^{n+1}})$ is a polynomial in n of degree d for $n \gg 0$:

$$\lambda(R/\overline{I^{n+1}}) = \overline{e}_0 \binom{n+d}{d} - \overline{e}_1 \binom{n+d-1}{d-1} + \cdots + (-1)^d \overline{e}_d.$$

The above polynomial is referred to as the normal Hilbert polynomial of I and the $\overline{e}_i = \overline{e}_i(I)$'s are the normal Hilbert coefficients. We note that $e_0 = \overline{e}_0$ is the multiplicity of I. As in the case of the I-adic filtration, there has been quite some interest in relating bounds among the normalized Hilbert coefficients and the depth of the associated graded ring of the normal filtration of I, denoted $\overline{\mathcal{G}}$. The forerunners of results along this line of investigation are Huneke (see [5]) and Itoh (see [6, 7]). We refer again to [17] for a detailed account of the results, which typically deal with the first few normal Hilbert coefficients, as in the I-adic case.

We now describe the contents of our paper. In Section 2 we first introduce a version of the Sally module for the normal filtration. We prove in Proposition 2.2 analogous homological properties to the ones of the classical Sally module (see [23]). This allows us to recover, in Proposition 2.4, the bound $\overline{e}_1 \geq e_0 - \lambda(R/\overline{I})$, established by Huneke [5, Theorem 4.5] and Itoh [7, Corollary]. That the equality in the bound is equivalent to the Cohen-Macaulayness of $\overline{\mathcal{G}}$ translates in our setting to the vanishing of the variant of the Sally module. We then show in Theorem 2.6 the main result of this section. Namely, we show that if the previous bound is almost extremal, that is, $\overline{e}_1 \leq e_0 - \lambda(R/\overline{I}) + 1$, then the depth of $\overline{\mathcal{G}}$ is at least $d - 1$.

In [7], Itoh already established lower bounds on $\overline{e}_2$ and $\overline{e}_3$. More precisely, he showed that $\overline{e}_2 \geq \overline{e}_1 - \lambda(\overline{I}/J)$ with equality if and only if the normal filtration of I has reduction number two (see [7, Theorem 2(2)]). In particular $\overline{\mathcal{G}}$ is Cohen-Macaulay. He also showed that $\overline{e}_3 \geq 0$ (see [7, Theorem 3(1)]). When R is Gorenstein and $\overline{I} = \mathfrak{m}$ he was able to conclude that the vanishing of $\overline{e}_3$ is equivalent to the normal filtration of I having reduction number two (see [7, Theorem 3(2)]). Again this implies that $\overline{\mathcal{G}}$ is Cohen-Macaulay.

In Section 3 we generalize Itoh's result on the vanishing of $\overline{e}_3$ by considering arbitrary Cohen-Macaulay rings of type $t(R)$ together with the assumption $\lambda(\overline{I^2}/J\overline{I}) \geq t(R) - 1$ (see Theorem 3.3). Obviously the latter is a vacuous assumption when R is Gorenstein. We also extend Itoh's result with no further assumptions to rings of type at most two (see Theorem 3.6). We note that a condition on the type of the ring is not unexpected as it is reminiscent of the celebrated result of Sally [22, Theorem 3.1] on Cohen-Macaulay rings of type $e + d - 2$.

2. The Sally module and Hilbert coefficients
of the normal filtration

One of the first inequalities involving the Hilbert coefficients of an $\mathfrak{m}$-primary ideal I goes back to 1960, when Northcott [10] showed that $e_1 - e_0 + \lambda(R/I) \geq 0$. Later it was shown by Huneke [5] and Ooishi [12] that equality is equivalent to the ideal having reduction number one for any minimal reduction J of I, that is, $I^2 = JI$. Hence $\mathcal{G}$ is Cohen-Macaulay.

An elegant and theoretical explanation of the results by Northcott, Huneke and Ooishi was captured by Vasconcelos [23] with the introduction of a new graded object: the so-called Sally module. As noted in the monograph [17], the Sally module can actually be defined for an arbitrary filtration. However, additional properties on the filtration are needed to be able to go the extra mile. This is the case in this article where we consider the normal filtration of an ideal.

We denote by $\overline{\mathcal{R}}$ the integral closure of the Rees algebra $\mathcal{R}$ of I inside the polynomial ring $R[t]$. Notice that

$$\overline{\mathcal{R}} = R + \overline{I}t + \overline{I^2}t^2 + \cdots.$$

In the following lemma we show that under mild assumptions the graded algebra $\overline{\mathcal{R}}$ satisfies the property S_2 of Serre.

Lemma 2.1. *Let $(R, \mathfrak{m})$ be an analytically unramified, Cohen-Macaulay local ring with a canonical module ω_R and let I be an ideal of height at least two. Then the integral closure $\overline{\mathcal{R}}$ of the Rees algebra $\mathcal{R} = R[It]$ of I in $R[t]$ satisfies the property S_2 of Serre.*

Proof. Let B denote the polynomial ring $R[t]$ and let A denote the integral closure of the Rees algebra $\mathcal{R}$ of I inside B. Let K be the total ring of quotients of A and of B. Write ω_A for a canonical module of A; it exists since R has a canonical module. The S_2-ification $\text{End}_A(\omega_A)$ of A can be identified with the subring D of K that contains A (see [2]). To show that A satisfies the property S_2 of Serre it suffices to prove the equality $A = D$. This follows once we show that D is contained in B. Indeed the inclusions $A \hookrightarrow D \hookrightarrow B$ together with the fact that D is a module finite extension of A force $A = D$, as A is the integral closure of $\mathcal{R}$ in B.

It suffices to show the inclusion $D \hookrightarrow B$ at the height one prime ideals of B. Indeed, consider the short exact sequence

$$0 \to B \longrightarrow B[D] \longrightarrow C \to 0,$$

where C is the cokernel of the inclusion map. We observe that any associated prime P of C as a B-module has height at most one. Otherwise the depth of B_P is at least two, since B is Cohen-Macaulay, whereas the depth of $B[D]_P$ is at least one, as all our rings are unmixed. These facts force the depth of C_P to be at least one, thus contradicting the fact that P is an associated prime of C.

Finally, suppose that P is a height one prime of B and let Q be its contraction to A. We claim that Q is still a prime ideal of height one. The contraction of P to R, say $\mathfrak{p}$, is a prime ideal of height one, given that B is a polynomial ring over R. Thus $\mathfrak{p}$ does not contain I, as I has height at least two. When we localize $R[It]$ at $\mathfrak{p}$ it becomes $R[t]_{\mathfrak{p}}$, which forces $A_{\mathfrak{p}}$ to also be equal to $B_{\mathfrak{p}}$. In particular it follows that $A_Q = B_P$, because they are further localizations. Hence Q has height one, since the dimension of B_P is one.

Now $A_Q = D_Q$ since A is unmixed and Q has height one. Therefore $D_Q = B_P$ for every height one prime P of B. This proves our statement. $\qquad \square$

The Sally module of the normal filtration of an $\mathfrak{m}$-primary ideal I with minimal reduction J is defined by the short exact sequence of $\mathcal{R}(J)$-modules

$$(2.1) \qquad\qquad 0 \to \overline{I}\mathcal{R}(J) \longrightarrow \overline{\mathcal{R}}_+[+1] \longrightarrow \overline{\mathcal{S}} \to 0.$$

More explicitly, one has

$$\overline{\mathcal{S}} = \bigoplus_{n \geq 1} \overline{I^{n+1}}/J^n\overline{I}.$$

In Proposition 2.2 we establish a key homological property of $\overline{\mathcal{S}}$ that will be used in Theorem 2.6, the main theorem of this section. This is the same property of the classical Sally module. However it does not follow directly from the original result because the normal filtration is not multiplicative.

Proposition 2.2. *Let $(R, \mathfrak{m})$ be an analytically unramified, Cohen-Macaulay local ring with a canonical module ω_R, of dimension $d \geq 2$ and infinite residue field. Let I be an $\mathfrak{m}$-primary ideal, J a minimal reduction of I, and $\overline{\mathcal{S}}$ the Sally module of the normal filtration of I with respect to J, as defined in (2.1). Then $\overline{\mathcal{S}}$ is a nonzero module if and only if $\mathrm{Ass}_{\mathcal{R}(J)}(\overline{\mathcal{S}}) = \{\, \mathfrak{m}\mathcal{R}(J) \,\}$. In particular, $\overline{\mathcal{S}}$ has dimension d.*

Proof. Let us assume that $\overline{\mathcal{S}} \neq 0$, as the other implication is trivial. As $\mathrm{Ass}_{\mathcal{R}(J)}(\overline{\mathcal{S}}) \neq \emptyset$, let $P \in \mathrm{Ass}_{\mathcal{R}(J)}(\overline{\mathcal{S}})$ and write $P \cap R = \mathfrak{p}$. If $\mathfrak{p} \neq \mathfrak{m}$, it follows that $\overline{\mathcal{S}}_{\mathfrak{p}} = 0$, from (2.1) and the fact that $(\overline{I}\mathcal{R}(J))_{\mathfrak{p}} = (\overline{\mathcal{R}}_+[+1])_{\mathfrak{p}}$. Hence $\mathfrak{p} = \mathfrak{m}$ and $P \supseteq \mathfrak{m}\mathcal{R}(J)$.

We claim that $P = \mathfrak{m}\mathcal{R}(J)$. Notice that $\mathfrak{m}\mathcal{R}(J)$ is a prime of height one, thus if $P \supsetneq \mathfrak{m}\mathcal{R}(J)$, we have that P is a prime ideal of height at least two. A depth computation in the short exact sequence (2.1) shows that depth $\overline{\mathcal{S}}_P \geq 1$, which contradicts the fact that P is an associated prime of $\overline{\mathcal{S}}$.

The asserted depth estimate follows since $\overline{I}\mathcal{R}(J)$ is maximal Cohen-Macaulay and $\overline{\mathcal{R}}_+$ has property S_2 of Serre. The first assertion is a consequence of the short exact sequence

$$0 \to \overline{I}\mathcal{R}(J) \longrightarrow \mathcal{R}(J) \longrightarrow \mathcal{R}(J)/\overline{I}\mathcal{R}(J) \to 0$$

and the fact that the module $\mathcal{R}(J)/\overline{I}\mathcal{R}(J)$ is isomorphic to $R/\overline{I}[T_1, \ldots, T_d]$. Now, the short exact sequence

$$0 \to \overline{\mathcal{R}}_+ \longrightarrow \overline{\mathcal{R}} \longrightarrow R \to 0$$

establishes instead the second assertion, since $\overline{\mathcal{R}}$ has property S_2 of Serre, by Lemma 2.1, and R is Cohen-Macaulay of dimension $d \geq 2$. $\qquad \square$

Our next goal is to determine the relationship between the coefficients of the Hilbert polynomial of the Sally module $\overline{\mathcal{S}}$ and the ones of the Hilbert polynomial of the associated graded ring $\overline{\mathcal{G}}$ of the normal filtration of I. The construction we describe allows us, in particular, to give a concrete characterization of the *sectional*

normal genus $g_s = \overline{e}_1 - e_0 + \lambda(R/\overline{I})$, defined by Itoh in [7], as the multiplicity of the Sally module $\overline{\mathcal{S}}$. We then use this characterization to give an estimate of the depth of the associated graded ring $\overline{\mathcal{G}}$.

Discussion 2.3. Following the construction of [17, Proposition 6.1], there exists a graded module N which fits in the short exact sequences

$$(2.2) \qquad 0 \to \mathcal{G}(\mathbb{E}) \longrightarrow N \longrightarrow \overline{\mathcal{S}}[-1] \to 0,$$

$$(2.3) \qquad 0 \to \overline{\mathcal{S}} \longrightarrow N \longrightarrow \overline{\mathcal{G}} \to 0.$$

More precisely, $\mathbb{E}$ denotes the J-good filtration $\{E_0 = R, E_n = J^{n-1}\overline{I} \text{ for all } n \geq 1\}$ induced by the R-ideal $\overline{I}$, $\mathcal{G}(\mathbb{E})$ denotes the associated graded ring of $\mathbb{E}$, $\overline{\mathcal{G}}$ denotes the associated graded ring of the normal filtration of I, and, finally, N denotes the graded module $\bigoplus_{n \geq 0} \overline{I^n}/J^n\overline{I}$.

From the results of [17, Sections 6.1 and 6.2] modified to suit our situation, we have that the Hilbert series of the graded modules appearing in (2.2) and (2.3) are related by the equation

$$(2.4) \qquad (1 - z)HS_{\overline{\mathcal{S}}}(z) = HS_{\mathcal{G}(\mathbb{E})}(z) - HS_{\overline{\mathcal{G}}}(z).$$

Furthermore, $\mathcal{G}(\mathbb{E})$ is Cohen-Macaulay with minimal multiplicity since $E_{n+1} = JE_n$ for all $n \geq 1$. Hence its Hilbert series is given by [17, (6.1)]

$$HS_{\mathcal{G}(\mathbb{E})}(z) = \frac{\lambda(R/\overline{I}) + (e_0 - \lambda(R/\overline{I}))z}{(1-z)^d}.$$

By combining all this information we obtain the following result.

Proposition 2.4. *Let* $(R, \mathfrak{m})$ *be an analytically unramified, Cohen-Macaulay local ring with a canonical module* ω_R, *of dimension* $d \geq 2$ *and infinite residue field. Let* I *be an* $\mathfrak{m}$-*primary ideal,* J *a minimal reduction of* I, *and* $\overline{\mathcal{S}}$ *the Sally module of the normal filtration of* I *with respect to* J. *Let* $\overline{\mathcal{G}}$ *denote the associated graded ring of the normal filtration of* I. *Let* $\overline{s}_i$ *and* $\overline{e}_i$ *denote the normalized Hilbert coefficients of* $\overline{\mathcal{S}}$ *and* $\overline{\mathcal{G}}$, *respectively. The following properties hold:*

    (a) *if* $\overline{\mathcal{S}} = 0$, *then* $\overline{\mathcal{G}}$ *is Cohen-Macaulay;*
    (b) *if* $\overline{\mathcal{S}} \neq 0$, *then* $\text{depth}\,\overline{\mathcal{G}} \geq \text{depth}\,\overline{\mathcal{S}} - 1$;
    (c) $\overline{s}_0 = \overline{e}_1 - e_0 + \lambda(R/\overline{I})$ *and* $\overline{s}_i = \overline{e}_{i+1}$ *for all* $1 \leq i \leq d - 1$.

Proof. (a) follows from (2.2) and (2.3) because $\overline{\mathcal{G}} \cong N \cong \mathcal{G}(\mathbb{E})$, whenever $\overline{\mathcal{S}} = 0$, and $\mathcal{G}(\mathbb{E})$ is Cohen-Macaulay. (b) follows from depth chase in (2.2) and (2.3). Finally, (c) follows from (2.4) and the fact, shown in Proposition 2.2, that if $\overline{\mathcal{S}} \neq 0$, then its dimension is d. $\qquad\square$

Remark 2.5. Proposition 2.4(c) provides a simple proof of the bounds

$$\overline{e}_1 \geq e_0 - \lambda(R/\overline{I}) = \lambda(\overline{I}/J) \geq 0.$$

Moreover, Proposition 2.4(a) is equivalent to the equality $\overline{e}_1 = e_0 - \lambda(R/\overline{I})$; it is also equivalent to the isomorphism $\overline{\mathcal{G}} \cong \mathcal{G}(\mathbb{E})$; it is also equivalent to the fact that the reduction number of the integral closure filtration is at most one, see [5, Theorem 4.5] and [7, Corollary 6].

In the main theorem of this section we study the relation between an upper bound on $\overline{e}_1$ and the depth of $\overline{\mathcal{G}}$. That is, we investigate the depth property of $\overline{\mathcal{G}}$ whenever $\overline{e}_1 \leq e_0 - \lambda(R/\overline{I}) + 1$. This is equivalent to assuming that the multiplicity of the Sally module $\overline{\mathcal{S}}$ is at most one. (See also [23, Proposition 3.5] and [23, Corollary 3.7] in the I-adic case.)

Theorem 2.6. *Under the same assumptions as in Proposition* 2.4*, if* $\overline{e}_1 \leq e_0 - \lambda(R/\overline{I}) + 1$*, then* $\operatorname{depth} \overline{\mathcal{G}} \geq d - 1$.

Proof. By Proposition 2.4(c), our assumption on $\overline{e}_1$ implies that either $\overline{e}_1 = e_0 - \lambda(R/\overline{I})$ or $\overline{e}_1 = e_0 - \lambda(R/\overline{I}) + 1$. In the first case we have that the Sally module $\overline{\mathcal{S}}$ is zero and hence $\overline{\mathcal{G}}$ is Cohen-Macaulay by Proposition 2.4(a). Thus we are left to consider the second case. From $\overline{e}_1 = e_0 - \lambda(R/\overline{I}) + 1$ we obtain that $\overline{\mathcal{S}}$ is a nonzero module of multiplicity one. By Proposition 2.2 we have that $\operatorname{Ass}_{\mathcal{R}(J)}(\overline{\mathcal{S}}) = \{\mathfrak{m}\mathcal{R}(J)\}$. Thus $\overline{\mathcal{S}}$ is a torsion free B-module of rank one, where $B = \mathcal{R}(J)/\mathfrak{m}\mathcal{R}(J)$ is a polynomial ring in d variables over the residue field. We claim that $\overline{\mathcal{S}}$ is a reflexive B-module, hence it is free since B is a UFD. In particular, $\operatorname{depth} \overline{\mathcal{S}} = d$. Hence, by Proposition 2.4(b), we conclude that $\operatorname{depth} \overline{\mathcal{G}} \geq d - 1$. Our claim is equivalent to the fact that $\overline{\mathcal{S}}$ has the property S_2 of Serre as a B-module. As $\operatorname{Ass}_B(\overline{\mathcal{S}}) = \{0\}$ it suffices to show $\operatorname{depth} \overline{\mathcal{S}}_P \geq 2$ for each $P \in \operatorname{Spec}(B)$ with height at least two. Let $Q \in \operatorname{Spec}(\mathcal{R}(J))$ be such that $P = Q/\mathfrak{m}\mathcal{R}(J)$.

As we observed in the proof of Proposition 2.2, $\overline{I}\mathcal{R}(J)$ is maximal Cohen-Macaulay and $\overline{\mathcal{R}}_+$ has property S_2 of Serre, hence depth chasing in (2.1) yields $\operatorname{depth} \overline{\mathcal{S}}_Q \geq 2$. Thus $\operatorname{depth} \overline{\mathcal{S}}_P = \operatorname{depth} \overline{\mathcal{S}}_Q \geq 2$. $\square$

Remark 2.7. Theorem 2.6 would be a consequence of [8, Proposition 4.9]. However the proof of [8, Proposition 4.9] is not correct as it relies on [9, Theorem 3.26] which is incorrectly stated. Indeed in dimension $d > 1$ it is not clear that $\overline{I^2} = J\overline{I}$ implies that the normal filtration has reduction number at most one, that is, $\overline{I^{n+1}} = J\overline{I^n}$ for all $n \geq 1$.

3. ON THE VANISHING OF $\overline{e}_3$

As we mentioned in the introduction Itoh showed that the vanishing of $\overline{e}_3$ is equivalent to the normal filtration of I having reduction number two, provided R is Gorenstein and $\overline{I} = \mathfrak{m}$ (see [7, Theorem 3(2)]). Again this implies that $\overline{\mathcal{G}}$ is Cohen-Macaulay. We now generalize Itoh's result on the vanishing of $\overline{e}_3$ by considering an arbitrary Cohen-Macaulay ring R and imposing a condition on the type $t(R)$ of the ring.

Proposition 3.1. *Let* $(R, \mathfrak{m})$ *be an analytically unramified, Cohen-Macaulay local ring of dimension* $d \geq 3$*, type* $t(R)$ *and infinite residue field. Let* I *be an* R*-ideal with* $\overline{I} = \mathfrak{m}$ *and* J *a minimal reduction of* I*. Assume that* $\overline{e}_3 = 0$*. Then*

$$\lambda(\overline{I^{n+1}}/J^n \overline{I}) \leq t(R) \binom{n + d - 2}{d - 1}$$

for all $n \geq 1$*. In particular,* $\lambda(\overline{I^2}/J\overline{I}) \leq t(R)$.

Proof. By [7, Theorem 3(1)] the assumption $\overline{e}_3 = 0$ yields $\overline{I^{n+2}} \subset J^n$ for all $n \geq 0$. By assumption we have that $\overline{I} = \mathfrak{m}$, hence

$$\mathfrak{m}\overline{I^{n+1}} = \overline{I}\,\overline{I^{n+1}} \subset \overline{I^{n+2}} \subset J^n.$$

This implies that $\overline{I^{n+1}} \subset J^n : \mathfrak{m}$, thus

$$
\begin{aligned}
\lambda(\overline{I^{n+1}}/J^n\overline{I}) &= \lambda(\overline{I^{n+1}}/(J^n \cap \overline{I^{n+1}})) = \lambda(\overline{I^{n+1}} + J^n/J^n) \\
&\leq \lambda(J^n : \mathfrak{m}/J^n) = t(R)\binom{n+d-2}{d-1}.
\end{aligned}
$$

We observe that in the first equality we used [6, Theorem 1] or [5, Theorem 4.7 and Appendix] whereas the last equality holds because $\lambda(J^n : \mathfrak{m}/J^n)$ is the dimension of the socle of the ring R/J^n, which can be computed from the Eagon-Northcott resolution of R/J^n.

The second inequality asserted in the theorem follows by setting $n = 1$ in the general inequality. $\qquad\square$

In the next result we present both a lower and an upper bound on $\overline{e}_1$ under the running assumptions of this section, namely, $\overline{I} = \mathfrak{m}$ and $\overline{e}_3 = 0$. The lower bound appears already in [7, Theorem 2(1)]. We also establish a condition that assures that the upper bound is tight.

Proposition 3.2. *Let $(R, \mathfrak{m})$ be an analytically unramified, Cohen-Macaulay local ring of dimension $d \geq 3$, type $t(R)$ and infinite residue field. Let I be an R-ideal with $\overline{I} = \mathfrak{m}$ and J a minimal reduction of I. Assume that $\overline{e}_3 = 0$. Then*

$$
e_0 - 1 + \lambda(\overline{I^2}/J\overline{I}) \leq \overline{e}_1 \leq e_0 - 1 + t(R).
$$

Moreover, if $t(R) \neq \lambda(\overline{I^2}/J\overline{I})$, then $\overline{e}_1 < e_0 - 1 + t(R)$.

Proof. Notice that $e_0 - 1 = \lambda(\overline{I}/J)$, since $\overline{I} = \mathfrak{m}$. By [6, Proposition 10] we have for any $n \geq 0$

$$
\begin{aligned}
(3.1) \quad \lambda(R/\overline{I^{n+1}}) \leq e_0\binom{n+d}{d} &- [\lambda(\overline{I}/J) + \lambda(\overline{I^2}/J\overline{I})]\binom{n+d-1}{d-1} \\
&+ \lambda(\overline{I^2}/J\overline{I})\binom{n+d-2}{d-2}.
\end{aligned}
$$

Now

$$
\begin{aligned}
\lambda(R/\overline{I^{n+1}}) &= \lambda(R/J^{n+1}) - \lambda(J^n\overline{I}/J^{n+1}) - \lambda(\overline{I^{n+1}}/J^n\overline{I}) \\
&= \lambda(R/J^{n+1}) - \lambda(J^n/J^{n+1}) + \lambda(J^n/J^n\overline{I}) - \lambda(\overline{I^{n+1}}/J^n\overline{I}) \\
&= e_0\binom{n+d}{d} - e_0\binom{n+d-1}{d-1} + \lambda(R/\overline{I})\binom{n+d-1}{d-1} - \lambda(\overline{I^{n+1}}/J^n\overline{I})
\end{aligned}
$$

where $\lambda(J^n/J^n\overline{I}) = \lambda(R/\overline{I})\binom{n+d-1}{d-1}$ follows since

$$
\begin{aligned}
J^n/J^n\overline{I} &\cong J^n/J^{n+1} \otimes R/\overline{I} \cong [\mathcal{G}(J)]_n \otimes R/\overline{I} \cong [\mathcal{G}(J) \otimes R/\overline{I}]_n \\
&\cong [R/J[T_1, \ldots, T_d] \otimes R/\overline{I}]_n \cong [R/\overline{I}[T_1, \ldots, T_d]]_n.
\end{aligned}
$$

By Proposition 3.1 it follows that for any $n \geq 0$ we have that

$$(3.2) \qquad \lambda(R/\overline{I^{n+1}}) \geq e_0 \binom{n+d}{d} - e_0 \binom{n+d-1}{d-1} + \lambda(R/\overline{I}) \binom{n+d-1}{d-1}$$
$$- t(R) \binom{n+d-2}{d-1}$$
$$= e_0 \binom{n+d}{d} - [\lambda(\overline{I}/J) + t(R)] \binom{n+d-1}{d-1}$$
$$+ t(R) \binom{n+d-2}{d-2}.$$

Note that $\binom{n+d-2}{d-1} = \binom{n+d-1}{d-1} - \binom{n+d-2}{d-2}$ and recall that for all $n \gg 0$ we have

$$(3.3) \qquad \lambda(R/\overline{I^{n+1}}) = e_0 \binom{n+d}{d} - \overline{e}_1 \binom{n+d-1}{d-1} + \overline{e}_2 \binom{n+d-2}{d-2} + \text{ lower terms.}$$

Comparing (3.1), (3.2) and (3.3) we obtain immediately that

$$\lambda(\overline{I}/J) + \lambda(\overline{I^2}/J\overline{I}) \leq \overline{e}_1 \leq \lambda(\overline{I}/J) + t(R).$$

Now assume that $t(R) \neq \lambda(\overline{I^2}/J\overline{I})$. If in (3.2) the inequality is strict for at least one $n \gg 0$, then comparing (3.2) and (3.3) we obtain the desired conclusion, that is,

$$\overline{e}_1 < \lambda(\overline{I}/J) + t(R).$$

Otherwise in (3.2) the equality holds for all $n \gg 0$. Again comparing (3.2) and (3.3) we obtain

$$\overline{e}_1 = \lambda(\overline{I}/J) + t(R), \qquad \overline{e}_2 = t(R).$$

Hence $\overline{e}_2 = \overline{e}_1 - \lambda(\overline{I}/J)$, which implies $\overline{I^{n+1}} = J^{n-1}\overline{I^2}$ for all $n \geq 1$ by [7, Theorem 2(2)]. Now by [6, Proposition 10] we have that equality holds in (3.1), hence

$$t(R) = \overline{e}_2 = \lambda(\overline{I^2}/J\overline{I}),$$

which is a contradiction. $\qquad \square$

In the following theorem we analyze the case when $\lambda(\overline{I^2}/J\overline{I})$ is maximal or almost maximal. In accordance with the classical philosophy we prove that if the bound is attained, then the associated graded ring of the normal filtration is Cohen-Macaulay and furthermore the normal filtration has reduction number two.

Theorem 3.3. *Let $(R, \mathfrak{m})$ be an analytically unramified, Cohen-Macaulay local ring of dimension $d \geq 3$, type $t(R)$ and infinite residue field. Let I be an R-ideal with $\overline{I} = \mathfrak{m}$ and J a minimal reduction of I. Assume that $\overline{e}_3 = 0$ and $\lambda(\overline{I^2}/J\overline{I}) \geq t(R) - 1$. Then $\overline{\mathcal{G}}$ is Cohen-Macaulay and $\overline{I^{n+1}} = J^{n-1}\overline{I^2}$ for all $n \geq 1$.*

Proof. Using Proposition 3.1 and our assumption we have that

$$\lambda(\overline{I^2}/J\overline{I}) \leq t(R) \leq \lambda(\overline{I^2}/J\overline{I}) + 1.$$

If $t(R) = \lambda(\overline{I^2}/J\overline{I})$, then by Proposition 3.2 we have $\overline{e}_1 = \lambda(\overline{I}/J) + \lambda(\overline{I^2}/J\overline{I})$. Hence by [7, Theorem 2.(1)], we have $\overline{I^{n+1}} = J^{n-1}\overline{I^2}$ for every $n \geq 1$.

If $t(R) = \lambda(\overline{I^2}/J\overline{I}) + 1$, then again by Proposition 3.2 we have

$$\lambda(\overline{I}/J) + \lambda(\overline{I^2}/J\overline{I}) \leq \overline{e}_1 < \lambda(\overline{I}/J) + \lambda(\overline{I^2}/J\overline{I}) + 1.$$

Thus $\overline{e}_1 = \lambda(\overline{I}/J) + \lambda(\overline{I^2}/J\overline{I})$ and we conclude as before.

Notice that if $\overline{I^{n+1}} = J^{n-1}\overline{I^2}$, then $\overline{\mathcal{G}}$ is Cohen-Macaulay by the Valabrega-Valla criterion (see [17, Theorem 1.1]), since $\overline{I^2} \cap J = J\overline{I}$ by [6, Theorem 1] or [5, Theorem 4.7 and Appendix]. $\square$

If we strengthen the assumptions in Theorem 2.6 by adding the vanishing of $\overline{e}_3$, we obtain the Cohen-Macaulayness of $\overline{\mathcal{G}}$. Furthermore, the normal filtration of I has reduction number at most two.

Proposition 3.4. *Let $(R, \mathfrak{m})$ be an analytically unramified, Cohen-Macaulay local ring with a canonical module ω_R, of dimension $d \geq 3$ and infinite residue field. Let I be an $\mathfrak{m}$-primary ideal and J a minimal reduction of I. Assume $\overline{e}_1 = e_0 - \lambda(R/\overline{I}) + 1$ and $\overline{e}_3 = 0$. Then $\overline{\mathcal{G}}$ is Cohen-Macaulay and the normal filtration of I has reduction number at most two.*

Proof. We have that

$$\lambda(\overline{I}/J) + 1 = \overline{e}_1 \geq \lambda(\overline{I}/J) + \sum_{n \geq 1} \lambda(\overline{I^{n+1}}/J \cap \overline{I^{n+1}}),$$

where the equality holds by assumption and the inequality is given by [4, Corollary 4.8]. Hence $\sum_{n \geq 1} \lambda(\overline{I^{n+1}}/J \cap \overline{I^{n+1}}) \leq 1$. Again by [4, Corollary 4.8] equality holds if and only if $\overline{\mathcal{G}}$ is Cohen-Macaulay and the reduction number of the normal filtration is at most two.

Thus we may assume that $\sum_{n \geq 1} \lambda(\overline{I^{n+1}}/J \cap \overline{I^{n+1}}) = 0$. In particular, $\overline{I^2} = J \cap \overline{I^2} = J\overline{I}$ by [6, Theorem 1] or [5, Theorem 4.7 and Appendix].

By Theorem 2.6 depth $\overline{\mathcal{G}} \geq d - 1$. According to [4, Proposition 4.6] we have

$$\overline{e}_3 = \sum_{j \geq 2} \binom{j}{2} \lambda(\overline{I^{j+1}}/J\overline{I^j}).$$

As $\overline{e}_3 = 0$, we obtain that $\overline{I^{j+1}} = J\overline{I^j}$ for all $j \geq 1$. Thus $\overline{\mathcal{G}}$ is Cohen-Macaulay by the Valabrega-Valla criterion (see [17, Theorem 1.1]). $\square$

Remark 3.5. Notice that $\sum_{n \geq 1} \lambda(\overline{I^{n+1}}/J \cap \overline{I^{n+1}}) = 0$ in the proof of Proposition 3.4 cannot happen. In fact one would have that $\overline{e}_1 > \lambda(\overline{I}/J) + \sum_{n \geq 1} \lambda(\overline{I^{n+1}}/J \cap \overline{I^{n+1}})$ and $\overline{\mathcal{G}}$ is Cohen-Macaulay, thus contradicting [4, Corollary 4.8].

Theorem 3.6. *Let $(R, \mathfrak{m})$ be an analytically unramified, Cohen-Macaulay local ring with a canonical module ω_R, of dimension $d \geq 3$, type $t(R) \leq 2$ and infinite residue field. Let I be an R-ideal with $\overline{I} = \mathfrak{m}$ and J a minimal reduction of I. Assume that $\overline{e}_3 = 0$. Then*

(a) *$\overline{\mathcal{G}}$ is Cohen-Macaulay;*

(b) *$\mathcal{G}(\mathfrak{m})$ is Cohen-Macaulay, except in the case $t(R) = \lambda(\overline{I^2}/J\mathfrak{m}) = \mu(\mathfrak{m}) - d = 2$ and $\lambda(\mathfrak{m}^2/J\mathfrak{m}) = 1$. In this latter situation, though, depth $\mathcal{G}(\mathfrak{m}) \geq d - 1$.*

Proof. Assume $t(R) = 1$. From Theorem 3.3 it follows that $\overline{\mathcal{G}}$ is Cohen-Macaulay (see also [7, Theorem 3]). Now $\lambda(\mathfrak{m}^2/J\mathfrak{m}) = \lambda(\mathfrak{m}^2/J\overline{I}) \leq \lambda(\overline{I^2}/J\overline{I}) \leq 1$ by Proposition 3.1. If $\mathfrak{m}^2 = J\mathfrak{m}$, then $\mathcal{G}(\mathfrak{m})$ is Cohen-Macaulay (see [18, Theorems 1 and 2]).

If $\lambda(\mathfrak{m}^2/J\mathfrak{m}) = \lambda(\overline{I^2}/J\mathfrak{m}) = 1$, then $\mathfrak{m}^2 = \overline{I^2}$ and hence by Theorem 3.3 we have that for all $n \geq 1$

$$\mathfrak{m}^{n+1} = \overline{I}^{n+1} \subseteq \overline{I^{n+1}} = J^{n-1}\overline{I^2} = J^{n-1}\mathfrak{m}^2 \subseteq \mathfrak{m}^{n+1}.$$

Thus $\overline{\mathcal{G}} = \mathcal{G}(\mathfrak{m})$ and so $\mathcal{G}(\mathfrak{m})$ is Cohen-Macaulay as well (see also [20, Proposition 3.3 and Theorem 3.4]).

Assume now $t(R) = 2$. By Proposition 3.1 we have $\lambda(\overline{I^2}/J\overline{I}) \leq t(R) = 2$ and, by Theorem 3.3, $\overline{\mathcal{G}}$ is Cohen-Macaulay whenever $\overline{I^2} \neq J\overline{I}$. Assume $\overline{I^2} = J\overline{I}$. By Proposition 3.2 one has

$$\lambda(\overline{I}/J) \leq \overline{e}_1 \leq \lambda(\overline{I}/J) + 1.$$

If $\overline{e}_1 = \lambda(\overline{I}/J)$, then $\overline{\mathcal{G}}$ is Cohen-Macaulay by [7, Theorem 2(1)]. If $\overline{e}_1 = \lambda(\overline{I}/J)+1$, then the same conclusion follows from Proposition 3.4.

Now we are going to study $\mathcal{G}(\mathfrak{m})$. As before $\lambda(\mathfrak{m}^2/J\mathfrak{m}) \leq \lambda(\overline{I^2}/J\overline{I}) \leq 2$. If $\mathfrak{m}^2 = J\mathfrak{m}$, clearly $\mathcal{G}(\mathfrak{m})$ is Cohen-Macaulay (see [18, Theorems 1 and 2]).

If $\lambda(\mathfrak{m}^2/J\mathfrak{m}) = 2$, then $\mathfrak{m}^2 = \overline{I^2}$. Since $\overline{I^{n+1}} = J^{n-1}\overline{I^2}$ again by Theorem 3.3 we have that $\mathfrak{m}^n = \overline{I^n}$ for all n, as shown above. In particular $\overline{\mathcal{G}} = \mathcal{G}(\mathfrak{m})$ and $\mathcal{G}(\mathfrak{m})$ is Cohen-Macaulay as well.

Finally, if $\lambda(\mathfrak{m}^2/J\mathfrak{m}) = 1$ and $\mu(\mathfrak{m}) - d < t(R) = 2$, then by [22, Theorem 3.1] $\mathcal{G}(\mathfrak{m})$ is Cohen-Macaulay. Thus we are left to consider the case when $\mu(\mathfrak{m}) - d = 2$ and $1 = \lambda(\mathfrak{m}^2/J\mathfrak{m}) < \lambda(\overline{I^2}/J\overline{I})$. Otherwise, as before, $\mathfrak{m}^n = \overline{I^n}$ for all n, thus $\overline{\mathcal{G}} = \mathcal{G}(\mathfrak{m})$ and $\mathcal{G}(\mathfrak{m})$ is Cohen-Macaulay as well. By [16, Theorem 2.1] we can only conclude that $\operatorname{depth} \mathcal{G}(\mathfrak{m}) \geq d - 1$. $\qquad\square$

We conclude our paper by showing that $\mathcal{G}(\mathfrak{m})$ fails to be Cohen-Macaulay in the exceptional case described in Theorem 3.6(b).

Example 3.7. Consider first the one-dimensional Cohen-Macaulay local ring S of type 2 and multiplicity 4 given by the semigroup ring $k[[t^4, t^5, t^{11}]]$, which can be easily seen to be isomorphic to $k[[X, Y, Z]]/(Z^2 - X^3Y^2, Y^3 - XZ, X^4 - YZ)$. It was shown by J. Sally in [18] that the associated graded ring $\mathcal{G}(\mathfrak{m})$ is not Cohen-Macaulay. Notice now that $\overline{\mathfrak{m}^n} = (t^{4n})k[[t]] \cap R$ for all $n \geq 1$ and that the conductor of R is given by t^8. Thus, $\overline{\mathfrak{m}^2} = \mathfrak{m}^2 + (t^{11})$ whereas $\overline{\mathfrak{m}^n} = \mathfrak{m}^n$ for all $n \geq 3$. This shows that the associated graded ring $\overline{\mathcal{G}}$ of the normal filtration of $\mathfrak{m}$ and $\mathcal{G}(\mathfrak{m})$ have the same Hilbert polynomial. In particular, $e_0 = \overline{e}_0$ and $e_1 = \overline{e}_1$.

Consider now the ring R obtained by adjoining two indeterminates U and V. Thus $R \cong k[[x, y, z, U, V]]$, where x, y, and z denote the images of X, Y, and Z, respectively, modulo the ideal $(Z^2 - X^3Y^2, Y^3 - XZ, X^4 - YZ)$. Let $\mathfrak{n}$ denote the maximal (x, y, z, U, V) of R and observe that $J = (x, U, V)$ is a minimal reduction of $\mathfrak{n}$. In addition to the $\mathfrak{n}$-adic and the integral closure filtrations, $\mathbb{N} = \{\mathfrak{n}^n\}$ and $\mathbb{F} = \{\overline{\mathfrak{n}^n}\}$, respectively, we also consider the following filtration $\mathbb{G}$ defined by

$$G_0 = R, \quad G_1 = \mathfrak{n}, \quad G_2 = (\mathfrak{n}^2, z), \quad \text{and} \quad G_n = J^{n-2}G_2$$

for all $n \geq 3$. Observe that $G_2 \cap J = JG_1$ so that the associated graded ring of the filtration $\mathbb{G}$ is Cohen-Macaulay by the Valabrega-Valla criterion. Moreover, the Rees algebra $\mathcal{R}(\mathbb{G})$ of the filtration $\mathbb{G}$ is also Cohen-Macaulay since the reduction number of the filtration is two and it is strictly smaller than the dimension of R.

We now claim that the filtration $\mathbb{G}$ is actually the normal filtration $\mathbb{F}$. In fact, it is easy to observe that

$$G_n = \mathfrak{m}^n + z \cdot (U, V)^{n-2}$$

for $n \geq 3$. Thus, going modulo U and V, we obtain the equalities $G_n \cdot S = \mathfrak{m}^n = \overline{\mathfrak{m}^n}$ in the one-dimensional ring S. This gives us that $e_0 = e_0(\mathbb{G}) = \overline{e}_0$ and $e_1 = e_1(\mathbb{G}) = \overline{e}_1$. Since we have the inclusion of the Rees algebra $\mathcal{R}(\mathbb{G}) \subset \overline{\mathcal{R}}$ with $\mathcal{R}(\mathbb{G})$ Cohen-Macaulay and $e_1(\mathbb{G}) = \overline{e}_1$, by [14, Theorem 2.2] we conclude that the filtration $\mathbb{G}$ is the normal filtration. In particular $\overline{\mathcal{G}}$ is Cohen-Macaulay.

Finally, by [4, Proposition 4.6] (see also [1, Proposition 1.9] for a simpler proof), we obtain that $\overline{e}_3 = 0$, as $\overline{\mathcal{G}}$ is Cohen-Macaulay and the reduction number of the normal filtration is two. However $\mathcal{G}(\mathfrak{m})$ is not Cohen-Macaulay.

Acknowledgments

References

[1] Anna Guerrieri and Maria Evelina Rossi, *Hilbert coefficients of Hilbert filtrations*, J. Algebra **199** (1998), no. 1, 40–61, DOI 10.1006/jabr.1997.7194. MR1489353 (98i:13027)

[2] Melvin Hochster and Craig Huneke, *Indecomposable canonical modules and connectedness*, Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992), Contemp. Math., vol. 159, Amer. Math. Soc., Providence, RI, 1994, pp. 197–208, DOI 10.1090/conm/159/01509. MR1266184 (95e:13014)

[3] Sam Huckaba and Thomas Marley, *Depth formulas for certain graded rings associated to an ideal*, Nagoya Math. J. **133** (1994), 57–69. MR1266362 (95d:13015)

[4] Sam Huckaba and Thomas Marley, *Hilbert coefficients and the depths of associated graded rings*, J. London Math. Soc. (2) **56** (1997), no. 1, 64–76, DOI 10.1112/S0024610797005206. MR1462826 (98i:13028)

[5] Craig Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J. **34** (1987), no. 2, 293–318, DOI 10.1307/mmj/1029003560. MR894879 (89b:13037)

[6] Shiroh Itoh, *Integral closures of ideals generated by regular sequences*, J. Algebra **117** (1988), no. 2, 390–401, DOI 10.1016/0021-8693(88)90114-7. MR957448 (90g:13013)

[7] Shiroh Itoh, *Coefficients of normal Hilbert polynomials*, J. Algebra **150** (1992), no. 1, 101–117, DOI 10.1016/S0021-8693(05)80052-3. MR1174891 (93j:13021)

[8] Mousumi Mandal, Shreedevi Masuti, and J. K. Verma, *Normal Hilbert polynomials: a survey*, Commutative algebra and algebraic geometry (CAAG-2010), Ramanujan Math. Soc. Lect. Notes Ser., vol. 17, Ramanujan Math. Soc., Mysore, 2013, pp. 139–166. MR3155958

[9] Thomas John Marley, *Hilbert functions of ideals in Cohen-Macaulay rings*, ProQuest LLC, Ann Arbor, MI, 1989. Thesis (Ph.D.)–Purdue University. MR2637965

[10] D. G. Northcott, *A note on the coefficients of the abstract Hilbet function*, J. London Math. Soc. **35** (1960), 209–214. MR0110731 (22 #1599)

[11] D. G. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Cambridge Philos. Soc. **50** (1954), 145–158. MR0059889 (15,596a)

[12] Akira Ooishi, *Δ-genera and sectional genera of commutative rings*, Hiroshima Math. J. **17** (1987), no. 2, 361–372. MR909621 (89f:13033)

[13] Claudia Polini, *A filtration of the Sally module and the associated graded ring of an ideal*, Comm. Algebra **28** (2000), no. 3, 1335–1341, DOI 10.1080/00927870008826896. MR1742657 (2001g:13006)

[14] Claudia Polini, Bernd Ulrich, and Wolmer V. Vasconcelos, *Normalization of ideals and Briançon-Skoda numbers*, Math. Res. Lett. **12** (2005), no. 5-6, 827–842, DOI 10.4310/MRL.2005.v12.n6.a5. MR2189243 (2006m:13006)

[15] D. Rees, *A note on analytically unramified local rings*, J. London Math. Soc. **36** (1961), 24–28. MR0126465 (23 #A3761)

[16] Maria Evelina Rossi and Giuseppe Valla, *A conjecture of J. Sally*, Comm. Algebra **24** (1996), no. 13, 4249–4261, DOI 10.1080/00927879608825812. MR1414582 (97j:13021)

[17] Maria Evelina Rossi and Giuseppe Valla, *Hilbert functions of filtered modules*, Lecture Notes of the Unione Matematica Italiana, vol. 9, Springer-Verlag, Berlin; UMI, Bologna, 2010. MR2723038 (2012b:13042)

[18] Judith D. Sally, *On the associated graded ring of a local Cohen-Macaulay ring*, J. Math. Kyoto Univ. **17** (1977), no. 1, 19–21. MR0450259 (56 #8555)

[19] Judith D. Sally, *Cohen-Macaulay local rings of maximal embedding dimension*, J. Algebra **56** (1979), no. 1, 168–183, DOI 10.1016/0021-8693(79)90331-4. MR527163 (80e:14022)

[20] Judith D. Sally, *Tangent cones at Gorenstein singularities*, Compositio Math. **40** (1980), no. 2, 167–175. MR563540 (81e:14004)

[21] Judith D. Sally, *Good embedding dimensions for Gorenstein singularities*, Math. Ann. **249** (1980), no. 2, 95–106, DOI 10.1007/BF01351406. MR578716 (82c:13031)

[22] Judith D. Sally, *Cohen-Macaulay local rings of embedding dimension $e + d - 2$*, J. Algebra **83** (1983), no. 2, 393–408, DOI 10.1016/0021-8693(83)90226-0. MR714252 (85c:13017)

[23] Wolmer V. Vasconcelos, *Hilbert functions, analytic spread, and Koszul homology*, Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992), Contemp. Math., vol. 159, Amer. Math. Soc., Providence, RI, 1994, pp. 401–422, DOI 10.1090/conm/159/01520. MR1266195 (95a:13006)

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