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# ON CONJECTURES BY CSORDAS, CHARALAMBIDES AND WALEFFE

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Abstract. In the present note we obtain new results on two conjectures by Csordas et al. regarding the interlacing property of zeros of special polynomials. These polynomials came from the Jacobi tau methods for the Sturm-Liouville eigenvalue problem. Their coefficients are the successive even derivatives of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ evaluated at the point one. The first conjecture states that the polynomials constructed from $P_n^{(\alpha,\beta)}$ and $P_{n-1}^{(\alpha,\beta)}$ are interlacing when $-1 < \alpha < 1$ and $-1 < \beta$. We prove it in a range of parameters wider than that given earlier by Charalambides and Waleffe. We also show that within narrower bounds another conjecture holds. It asserts that the polynomials constructed from $P_n^{(\alpha,\beta)}$ and $P_{n-2}^{(\alpha,\beta)}$ are also interlacing.

## 1. Introduction

This study is devoted to properties of zeros of polynomials which originate from mathematical physics. The orthogonal polynomials proved to be a very helpful tool for the discretization of linear differential operators. The main feature of the tau methods is the adoption of a polynomial basis which does not automatically satisfy the boundary conditions. This induces a problem at the boundary (i.e., the points $\pm 1$ in the Jacobi case). The family of polynomials studied here is connected this way to the eigenproblem $u''(x) = \lambda u(x)$ on the interval $x \in (-1, 1)$ with various homogeneous boundary conditions (for the details see [3, 4]). We place our main emphasis on the analytic properties of the considered family itself, leaving aside the corresponding properties of the original differential operators. More information on the tau methods can be found in, e.g., [1, §10.4.2].

The *Jacobi polynomials* (see their definition and basic properties in, e.g., [7, Ch. IV])

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} {}_2F_1\left[\begin{matrix} -n, & n+\alpha+\beta+1 \\ \alpha+1 & \end{matrix}; \frac{1-x}{2}\right], \quad n = 1, 2, \ldots,$$

appear regularly in applications as classical orthogonal polynomials. They are more general than those of Chebyshev, Legendre and Gegenbauer. The Jacobi polynomials are orthogonal with respect to the measure $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$

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on the interval $(-1,1)$ whenever both the parameters $\alpha$ and $\beta$ are greater than $-1$:

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(x) w_{\alpha,\beta}(x)\,dx = 0 \quad \text{if} \quad k \neq n.$$

The usual normalization supposes that $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \frac{(\alpha+1)_n}{n!}$, where we applied the so-called *Pochhammer symbol* or the *rising factorial* defined as

$$(\alpha+1)_n := (\alpha+1)\cdot(\alpha+2)\cdots(\alpha+n).$$

In this notation we have

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (n+\alpha+\beta+1)_k}{k!\,(\alpha+1)_k} \left(\frac{1-x}{2}\right)^k, \quad n = 1,2,\ldots. \tag{1}$$

**Definition** (see [2, p. 17])**.** We say that the zeros of the polynomials $g(x)$ and $h(x)$ *interlace* (or *interlace strictly*) if the following conditions hold simultaneously:

- all zeros of $g(x)$ and $h(x)$ are simple, real and distinct (i.e., the polynomials are coprime),
- between each two consecutive zeros of $g(x)$ there is exactly one zero of the polynomial $h(x)$, and
- between each two consecutive zeros of $h(x)$ there is exactly one zero of the polynomial $g(x)$.

We say that the zeros of the polynomials $g(x)$ and $h(x)$ *interlace non-strictly* if their zeros are real and become strictly interlacing after dividing both polynomials by the greatest common divisor $\gcd(g,h)$. Roughly speaking, the zeros of two polynomials interlace non-strictly if they can meet but never pass through each other when changing continuously from a strictly interlacing state.

**Definition.** A pair $(g(x),h(x))$ is called *real* if for any real numbers $A,B$ the combination $Ag(x)+Bh(x)$ has only real zeros. This is equivalent to the non-strict interlacing property of $g(x)$ and $h(x)$, which is shown in, e.g., [2, Chapter I].

*Remark.* The phrases *"$g(x)$ and $h(x)$ interlace"*, *"$g(x)$ and $h(x)$ possess the interlacing property"*, *"$g(x)$ interlaces $h(x)$"*, *"$g(x)$ and $h(x)$ have interlacing zeros"* and *"the zeros of $g(x)$ and $h(x)$ are interlacing"* we use synonymously.

It is well known that the orthogonal polynomials on the real line have real interlacing zeros (due to the so-called three-term recurrence; see, e.g., [7, pp. 42–47, Sections 3.2–3.3]). That is, in particular, the zeros of $P_n^{(\alpha,\beta)}$ and $P_{n-1}^{(\alpha,\beta)}$ interlace for all natural $n$. In the present note we study zeros of polynomials that do not satisfy the three-term recurrence. More specifically, we consider

$$\begin{aligned} \phi_n^{(\alpha,\beta)}(\mu) &:= \sum_{k=0}^{[n/2]} \frac{d^{2k}}{dx^{2k}} P_n^{(\alpha,\beta)}(x)\bigg|_{x=1} \cdot \mu^k \\ &= \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{[n/2]} \frac{(-n)_{2k}(n+\alpha+\beta+1)_{2k}}{(\alpha+1)_{2k}} \left(\frac{\mu}{4}\right)^k, \quad n = 1,2,\ldots, \end{aligned} \tag{2}$$

where the notation $[a]$ stands for the integer part of the number $a$.

**Theorem 1** (Csordas, Charalambides and Waleffe [4])**.** *For every positive integer $n \geqslant 2$ the polynomial $\phi_n^{(\alpha,\beta)}(\mu)$, $-1<\alpha<1$, $-1<\beta$, has only real negative zeros.*

The proof of this theorem given in [4] rests on the Hermite-Biehler theory (see Theorem 3 herein).

*Remark* (to Theorem 1). In fact, the authors have shown that $\phi_n^{(\alpha,\beta)}(\mu)$ interlaces $\phi_{n-1}^{(\alpha+1,\beta+1)}(\mu)$. As a result, *the theorem remains valid when* $1 \leqslant \alpha < 2$ *and* $0 < \beta$. Note that the interlacing property here is strict, so it implies the simplicity of the zeros. Furthermore, *the polynomial* $\phi_n^{(\alpha,\beta)}(\mu)$ *has only simple negative zeros for* $-1 < \alpha < 0$ *and* $-2 < \beta$, as well. This follows as a straightforward consequence of Theorem 7 and Lemma 4 of the present study.

Based on Theorem 1 the authors of [4] conjectured that these polynomials also have the following property.

**Conjecture 1** ([4, p. 3559])**.** *For* $-1 < \alpha < 1$, $-1 < \beta$ *and* $n \geqslant 4$ *the zeros of the polynomials* $\phi_n^{(\alpha,\beta)}(\mu)$ *and* $\phi_{n-1}^{(\alpha,\beta)}(\mu)$ *interlace.*

In particular, this assertion would imply that the spectra of polynomial approximations to the corresponding differential operator are negative and simple (see [3]). In [3], Conjecture 1 was proved for $-1 < \alpha, \beta < 0$ and $0 < \alpha, \beta < 1$; see Theorem 6 below. In fact, the upper bound on $\beta$ is redundant. Theorem 7 with a shorter proof states that the conjecture holds true for

$$-1 < \alpha < 0,\ -1 < \beta \quad \text{or} \quad 0 \leqslant \alpha < 1,\ 0 < \beta \quad \text{or} \quad 1 \leqslant \alpha < 2,\ 1 < \beta. \tag{3}$$

Additionally, we study another assertion about the same polynomials.

**Conjecture 2** ([4, p. 3559])**.** *For* $\phi_n^{(\alpha,\beta)}(\mu)$ *as in Theorem* 1 *and for all* $n \geqslant 5$, *the zeros of the polynomials* $\phi_n^{(\alpha,\beta)}(\mu)$ *and* $\phi_{n-2}^{(\alpha,\beta)}(\mu)$ *interlace.*

Originally, this conjecture was stated for $-1 < \alpha < 1$ and $\beta > -1$. However, numerical calculations show that it fails for some values satisfying $-1 < \beta < 0 < \alpha < 1$. Our (partial) solution to Conjecture 2 is given in Theorem 14: it holds true for $-1 < \alpha < 0 < \beta$ or $0 < \alpha < 1 < \beta$. We approach by extending the idea of [3] to another pair of auxiliary polynomials. Certainly, there exists a relation between Conjecture 1 and Conjecture 2 as discussed in Section 5.

Vieta's formulae imply that the sum of all zeros of $\phi_n^{(\alpha,\beta)}(\mu)$ tends to $-1/2$ for even $n$ and to $-1/6$ for odd $n$ as $n \to \infty$. Thus, the assertion of Conjecture 2 gives that the zero points of $\phi_n^{(\alpha,\beta)}(\mu)$ converge monotonically in $n$ outside of any fixed interval containing the origin. If the assertions of both conjectures hold, then the fraction $\phi_{2n-1}^{(\alpha,\beta)}(\mu)/\phi_{2n}^{(\alpha,\beta)}(\mu)$ maps the upper half of the complex plane into itself and converges to a function meromorphic outside of any disk centered at the origin. This situation resembles how the quotients of orthogonal polynomials of the first and second kinds behave.

Section 2 of the present paper introduces connections between polynomials $\phi_n^{(\alpha,\beta)}(\mu)$ with different $n$, $\alpha$ and $\beta$. These connections allow us to extend and clarify the result [3] in Section 3 (see Theorem 7). We show that Conjecture 1 holds true under the conditions (3). Section 4 contains the proof of Conjecture 2 for $-1 < \alpha < 0 < \beta$ and $0 < \alpha < 1 < \beta$ (see Theorem 14). In the last section we show that the studied conjectures are actually related.

## 2. Basic relations between the polynomials $\phi_n^{(\alpha,\beta)}$ for various $\alpha$ and $\beta$

Being connected with the Jacobi polynomials, the family $\left(\phi_n^{(\alpha,\beta)}\right)_n$, where $n = 2, 3, \dots$, inherits some of their properties. The formulae induced by the corresponding relations for the Jacobi case include (we omit the argument $\mu$ of $\phi_n^{(\alpha,\beta)}$ for brevity's sake):

$$(2n+\alpha+\beta)\phi_n^{(\alpha,\beta-1)} = (n+\alpha+\beta)\phi_n^{(\alpha,\beta)} + (n+\alpha)\phi_{n-1}^{(\alpha,\beta)}, \tag{4}$$

$$(2n+\alpha+\beta)\phi_n^{(\alpha-1,\beta)} = (n+\alpha+\beta)\phi_n^{(\alpha,\beta)} - (n+\beta)\phi_{n-1}^{(\alpha,\beta)}, \tag{5}$$

$$(n+\alpha+\beta)\phi_n^{(\alpha,\beta)} \overset{(4)+(5)}{=\!=\!=} (n+\beta)\phi_n^{(\alpha,\beta-1)} + (n+\alpha)\phi_n^{(\alpha-1,\beta)}, \tag{6}$$

$$\phi_{n-1}^{(\alpha,\beta)} \overset{(4)-(5)}{=\!=\!=} \phi_n^{(\alpha,\beta-1)} - \phi_n^{(\alpha-1,\beta)}. \tag{7}$$

The latter two identities contain labels of equations above the equality sign: we use the convention that this explains how the equalities can be obtained (up to some proper coefficients). For example, $2n+\alpha+\beta$ times (6) is the sum of $n+\beta$ times (4) and $n+\alpha$ times (5), whereas $2n+\alpha+\beta$ times (7) is the difference (4) − (5). The identities (4) and (5) can be checked by applying the formulae (see, e.g., [6, p. 737])

$$(2n+\alpha+\beta)P_n^{(\alpha,\beta-1)}(x) = (n+\alpha+\beta)P_n^{(\alpha,\beta)}(x) + (n+\alpha)P_{n-1}^{(\alpha,\beta)}(x), \tag{8}$$

$$(2n+\alpha+\beta)P_n^{(\alpha-1,\beta)}(x) = (n+\alpha+\beta)P_n^{(\alpha,\beta)}(x) - (n+\beta)P_{n-1}^{(\alpha,\beta)}(x), \tag{9}$$

respectively, to the left-hand side of the definition (2). The relations involving derivatives (not surprisingly) differ from those for the Jacobi polynomials:

$$(n+\alpha)\phi_{n-1}^{(\alpha,\beta+1)} = n\phi_n^{(\alpha,\beta)} - 2\mu\left(\phi_n^{(\alpha,\beta)}\right)', \tag{10}$$

$$(n+\alpha+\beta+1)\phi_n^{(\alpha,\beta+1)} \overset{(10)\text{ and }(4)}{=\!=\!=\!=} (n+\alpha+\beta+1)\phi_n^{(\alpha,\beta)} + 2\mu\left(\phi_n^{(\alpha,\beta)}\right)', \tag{11}$$

$$(n+\alpha)\phi_n^{(\alpha-1,\beta+1)} \overset{(5),\text{ then }(11)-(10)}{=\!=\!=\!=} \alpha\phi_n^{(\alpha,\beta)} + 2\mu\left(\phi_n^{(\alpha,\beta)}\right)', \tag{12}$$

$$\frac{n+\beta}{n+\alpha+\beta}2\mu\left(\phi_n^{(\alpha,\beta-1)}\right)' \overset{(11)\text{ and }(6)}{=\!=\!=\!=} (n+\alpha)\phi_n^{(\alpha-1,\beta)} - \alpha\phi_n^{(\alpha,\beta)}, \tag{13}$$

$$2\mu\left(\phi_n^{(\alpha,\beta-1)}\right)' \overset{(10)\text{ and }(7)}{=\!=\!=\!=} n\phi_n^{(\alpha-1,\beta)} - \alpha\phi_{n-1}^{(\alpha,\beta)}. \tag{14}$$

So we see that the presented identities are not independent in the sense that they can be obtained as combinations of others with various $\alpha$ and $\beta$. We derive

the formula (10) with the help of the right-hand side of (2).

$$\begin{aligned}
&n\phi_n^{(\alpha,\beta)} - 2\mu\left(\phi_n^{(\alpha,\beta)}\right)' \\
&\quad = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{[n/2]}\frac{(-n)_{2k}(n+\alpha+\beta+1)_{2k}}{(\alpha+1)_{2k}}\left(\frac{\mu}{4}\right)^k(n-2k) \\
&\quad = (n+\alpha)\frac{(\alpha+1)_{n-1}}{(n-1)!} \\
&\qquad \times \sum_{k=0}^{[n/2]}(-n+2k)\frac{(-n)(-n+1)\cdots(-n+2k-1)(n+\alpha+\beta+1)_{2k}}{(-n)(\alpha+1)_{2k}}\left(\frac{\mu}{4}\right)^k \\
&\quad = (n+\alpha)\phi_{n-1}^{(\alpha,\beta+1)}.
\end{aligned}$$

Certainly, we can combine formulae further obtaining, e.g.,

$$\begin{aligned}
n\phi_n^{(\alpha,\beta)} - 2\mu\left(\phi_n^{(\alpha,\beta)}\right)' &\overset{(10)}{=\!=\!=} (n+\alpha)\phi_{n-1}^{(\alpha,\beta+1)} \\
&\overset{(11)}{=\!=\!=} (n+\alpha)\phi_{n-1}^{(\alpha,\beta)} + \frac{n+\alpha}{n+\alpha+\beta}2\mu\left(\phi_{n-1}^{(\alpha,\beta)}\right)',
\end{aligned} \tag{15}$$

$$\begin{aligned}
n\phi_n^{(\alpha,\beta)} - (n+\alpha)\phi_{n-1}^{(\alpha,\beta)} &\overset{(11)-(10)}{=\!=\!=\!=} \frac{2n+\alpha+\beta}{n+\alpha+\beta}2\mu\left(\phi_n^{(\alpha,\beta-1)}\right)' \\
&\overset{\frac{d}{d\mu}(4)}{=\!=\!=\!=} 2\mu\left(\phi_n^{(\alpha,\beta)}\right)' + \frac{n+\alpha}{n+\alpha+\beta}2\mu\left(\phi_{n-1}^{(\alpha,\beta)}\right)'.
\end{aligned} \tag{16}$$

The key role further plays the following combination of (10) and (11), which is valid for an arbitrary real $A$,

$$\frac{n+\alpha+\beta}{n+\alpha}\phi_n^{(\alpha,\beta)} + A\phi_{n-1}^{(\alpha,\beta)} = \frac{(1+A)n+\alpha+\beta}{n+\alpha}\phi_n^{(\alpha,\beta-1)} + 2\mu\frac{1-A}{n+\alpha}\left(\phi_n^{(\alpha,\beta-1)}\right)'. \tag{17}$$

The next identity stands apart and can be checked explicitly with the help of (2)

$$\phi_n^{(\alpha,\beta)}(\mu) - \phi_n^{(\alpha,\beta)}(0) = \tfrac{1}{4}(n+\alpha+\beta+1)_2\cdot\mu\phi_{n-2}^{(\alpha+2,\beta+2)}(\mu).$$

It reflects the standard formula for the derivative of the Jacobi polynomial (e.g., [6, p. 737]):

$$\left(P_n^{(\alpha,\beta)}(x)\right)^{(m)} = 2^{-m}(n+\alpha+\beta+1)_m\,P_{n-m}^{(\alpha+m,\beta+m)}(x). \tag{18}$$

*Remark.* Note that the equalities (4)–(18) are of formal nature, and therefore their validity requires no orthogonality from the Jacobi polynomials. That is, these equalities hold true if all coefficients are defined, not only for $\alpha,\beta > -1$.

*Remark.* A polynomial with only real zeros interlaces its derivative by Rolle's theorem.[1] Consequently, they both interlace any real combination of them.[2] So if in one of the formulae (10)–(12) the first term on the right-hand side has only real zeros, then all three involved polynomials are pairwise interlacing. For example, if $\phi_n^{(\alpha,\beta)}$ has only real zeros, then $\phi_{n-1}^{(\alpha,\beta+1)}$, $\phi_n^{(\alpha,\beta)}$ and $2\mu\big(\phi_n^{(\alpha,\beta)}\big)'$ are pairwise interlacing which is provided by (10).

[1]Non-strictly whenever the polynomial has a multiple zero.

[2]See the definition of a real pair on page 2038.

*Remark.* The identities (6) and (7) show that the interlacing property of $\phi_n^{(\alpha,\beta)}(\mu)$ and $\phi_{n-1}^{(\alpha,\beta)}(\mu)$ can also be expressed as the interlacing property of $\phi_n^{(\alpha,\beta-1)}(\mu)$ and $\phi_n^{(\alpha-1,\beta)}(\mu)$. Analogously, from the relations (13) and (14) it can be seen that this is also equivalent to the interlacing property of $\big(\phi_n^{(\alpha,\beta-1)}(\mu)\big)'$ and $\phi_n^{(\alpha-1,\beta)}(\mu)$.

**Lemma 2.** *If $-1<\alpha<1$, $-1<\beta$ or if $1\leqslant\alpha<2$, $0<\beta$, then the polynomials $\big(\phi_n^{(\alpha,\beta)}(\mu)\big)'$, $\big(\phi_{n-1}^{(\alpha,\beta)}(\mu)\big)'$ and $\big(\phi_n^{(\alpha,\beta-1)}(\mu)\big)'$ are pairwise interlacing.*

*Proof.* By Theorem 1 the polynomials $\phi_n^{(\alpha,\beta)}(\mu)$, $\phi_{n-1}^{(\alpha,\beta)}(\mu)$ have only (simple) negative zeros. Then the relation (10) shows that the polynomial $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$ with negative zeros interlaces (strictly) both $\phi_n^{(\alpha,\beta)}(\mu)$ and $\mu\big(\phi_n^{(\alpha,\beta)}(\mu)\big)'$. Moreover, the sign of $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$ at the origin and at the rightmost zero of $\phi_n^{(\alpha,\beta)}(\mu)$ is the same, and therefore $\Big(\phi_n^{(\alpha,\beta)}(\mu),\mu\phi_{n-1}^{(\alpha,\beta+1)}(\mu)\Big)$ is a real coprime pair. A similar consideration of the relation (11) gives that the pair $\Big(\mu\phi_{n-1}^{(\alpha,\beta)}(\mu),\phi_{n-1}^{(\alpha,\beta+1)}(\mu)\Big)$ is also real and coprime.

In particular, each interval between two consequent zeros of $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$ contains exactly one zero of $\phi_n^{(\alpha,\beta)}(\mu)$ as well as of $\phi_{n-1}^{(\alpha,\beta)}(\mu)$. Taking the signs of the last two polynomials at the ends of these intervals into account shows that the difference in the left-hand side of (16), and hence $\big(\phi_n^{(\alpha,\beta-1)}(\mu)\big)'$, changes its sign between consecutive zeros of $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$. Since $\deg\big(\phi_n^{(\alpha,\beta-1)}\big)'\leqslant\deg\phi_{n-1}^{(\alpha,\beta+1)}$, the polynomial $\mu\big(\phi_n^{(\alpha,\beta-1)}(\mu)\big)'$ necessarily interlaces $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$. Then the right-hand side of (16) shows that

$$\Big(\phi_n^{(\alpha,\beta)}(\mu)\Big)'+\frac{n+\alpha}{n+\alpha+\beta}\Big(\phi_{n-1}^{(\alpha,\beta)}(\mu)\Big)' \tag{19}$$

and $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$ have interlacing zeros. At the same time, by differentiating the equality (16) we obtain that

$$n\Big(\phi_n^{(\alpha,\beta)}(\mu)\Big)'-(n+\alpha)\Big(\phi_{n-1}^{(\alpha,\beta)}(\mu)\Big)' \tag{20}$$

is proportional to $\Big(\mu\big(\phi_n^{(\alpha,\beta-1)}(\mu)\big)'\Big)'$ and, hence, interlaces $\mu\big(\phi_n^{(\alpha,\beta-1)}(\mu)\big)'$. Put in other words, the polynomials (19) and (20) are interlacing. With appropriate factors, their sum gives $\big(\phi_n^{(\alpha,\beta)}(\mu)\big)'$ and their difference gives $\big(\phi_{n-1}^{(\alpha,\beta)}(\mu)\big)'$. This yields the lemma. □

## 3. Results on Conjecture 1

**Theorem 3** (Hermite-Biehler, for real polynomials)**.** *A real polynomial $f(z):=p(z^2)+zq(z^2)$ is stable*[3] *if and only if $p(0)\cdot q(0)>0$ and all zeros of $p(z)$ and $zq(z)$ are non-positive and strictly interlacing.*

This well-known fact plays a crucial role in [4] for proving Theorem 1. Here the statement of the Hermite-Biehler theorem is expressed closely to the one given in [5, p. 228]. The replacement of $q(z)$ with $zq(z)$ provides the desired order of zeros. That is, the zero of $p(z)$ and $q(z)$ closest to the origin belongs to the former

[3]That is, Hurwitz stable: $f(z)=0\implies\Re z<0$.

polynomial. The more general statement can be found in, e.g., [2, p. 21]. The present study also uses Theorem 3 as a main tool.

Some bounds on the parameters $\alpha$ and $\beta$ are necessary even for Theorem 1 (i.e., are satisfied if all zeros of $\phi_n^{(\alpha,\beta)}(\mu)$ are negative for every $n \geqslant 2$). The restriction $\alpha > -1$ corresponds to positivity of the coefficients (and to negativity of all zeros). The parameter $\alpha$ is bounded from above by $3.37228\ldots$. Indeed, if the polynomial $\phi_n^{(\alpha,\beta)}(\mu) =: \sum_{k=0}^{m} b_k\mu^k$, $m = [n/2]$, has only negative[4] zeros, then by Rolle's theorem, the zeros of $p(\mu) := \mu^m\phi_n^{(\alpha,\beta)}(\mu^{-1})$ and $p'(\mu)$ are interlacing. So, Theorem 3 then implies that the polynomial $p(\mu^2) + \mu p'(\mu^2)$ is stable. Therefore, when $b_0 > 0$ the Hurwitz criterion (see, e.g., [2, Chapter I]) gives us the easy-to-check conditions $b_1 > 0$ and

$$\begin{vmatrix} mb_0 & (m-1)b_1 & (m-2)b_2 \\ b_0 & b_1 & b_2 \\ 0 & mb_0 & (m-1)b_1 \end{vmatrix} \geqslant 0, \quad \text{that is,} \quad \frac{b_1^2}{b_0b_2} \geqslant \frac{2m}{m-1} > 2, \tag{21}$$

which are necessarily true when the polynomial $p(\mu)$ has only real zeros. In our case we have

$$\frac{b_1^2}{b_0b_2} = \frac{n(n-1)(\alpha+3)_2(n+\alpha+\beta+1)_2}{(n-2)(n-3)(\alpha+1)_2(n+\alpha+\beta+3)_2} \xrightarrow{n\to\infty} \frac{(\alpha+3)(\alpha+4)}{(\alpha+1)(\alpha+2)},$$

so the condition (21) fails to be true (along with the assertion of Theorem 1) for every $n$ big enough and every $\beta$ when $\alpha > \frac{1+\sqrt{33}}{2} = 3.37228\ldots$ or $\alpha < \frac{1-\sqrt{33}}{2} = -2.37228\ldots$. Computer experiments show that the polynomials $\phi_n^{(\alpha,\beta)}$ with *positive coefficients* can have zeros outside the real axis for a (large enough) negative $\beta$. So, some lower constraint on the parameter $\beta$ is also required.

**Definition.** Denote the $i$th zero of a polynomial $p$ with respect to the distance from the origin by $\mathbf{zr}_i(p)$. Put $\mathbf{zr}_i(p)$ equal to $-\infty$ if $\deg p < i$ and to zero if $i = 0$ (it is convenient since all coefficients of the polynomials we deal with are non-negative).

**Lemma 4.** *Let $\alpha > -1$ and $n + \alpha + \beta > 0$, $n = 4, 5, \ldots$. The zeros of the polynomials $\phi_n^{(\alpha,\beta)}$ and $\phi_{n-1}^{(\alpha,\beta)}$ are negative and interlace non-strictly (strictly) if and only if the polynomial $\phi_n^{(\alpha,\beta-1)}$ has only real zeros (only simple real zeros, respectively). Moreover, if $\phi_n^{(\alpha,\beta-1)}$ has only real zeros, then $\mathbf{zr}_1\left(\phi_{n-1}^{(\alpha,\beta)}\right) \leqslant \mathbf{zr}_1\left(\phi_n^{(\alpha,\beta)}\right)$.*

*Proof.* The relation (17) with $A = 1$ and $A = -(n+\alpha+\beta)/n$ implies that each common zero of the polynomials $\phi_{n-1}^{(\alpha,\beta)}$ and $\phi_n^{(\alpha,\beta)}$ is a multiple zero of $\phi_n^{(\alpha,\beta-1)}$. The converse result is given by (10) and (11).

Suppose that $\phi_n^{(\alpha,\beta-1)}$ has only real zeros. The coefficients of this polynomial are positive under the assumptions of the lemma, and hence all of its zeros are negative. By Rolle's theorem, the pair $\left(\phi_n^{(\alpha,\beta-1)}, \mu\left(\phi_n^{(\alpha,\beta-1)}\right)'\right)$ is real. Therefore, the polynomials $\phi_{n-1}^{(\alpha,\beta)}$ and $\phi_n^{(\alpha,\beta)}$ have only real zeros by the formulae (10) and (11), respectively. The zeros are negative automatically since the coefficients of polynomials are positive. Moreover, we have that the first zero of $\phi_n^{(\alpha,\beta)}$ is closer to the origin than that of $\phi_{n-1}^{(\alpha,\beta)}$. The relation (17) holds for all real $A$, which yields

[4]Here we suppose that $\phi_n^{(\alpha,\beta)}(\mu)$ has only simple zeros; the case of multiple zeros follows by continuity.

that the polynomials $\phi_{n-1}^{(\alpha,\beta)}$ and $\phi_n^{(\alpha,\beta)}$ form a real pair and thus have (non-strictly) interlacing zeros.

Let $\phi_{n-1}^{(\alpha,\beta)}$ and $\phi_n^{(\alpha,\beta)}$ have negative interlacing zeros. Then any of their real combinations only has real zeros. This is true for $\phi_n^{(\alpha,\beta-1)}$ according to the identity (4). □

**Corollary 5.** *For $-1 < \alpha < 1$, $\beta > 0$ or $1 \leqslant \alpha < 2$, $\beta > 1$ the zeros of the polynomials $\phi_n^{(\alpha,\beta)}$ and $\phi_{n-1}^{(\alpha,\beta)}$, $n = 4, 5, \dots$, interlace. (Furthermore, the polynomials $\phi_n^{(\alpha,\beta)}$ and $\mu\phi_{n-1}^{(\alpha,\beta)}$ interlace.)*

*Proof.* This corollary is provided by Theorem 1 (see also the remark on it) and Lemma 4. □

**Theorem 6** (Theorem 3.10 in Charalambides et al. [3])**.** *Conjecture* 1 *holds true for $-1 < \alpha, \beta < 0$ and for $0 < \alpha, \beta < 1$.*

This theorem relies on Theorem 3.8 and Theorem 3.9 of the same work and on Theorem 1. In fact, the original proof (which we extend in the next section to treat Conjecture 2) does not need any upper bound on $\beta$. It becomes more evident on recalling that the region of positive $\beta$ is covered by Corollary 5 (i.e., is a straightforward consequence of Lemma 4 and Theorem 1).

**Theorem 7.** *Conjecture* 1 *holds true for $-1 < \alpha < 0$, $-1 < \beta$ or $0 \leqslant \alpha < 1$, $0 < \beta$ or $1 \leqslant \alpha < 2$, $1 < \beta$.*

*Proof.* Corollary 5 suits the case of positive $\alpha$. For the region with negative $\alpha$ it is enough to prove only Theorem 8 (which is stated below) and Theorem 3.9 is not needed. Indeed, according to Theorem 8 and Theorem 3 we have that all zeros of $\phi_n^{(\alpha,\beta-1)}$ are simple and real for all $n$, so Lemma 4 is applicable. □

**Theorem 8** (Equivalent to Theorem 3.8, Charalambides et al. [3])**.** *If $-1 < \alpha < 0$, $-1 < \beta$ and $n = 4, 5, \dots$, then the zeros of the polynomial $\Phi_n(1;\mu)$, where*

$$\Phi_n(x;\mu) := \sum_{k=0}^{n} \frac{d^k}{dx^k} P_n^{(\alpha,\beta-1)}(x)\mu^k,$$

*lie in the open left half of the complex plane.*

*Remark.* It is worth noting that this theorem cannot be extended to the full range $-1 < \beta < 0 < \alpha < 1$. According to computer experiments, Conjecture 1 seems to hold in this range, while Theorem 8 fails, e.g., for $n = 12$ when $\beta = -0.8$ and $\alpha \gtrapprox 0.97842$, or when $\beta = -0.9$ and $\alpha \gtrapprox 0.97140$. The reason is that proving Conjecture 1 only requires negative simple zeros of the polynomial $\phi_n^{(\alpha,\beta-1)}(\mu)$, Theorem 8 nevertheless asserts additional properties of $\phi_{n-1}^{(\alpha+1,\beta)}(\mu)$ as given by the Hermite-Biehler theorem.

*Proof.* This proof is akin to [3, Theorem 3.8] but uses other relations for the Jacobi polynomials. The polynomial $\Phi_n(x;\mu)$ satisfies the differential equation (here we consider $\mu$ as a parameter)

$$\Phi_n(x;\mu) = P_n^{(\alpha,\beta-1)}(x) + \mu\frac{d\Phi_n(x;\mu)}{dx}.$$

Let $\Phi_n := \Phi_n(x;\mu)$ for brevity and let $\overline{\frac{d\Phi_n}{dx}}$ denote a complex conjugate of $\frac{d\Phi_n}{dx}$. Multiplying by $\overline{\frac{d\Phi_n}{dx}} w_{\alpha,\beta+1}$, where $w_{\alpha,\beta+1} := w_{\alpha,\beta+1}(x) = (1-x)^\alpha(1+x)^{\beta+1}$, and integration over the interval $(-1,1)$ gives us

$$\int_{-1}^{1} \Phi_n \overline{\frac{d\Phi_n}{dx}} w_{\alpha,\beta+1}\,dx = \int_{-1}^{1} P_n^{(\alpha,\beta-1)} \overline{\frac{d\Phi_n}{dx}} w_{\alpha,\beta+1}\,dx + \mu \int_{-1}^{1} \left|\frac{d\Phi_n}{dx}\right|^2 w_{\alpha,\beta+1}\,dx.$$

Select $\mu$ so that $\Phi_n(1;\mu) = 0$. To estimate the real part of $\mu$ we add the last equation to its complex conjugate and obtain

$$\begin{aligned} &\int_{-1}^{1} \frac{d\left(|\Phi_n|^2\right)}{dx} w_{\alpha,\beta+1}\,dx \\ &\quad = \int_{-1}^{1} P_n^{(\alpha,\beta-1)} \cdot 2\Re \frac{d\Phi_n}{dx} \cdot w_{\alpha,\beta+1}\,dx + 2\Re\mu \int_{-1}^{1} \left|\frac{d\Phi_n}{dx}\right|^2 w_{\alpha,\beta+1}\,dx. \end{aligned} \tag{22}$$

Since $w_{\alpha,\beta+1}$ increases on $(-1,1)$ and

$$\lim_{x\to -1+} \Phi_n w_{\alpha,\beta+1} = \lim_{x\to 1-} \Phi_n w_{\alpha,\beta+1} = 0,$$

the left-hand side satisfies

$$\int_{-1}^{1} \frac{d\left(|\Phi_n|^2\right)}{dx} w_{\alpha,\beta+1}\,dx = -\int_{-1}^{1} |\Phi_n|^2 w'_{\alpha,\beta+1}\,dx < 0.$$

Applying the relation (8) to the polynomial $P_n^{(\alpha,\beta-1)}$ three times gives us

$$\begin{aligned} P_n^{(\alpha,\beta-1)} &= \frac{n+\alpha+\beta}{2n+\alpha+\beta} P_n^{(\alpha,\beta)} + \frac{n+\alpha}{2n+\alpha+\beta} P_{n-1}^{(\alpha,\beta)} \\ &= \frac{(n+\alpha+\beta)_2}{(2n+\alpha+\beta)_2} P_n^{(\alpha,\beta+1)} + \frac{(n+\alpha+\beta)(n+\alpha)}{(2n+\alpha+\beta)_2} P_{n-1}^{(\alpha,\beta+1)} \\ &\quad + \frac{(n+\alpha)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)_2} P_{n-1}^{(\alpha,\beta+1)} + \frac{(n+\alpha-1)_2}{(2n+\alpha+\beta-1)_2} P_{n-2}^{(\alpha,\beta+1)}, \end{aligned}$$

that is,

$$\begin{aligned} P_n^{(\alpha,\beta-1)} &= \frac{(n+\alpha+\beta)_2}{(2n+\alpha+\beta)_2} P_n^{(\alpha,\beta+1)} + \frac{2(n+\alpha)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta+1)} P_{n-1}^{(\alpha,\beta+1)} \\ &\quad + \frac{(n+\alpha-1)_2}{(2n+\alpha+\beta-1)_2} P_{n-2}^{(\alpha,\beta+1)}. \end{aligned}$$

By the definition of $\Phi_n$ and the formula (18),

$$\Re \frac{d\Phi_n}{dx} = 2^{-1}(n+\alpha+\beta) P_{n-1}^{(\alpha+1,\beta)} + \Re\mu \cdot 2^{-2}(n+\alpha+\beta)_2 P_{n-2}^{(\alpha+2,\beta+1)} + \psi,$$

where $\psi$ is a polynomial of degree at most $n-3$. The difference (8) − (9) induces the identity $P_{n-1}^{(\alpha+1,\beta)} = P_{n-1}^{(\alpha,\beta+1)} + P_{n-2}^{(\alpha+1,\beta+1)}$, so we finally have

$$\int_{-1}^{1} P_n^{(\alpha,\beta-1)} \cdot 2\Re \frac{d\Phi_n}{dx} \cdot w_{\alpha,\beta+1}\,dx = \eta + \zeta\Re\mu, \quad \text{where} \quad \eta,\zeta > 0.$$

Now the terms of the relation (22) are estimated, and it yields $0 > \Re\mu$. □

## 4. Results on Conjecture 2

We just have shown that the result on Conjecture 1 in [3] can be obtained in a shorter way if we consider polynomials with shifted parameter values (we used $\phi_n^{(\alpha,\beta-1)}$) instead of real combinations of the polynomials $\phi_n^{(\alpha,\beta)}$ and $\phi_{n-1}^{(\alpha,\beta)}$. At the same time, to verify Conjecture 2 we can combine both these ideas.

For any fixed $n > 3$ consider the intermediary polynomial

$$f(x;\mu) \coloneqq \sum_{k=0}^{n} \mu^k \left( A \frac{d^k}{dx^k} P_n^{(\alpha,\beta)}(x) + \mu \frac{d^k}{dx^k} P_{n-1}^{(\alpha,\beta)}(x) \right).$$

**Lemma 9.** *The polynomial $f(1;\mu)$ is Hurwitz-stable provided that $-1 < \alpha < 1$ and $\beta, A > 0$.*

*Proof.* From the definition of $f(x,\mu)$ the differential equation

$$\mu \frac{d}{dx} f(x;\mu) + A P_n^{(\alpha,\beta)}(x) + \mu P_{n-1}^{(\alpha,\beta)}(x) = f(x;\mu)$$

follows. Multiplication by $\overline{f(x;\mu)} w_{\alpha-1,\beta}(x)$ gives us

$$\overline{f} \frac{df}{dx} w_{\alpha-1,\beta} + A P_n^{(\alpha,\beta)}(x) \frac{\overline{f}}{\mu} w_{\alpha-1,\beta} + P_{n-1}^{(\alpha,\beta)}(x) \overline{f} w_{\alpha-1,\beta} = \frac{1}{\mu} |f|^2 w_{\alpha-1,\beta},$$

where we put $f \coloneqq f(x;\mu)$ and $w_{\alpha-1,\beta} \coloneqq w_{\alpha-1,\beta}(x) = (1-x)^{\alpha-1}(1+x)^{\beta}$ for brevity. Adding to this equality its complex conjugate and integrating yields

$$\begin{aligned} \int_{-1}^{1} \frac{d\left(|f|^2\right)}{dx} w_{\alpha-1,\beta}\, dx + A \int_{-1}^{1} P_n^{(\alpha,\beta)}(x) \left( \frac{\overline{f}}{\mu} + \frac{f}{\overline{\mu}} \right) w_{\alpha-1,\beta}\, dx \\ + \int_{-1}^{1} P_{n-1}^{(\alpha,\beta)}(x) \left( \overline{f} + f \right) w_{\alpha-1,\beta}\, dx = \left( \frac{1}{\mu} + \frac{1}{\overline{\mu}} \right) \int_{-1}^{1} |f|^2 w_{\alpha-1,\beta}\, dx. \end{aligned} \tag{23}$$

Take $\mu$ so that $f(1;\mu) = 0$. Then the polynomial $f(x;\mu)$ can be represented as

$$f(x;\mu) = (1-x) \sum_{k=0}^{n-1} c_k P_k^{(\alpha,\beta)}(x)$$

with some complex constants $c_k$ depending on $\mu$ (generally speaking). Observe that $c_{n-1} < 0$: denoting the leading coefficient in $x$ by $\mathbf{lc}$, we obtain

$$\begin{aligned} c_{n-1} &= \frac{\mathbf{lc}\left(f(x;\mu)\right)}{\mathbf{lc}\left(-x P_{n-1}^{(\alpha,\beta)}(x)\right)} = -\frac{A \cdot \mathbf{lc}\left(P_n^{(\alpha,\beta)}(x)\right)}{\mathbf{lc}\left(P_{n-1}^{(\alpha,\beta)}(x)\right)} \\ &\overset{(1)}{=\!=} -A \frac{(n+\alpha+\beta+1)_n \cdot (n-1)!\, 2^{n-1}}{n!\, 2^n \cdot (n+\alpha+\beta)_{n-1}} = -A \frac{(2n+\alpha+\beta-1)_2}{2n(n+\alpha+\beta)} < 0. \end{aligned}$$

Then we have

$$\begin{aligned} \int_{-1}^{1} P_n^{(\alpha,\beta)}(x) f w_{\alpha-1,\beta}\, dx &= \int_{-1}^{1} P_n^{(\alpha,\beta)}(x) \sum_{k=0}^{n-1} c_k P_k^{(\alpha,\beta)}(x) w_{\alpha,\beta}\, dx = 0, \\ \int_{-1}^{1} P_{n-1}^{(\alpha,\beta)}(x) f w_{\alpha-1,\beta}\, dx &= c_{n-1} \int_{-1}^{1} \left( P_{n-1}^{(\alpha,\beta)}(x) \right)^2 w_{\alpha,\beta}\, dx < 0 \end{aligned} \tag{24}$$

as a consequence of orthogonality of the Jacobi polynomials. Note that if $-1 < \alpha < 1$ and $\beta > 0$, then $w'_{\alpha-1,\beta} = \big(\beta(1-x) - (\alpha-1)(1+x)\big) \cdot w_{\alpha-2,\beta-1} > 0$ and

$$\lim_{x\to -1+} |f|^2 w_{\alpha-1,\beta} = \lim_{x\to 1-} |f|^2 w_{\alpha-1,\beta} = 0.$$

Therefore, integrating by parts yields

$$\int_{-1}^{1} \frac{d\left(|f|^2\right)}{dx} w_{\alpha-1,\beta}\, dx = -\int_{-1}^{1} \left(|f|^2\right) w'_{\alpha-1,\beta}\, dx < 0. \tag{25}$$

By the formulae (24)–(25), the left-hand side of equation (23) is negative, thus necessarily $\Re\mu < 0$. □

Consider also another intermediary polynomial for a fixed $n > 3$, namely

$$g(x;\mu) \coloneqq \sum_{k=0}^{n} \mu^k \left(\mu \frac{d^k}{dx^k} P_n^{(\alpha,\beta)}(x) + A \frac{d^k}{dx^k} P_{n-1}^{(\alpha,\beta)}(x)\right).$$

**Lemma 10.** *The polynomial $g(1;\mu)$ is Hurwitz-stable provided that $-1 < \alpha < 0$ and $\beta, A > 0$.*

*Proof.* This proof is analogous to the proofs of Theorem 8 and Lemma 9. From the definition of $g(x,\mu)$ we have

$$\mu \frac{dg(x;\mu)}{dx} + \mu P_n^{(\alpha,\beta)}(x) + A P_{n-1}^{(\alpha,\beta)}(x) = g(x;\mu)$$

which gives us (we put $g \coloneqq g(x;\mu)$ and $g' \coloneqq \frac{dg(x;\mu)}{dx}$ for brevity's sake)

$$\mu g' \overline{g'} w_{\alpha,\beta} + \mu P_n^{(\alpha,\beta)}(x) \overline{g'} w_{\alpha,\beta} + A P_{n-1}^{(\alpha,\beta)}(x) \overline{g'} w_{\alpha,\beta} = g \overline{g'} w_{\alpha,\beta}$$

after multiplication by $\overline{g'} w_{\alpha,\beta}$. Adding to the equation its complex conjugate and integrating yields

$$\begin{aligned}(\mu + \overline{\mu}) \int_{-1}^{1} |g'|^2 w_{\alpha,\beta}\, dx + \int_{-1}^{1} P_n^{(\alpha,\beta)}(x) \left(\mu \overline{g'} + \overline{\mu} g'\right) w_{\alpha,\beta}\, dx \\ + A \int_{-1}^{1} P_{n-1}^{(\alpha,\beta)}(x) \left(\overline{g'} + g'\right) w_{\alpha,\beta}\, dx = \int_{-1}^{1} \left(g\overline{g'} + \overline{g} g'\right) w_{\alpha,\beta}\, dx.\end{aligned} \tag{26}$$

Observe that $g$ is a polynomial of degree $n$ in $x$ and its leading coefficient is given by $P_n^{(\alpha,\beta)}(x) \cdot \mu$. Consequently, substituting the explicit expression for $\mathbf{lc}\left(P_n^{(\alpha,\beta)}(x)\right)$ from the formula (1) gives

$$\begin{aligned}\mathbf{lc}(g') &= n\, \mathbf{lc}\left(P_n^{(\alpha,\beta)}(x)\right) \mu = \frac{(2n+\alpha+\beta-1)_2}{2(n+\alpha+\beta)} \cdot \frac{(n+\alpha+\beta)_{n-1}}{(n-1)!\, 2^{n-1}} \mu \\ &= \frac{(2n+\alpha+\beta-1)_2}{2(n+\alpha+\beta)}\, \mathbf{lc}\left(P_{n-1}^{(\alpha,\beta)}(x)\right) \mu.\end{aligned}$$

This allows us to calculate the third summand on the left-hand side of (26):

$$\begin{aligned}
&\int_{-1}^{1} P_{n-1}^{(\alpha,\beta)}(x)\left(\overline{g'}+g'\right) w_{\alpha,\beta}\,dx \\
&\quad= \int_{-1}^{1} P_{n-1}^{(\alpha,\beta)}(x)\frac{(2n+\alpha+\beta-1)_2}{2(n+\alpha+\beta)}\,\mathbf{lc}\left(P_{n-1}^{(\alpha,\beta)}(x)\right)(\overline{\mu}+\mu)\,x^{n-1}w_{\alpha,\beta}\,dx \qquad (27)\\
&\quad= (\overline{\mu}+\mu)\,\frac{(2n+\alpha+\beta-1)_2}{2(n+\alpha+\beta)}\int_{-1}^{1}\left(P_{n-1}^{(\alpha,\beta)}(x)\right)^2 w_{\alpha,\beta}\,dx.
\end{aligned}$$

Additionally, we have

$$\int_{-1}^{1} P_{n}^{(\alpha,\beta)}(x)g'w_{\alpha,\beta}\,dx = 0. \tag{28}$$

Take $\mu$ so that $g(1;\mu)=0$. Then $w'_{\alpha,\beta} = \big(\beta(1-x)-\alpha(1+x)\big)\cdot w_{\alpha-1,\beta-1} > 0$ and

$$\lim_{x\to -1+}|g|^2 w_{\alpha,\beta} = \lim_{x\to 1-}|g|^2 w_{\alpha,\beta} = 0$$

since $-1<\alpha<0$ and $\beta>0$. Integrating by parts we obtain

$$\int_{-1}^{1}\left(g\overline{g'}+\overline{g}g'\right)w_{\alpha,\beta}\,dx = -\int_{-1}^{1}|g|^2 w'_{\alpha,\beta}\,dx < 0. \tag{29}$$

Now let us bring together the relations (26)–(29):

$$(\mu+\overline{\mu})\int_{-1}^{1}\left(|g'|^2 + A\frac{(2n+\alpha+\beta-1)_2}{2(n+\alpha+\beta)}\left(P_{n-1}^{(\alpha,\beta)}(x)\right)^2\right)w_{\alpha,\beta}\,dx = -\int_{-1}^{1}|g|^2 w'_{\alpha,\beta}\,dx,$$

hence $2\Re\mu = \mu+\overline{\mu} < 0$. That is, any zero $\mu$ of the polynomial $g(1;\mu)$ resides in the left half of the complex plane. □

**Corollary 11.** *For any positive $A$, the zeros of the polynomials*

$$\begin{aligned}
&2A\phi_n^{(\alpha,\beta)}(\mu) + (n+\alpha+\beta)\,\mu\phi_{n-2}^{(\alpha+1,\beta+1)}(\mu) \quad \textit{and}\\
&A\,(n+\alpha+\beta+1)\,\phi_{n-1}^{(\alpha+1,\beta+1)}(\mu) + 2\phi_{n-1}^{(\alpha,\beta)}(\mu)
\end{aligned} \tag{30}$$

*are interlacing provided that $-1<\alpha<1$ and $\beta>0$. If in addition $-1<\alpha<0$, the zeros of the polynomials*

$$\begin{aligned}
&(n+\alpha+\beta+1)\,\mu\phi_{n-1}^{(\alpha+1,\beta+1)}(\mu) + 2A\phi_{n-1}^{(\alpha,\beta)}(\mu) \quad \textit{and}\\
&2\phi_n^{(\alpha,\beta)}(\mu) + A\,(n+\alpha+\beta)\,\phi_{n-2}^{(\alpha+1,\beta+1)}(\mu)
\end{aligned} \tag{31}$$

*are also interlacing.*

*Proof.* To get the assertion we apply the relation (18) to the even and odd parts of the polynomials $f(x;\mu)$ and $g(x;\mu)$. The even part of $f(x;\mu)$ is

$$\begin{aligned}
\frac{f(x;\mu)+f(x;-\mu)}{2} &= A\sum_{k=0}^{[n/2]}\mu^{2k}\frac{d^{2k}}{dx^{2k}}P_n^{(\alpha,\beta)}(x) + \mu^2\sum_{k=0}^{[n/2]}\mu^{2k}\frac{d^{2k+1}}{dx^{2k+1}}P_{n-1}^{(\alpha,\beta)}(x)\\
&= A\sum_{k=0}^{[n/2]}\mu^{2k}\frac{d^{2k}}{dx^{2k}}P_n^{(\alpha,\beta)}(x) + \frac{n+\alpha+\beta}{2}\mu^2\sum_{k=0}^{[n/2]-1}\mu^{2k}\frac{d^{2k}}{dx^{2k}}P_{n-2}^{(\alpha+1,\beta+1)}(x).
\end{aligned}$$

Analogously, for the odd part we have

$$\frac{f(x;\mu)-f(x;-\mu)}{2} = A\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \mu^{2k+1}\frac{d^{2k+1}}{dx^{2k+1}}P_n^{(\alpha,\beta)}(x) + \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \mu^{2k+1}\frac{d^{2k}}{dx^{2k}}P_{n-1}^{(\alpha,\beta)}(x)$$
$$= A\frac{n+\alpha+\beta+1}{2}\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \mu^{2k+1}\frac{d^{2k}}{dx^{2k}}P_{n-1}^{(\alpha+1,\beta+1)}(x) + \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \mu^{2k+1}\frac{d^{2k}}{dx^{2k}}P_{n-1}^{(\alpha,\beta)}(x).$$

The same manipulations with $g(x;\mu)$ give

$$\frac{g(x;\mu)+g(x;-\mu)}{2} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \mu^{2k+2}\frac{d^{2k+1}}{dx^{2k+1}}P_n^{(\alpha,\beta)}(x) + A\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \mu^{2k}\frac{d^{2k}}{dx^{2k}}P_{n-1}^{(\alpha,\beta)}(x)$$
$$= \frac{n+\alpha+\beta+1}{2}\mu^2\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \mu^{2k}\frac{d^{2k}}{dx^{2k}}P_{n-1}^{(\alpha+1,\beta+1)}(x) + A\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \mu^{2k}\frac{d^{2k}}{dx^{2k}}P_{n-1}^{(\alpha,\beta)}(x)$$

and

$$\frac{g(x;\mu)-g(x;-\mu)}{2} = \sum_{k=0}^{[n/2]} \mu^{2k+1}\frac{d^{2k}}{dx^{2k}}P_n^{(\alpha,\beta)}(x) + A\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \mu^{2k+1}\frac{d^{2k+1}}{dx^{2k+1}}P_{n-1}^{(\alpha,\beta)}(x)$$
$$= \mu\left(\sum_{k=0}^{[n/2]} \mu^{2k}\frac{d^{2k}}{dx^{2k}}P_n^{(\alpha,\beta)}(x) + A\frac{n+\alpha+\beta}{2}\sum_{k=0}^{[n/2]-1} \mu^{2k}\frac{d^{2k}}{dx^{2k}}P_{n-2}^{(\alpha+1,\beta+1)}(x)\right).$$

The polynomials $f(1;\mu)$ and $g(1;\mu)$ are stable by Lemma 9 and Lemma 10, respectively. Thus, the pairs of polynomials mentioned in (30) and (31) have interlacing zeros by Theorem 3. □

**Lemma 12** (see, e.g., [8, Lemma 3.4] or [3, Lemma 3.5])**.** *Let real polynomials $p(x)$ and $q(x)$ such that $p(0), q(0) > 0$ have only negative zeros. Then $\big(p(x), xq(x)\big)$ is a real pair if and only if the combinations $Ap(x) + Bxq(x)$ and $Cp(x) + Dq(x)$ are non-zero outside the real line for all $A, B, C, D > 0$.*

Recall that $p(x)$ and $xq(x)$ is a real pair whenever they interlace (non-strictly if $p(x)$ and $xq(x)$ have a common zero). For $p$ and $q$ as in this lemma we thus have $\deg q \leqslant \deg p \leqslant 1 + \deg q$ automatically.

The next corollary complements the interlacing property of the polynomials $\phi_n^{(\alpha,\beta)}(\mu)$ and $\phi_{n-1}^{(\alpha+1,\beta+1)}(\mu)$ (see the remark to Theorem 1).

**Corollary 13.** *If $-1 < \alpha < 0$ and $\beta > 0$ the pairs $\big(\phi_n^{(\alpha,\beta)}(\mu), \mu\phi_{n-2}^{(\alpha+1,\beta+1)}(\mu)\big)$ and $\big(\phi_n^{(\alpha,\beta)}(\mu), \mu\phi_n^{(\alpha+1,\beta+1)}(\mu)\big)$ possess the (strict) interlacing property.*

*Proof.* By Theorem 1, all involved polynomials have only real non-positive zeros. Corollary 11 adds that the polynomials in (30)–(31) have (strictly) interlacing zeros. Therefore, Lemma 12 assures the asserted fact. □

**Theorem 14.** *If $-1 < \alpha < 0 < \beta$ and $n = 5, 6, \ldots,$ then*

- *the polynomial $\phi_n^{(\alpha+1,\beta+1)}(\mu)$ interlaces $\phi_{n-2}^{(\alpha+1,\beta+1)}(\mu)$, and*
- *the polynomial $\phi_n^{(\alpha,\beta)}(x)$ interlaces $\phi_{n-2}^{(\alpha,\beta)}(x)$.*

*Proof.* According to Corollary 5, we have

$$\begin{aligned} &\mathbf{zr}_i\left(\phi_{n-2}^{(\alpha+1,\beta+1)}\right) < \mathbf{zr}_i\left(\phi_{n-1}^{(\alpha+1,\beta+1)}\right) < \mathbf{zr}_{i-1}\left(\phi_{n-2}^{(\alpha+1,\beta+1)}\right), \\ &\mathbf{zr}_i\left(\phi_{n-1}^{(\alpha+1,\beta+1)}\right) < \mathbf{zr}_i\left(\phi_{n}^{(\alpha+1,\beta+1)}\right) < \mathbf{zr}_{i-1}\left(\phi_{n-1}^{(\alpha+1,\beta+1)}\right) \end{aligned} \tag{32}$$

for any natural $i \leqslant n/2$. From Corollary 13 we obtain that

$$\mathbf{zr}_i\left(\phi_{n-2}^{(\alpha+1,\beta+1)}\right) < \mathbf{zr}_i\left(\phi_{n}^{(\alpha,\beta)}\right) < \mathbf{zr}_{i-1}\left(\phi_{n-2}^{(\alpha+1,\beta+1)}\right), \tag{33}$$

$$\mathbf{zr}_i\left(\phi_{n}^{(\alpha+1,\beta+1)}\right) < \mathbf{zr}_i\left(\phi_{n}^{(\alpha,\beta)}\right) < \mathbf{zr}_{i-1}\left(\phi_{n}^{(\alpha+1,\beta+1)}\right). \tag{34}$$

Bringing together the right inequality in (33) and the left inequalities in (34) and (32), we obtain

$$\begin{aligned} \mathbf{zr}_i\left(\phi_{n-2}^{(\alpha+1,\beta+1)}\right) &\overset{(32)}{<} \mathbf{zr}_i\left(\phi_{n-1}^{(\alpha+1,\beta+1)}\right) \overset{(32)}{<} \mathbf{zr}_i\left(\phi_{n}^{(\alpha+1,\beta+1)}\right) \\ &\overset{(34)}{<} \mathbf{zr}_i\left(\phi_{n}^{(\alpha,\beta)}\right) \overset{(33)}{<} \mathbf{zr}_{i-1}\left(\phi_{n-2}^{(\alpha+1,\beta+1)}\right) \end{aligned}$$

for all natural $i \leqslant n/2$. This relation implies that the zeros of $\phi_n^{(\alpha+1,\beta+1)}(\mu)$ and $\phi_{n-2}^{(\alpha+1,\beta+1)}(\mu)$ interlace.

By Corollary 5 we obtain (cf. (32))

$$\mathbf{zr}_i\left(\phi_{n-2}^{(\alpha,\beta)}\right) < \mathbf{zr}_i\left(\phi_{n-1}^{(\alpha,\beta)}\right) < \mathbf{zr}_i\left(\phi_{n}^{(\alpha,\beta)}\right), \quad i = 1, 2, \ldots.$$

This chain can be continued with the left inequality in (34) and the right inequality in (33) so that

$$\begin{aligned} \mathbf{zr}_i\left(\phi_{n-2}^{(\alpha+1,\beta+1)}(\mu)\right) &\overset{(34)}{<} \mathbf{zr}_i\left(\phi_{n-2}^{(\alpha,\beta)}\right) < \mathbf{zr}_i\left(\phi_{n-1}^{(\alpha,\beta)}\right) \\ &< \mathbf{zr}_i\left(\phi_{n}^{(\alpha,\beta)}\right) \overset{(33)}{<} \mathbf{zr}_{i-1}\left(\phi_{n-2}^{(\alpha+1,\beta+1)}\right) \end{aligned}$$

for each natural $i$. □

## 5. Conclusion: Relations between Conjecture 1 and Conjecture 2

Here we obtain two instructive facts giving an idea about the limits of the current approach. In the present note, we used a modification of the method applied in [3], so it has the same deficiency: the parameters $\alpha$ and $\beta$ are constrained to provide the positivity of $w'_{\alpha,\beta}(x)$ and the convergence of integrals. However, these sufficient conditions seem to be quite far from being necessary.

The following two lemmata coupled with Theorem 14 give no new parameter range for Conjecture 1 to hold. At the same time, the comparison to Lemma 4 clearly shows that this conjecture is less restrictive than Conjecture 2.

**Lemma 15.** *If the polynomials $\phi_n^{(\alpha,\beta)}(\mu)$, $\phi_{n-1}^{(\alpha,\beta)}(\mu)$ and $\phi_{n-2}^{(\alpha,\beta)}(\mu)$ are pairwise interlacing in such a way that*

$$\mathbf{zr}_1\left(\phi_{n-2}^{(\alpha,\beta)}\right) < \mathbf{zr}_1\left(\phi_{n-1}^{(\alpha,\beta)}\right) < \mathbf{zr}_1\left(\phi_{n}^{(\alpha,\beta)}\right),$$

*then $\phi_n^{(\alpha,\beta-1)}(\mu)$ interlaces $\phi_{n-1}^{(\alpha,\beta-1)}(\mu)$.*

*Proof.* By the formula (4), $\phi_n^{(\alpha,\beta-1)}(\mu)$ and $\phi_{n-1}^{(\alpha,\beta-1)}(\mu)$ have only real zeros. Furthermore,

$$\mathbf{zr}_{i+1}\left(\phi_n^{(\alpha,\beta)}\right) < \mathbf{zr}_i\left(\phi_{n-2}^{(\alpha,\beta)}\right) < \mathbf{zr}_i\left(\phi_{n-1}^{(\alpha,\beta-1)}\right)$$
$$< \mathbf{zr}_i\left(\phi_{n-1}^{(\alpha,\beta)}\right) < \mathbf{zr}_i\left(\phi_n^{(\alpha,\beta-1)}\right) < \mathbf{zr}_i\left(\phi_n^{(\alpha,\beta)}\right)$$

for natural $i = 1, \ldots, \left[\frac{n-1}{2}\right]$. This implies the interlacing property for the polynomials $\phi_n^{(\alpha,\beta-1)}(\mu)$ and $\phi_{n-1}^{(\alpha,\beta-1)}(\mu)$. □

As an intermediate result (Corollary 13) we had the interlacing property of $\phi_n^{(\alpha,\beta)}(\mu)$ and $\phi_n^{(\alpha+1,\beta+1)}(\mu)$ when the parameters satisfy $-1 < \alpha < 0 < \beta$. Such a fact allows us to get a relationship complementing Lemma 15.

**Lemma 16.** *Let the polynomial pairs*

$$\left(\phi_n^{(\alpha,\beta)}(\mu), \phi_n^{(\alpha+1,\beta+1)}(\mu)\right) \quad \textit{and} \quad \left(\phi_n^{(\alpha,\beta)}(\mu), \phi_{n-1}^{(\alpha+1,\beta+1)}(\mu)\right)$$

*be interlacing in such a way that*

$$\mathbf{zr}_1\left(\phi_{n-1}^{(\alpha+1,\beta+1)}\right) < \mathbf{zr}_1\left(\phi_n^{(\alpha,\beta)}\right), \quad \mathbf{zr}_1\left(\phi_n^{(\alpha+1,\beta+1)}\right) < \mathbf{zr}_1\left(\phi_n^{(\alpha,\beta)}\right). \tag{35}$$

*Then* $\phi_n^{(\alpha+1,\beta)}(\mu)$ *interlaces* $\phi_{n-1}^{(\alpha+1,\beta)}(\mu)$.

*Proof.* The identities (4) and (5) give

$$(n+\alpha+\beta+2)\phi_n^{(\alpha+1,\beta+1)} + (n+\alpha+1)\phi_{n-1}^{(\alpha+1,\beta+1)} = (2n+\alpha+\beta+2)\phi_n^{(\alpha+1,\beta)}, \tag{36}$$

$$(n+\alpha+\beta+1)\phi_n^{(\alpha+1,\beta)} - (2n+\alpha+\beta+1)\phi_n^{(\alpha,\beta)} = (n+\beta)\phi_{n-1}^{(\alpha+1,\beta)}. \tag{37}$$

From inequalities (35) we obtain that each interval $\left(\mathbf{zr}_{i+1}\left(\phi_n^{(\alpha,\beta)}\right), \mathbf{zr}_i\left(\phi_n^{(\alpha,\beta)}\right)\right)$, $i = 1, \ldots, [n/2]-1$, contains the points $\mathbf{zr}_i\left(\phi_{n-1}^{(\alpha+1,\beta+1)}\right)$ and $\mathbf{zr}_i\left(\phi_n^{(\alpha+1,\beta+1)}\right)$ and no other zeros of the polynomials $\phi_{n-1}^{(\alpha+1,\beta+1)}$ and $\phi_n^{(\alpha+1,\beta+1)}$. Thus, the left-hand side of (36) also has exactly one zero on each of the intervals. As a result, $\phi_n^{(\alpha+1,\beta)}$ interlaces $\phi_n^{(\alpha,\beta)}$, so the zeros of their difference appearing in (37) and hence of $\phi_{n-1}^{(\alpha+1,\beta)}$ are interlacing with the zeros of $\phi_n^{(\alpha+1,\beta)}$. □

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