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SQUARING A CONJUGACY CLASS AND COSETS OF NORMAL SUBGROUPS

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Abstract. Let G be a finite group and let K be the conjugacy class of $x \in G$. If K^2 is a conjugacy class of G, then $[x, G]$ is solvable. If the order of x is a power of prime, then $[x, G]$ has a normal p-complement. We also prove some related results on the solvability of certain normal subgroups when a non-trivial coset has certain properties.

1. Introduction

If G is a finite group and $\chi \in \mathrm{Irr}(G)$ is an irreducible complex character of G, then χ^2 is never irreducible unless $\chi(1) = 1$. On the other hand, if $K = x^G$ is the conjugacy class of x in G, it can occur that K^2 is a conjugacy class of G, even if x is not central in G. In fact, this occurs not so rarely but only if $[x, G]$ is solvable.

Theorem A. *Let G be a finite group, let $x \in G$, and let $K = x^G$ be the conjugacy class of x in G. Then the following are equivalent:*

(a) K^2 is a conjugacy class of G.
(b) $\chi(x) = 0$ or $|\chi(x)| = \chi(1)$ for every $\chi \in \mathrm{Irr}(G)$, and $\mathbf{C}_G(x) = \mathbf{C}_G(x^2)$.
(c) $K = x[x, G]$ and $\mathbf{C}_G(x) = \mathbf{C}_G(x^2)$.

In this case, $[x, G]$ is solvable. Furthermore, if x has order a power of a prime p, then $[x, G]$ has a normal p-complement.

One can find examples of elements satisfying the conclusions in Theorem A whenever x is an odd order fixed-point-free automorphism of a group N, and $G = N\langle x\rangle$ is the semidirect product. More generally, this holds if x is an automorphism of N and x^2 acts fixed point freely on N (In this case, $N = [x, G]$, and there are many results in the literature about N.) The odd order elements of the center of a Frobenius complement, or the odd order elements in the second center of any nilpotent group, also satisfy Theorem A.

There is a great number of references on the product of conjugacy classes in finite groups, and some related results. For instance, Arad and Herzog conjectured

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in [AH] that the product of two non-trivial conjugacy classes of a finite non-abelian simple group is never a conjugacy class, and our Theorem A is an easy case consistent with that conjecture. (It is a fact that simple cannot be replaced by almost simple in the Arad-Herzog conjecture.) See [GMT] for some results and examples. Theorem A is also related to the so-called *Camina pairs*.

Our proof of the fact that $[x, G]$ is solvable in Theorem A uses the Classification of Finite Simple Groups. As we will see, the key fact is that all the elements of $x[x, G]$ are G-conjugate. This inspired some related results below.

Theorem B. *Suppose that N is a normal subgroup of a finite group G. Let $x \in G$.*

(a) If all the elements of xN are G-conjugate, then N is solvable.
(b) If all the elements of xN are G-conjugate, and x is a p-element for some prime p, then N has a normal p-complement.
(c) If all the elements of xN have odd order, then N is solvable.

It is not true that if all the elements of xN have the same order, then N is solvable. (Take $G = \mathrm{Alt}_5 \times C$, where C is a cyclic group of order 30, $N = G'$ and $x \in \mathbf{Z}(G)$ of order 30.) Also, it is not true that if all the elements of xN are p-elements, then N has a normal p-complement (as shown by S_4 with $p = 2$ and one can construct similar examples for any p). It is also not true that if all elements of xN are 2-elements, then N is solvable. If $N = \mathrm{Alt}_6$ and $G = M_{10}$ (so G/N has order 2), every element in $G \setminus N$ has order either 4 or 8. This is likely the basis of any such counterexample.

The proofs of (a) and (c) require the Classification of Finite Simple Groups. However, (b) does not.

2. Proofs

If G is a group and $x \in G$, recall that

$$[x, G] = \langle [x, g] \mid g \in G \rangle .$$

Using that $[x, yz] = [x, z][x, y]^z$, we easily check that $[x, G] \triangleleft G$.

In the complex group algebra $\mathbb{C}G$, if $X \subseteq G$, we write $\hat{X} = \sum_{x \in X} x \in \mathbb{C}G$.

Lemma 2.1. *Let $x \in G$, where G is a finite group, and let $K = x^G$. Then the following are equivalent:*

(a) $\hat{K}x \in \mathbf{Z}(\mathbb{C}[G])$.
(b) $\hat{K}x^{-1} \in \mathbf{Z}(\mathbb{C}[G])$.
(c) For each character $\chi \in \mathrm{Irr}(G)$, either $\chi(x) = 0$ or $|\chi(x)| = \chi(1)$.

Proof. Let $\mathcal{X}$ be an irreducible representation of G. Then we know that $\mathcal{X}(\hat{K}) = \omega I$, where $\omega = \chi(x)|K|/\chi(1)$ and I is the identity matrix. Note that $\omega = 0$ if and only if $\chi(x) = 0$ and that $|\chi(x)| = \chi(1)$ if and only if $\mathcal{X}(x)$ is a scalar matrix, and this happens if and only if $\mathcal{X}(x^{-1})$ is a scalar matrix. (See Lemma (2.27) of [I2].)

If $\hat{K}x \in \mathbf{Z}(\mathbb{C}[G])$, then $\mathcal{X}(\hat{K}x)$ is a scalar matrix, and if $\hat{K}x^{-1} \in \mathbf{Z}(\mathbb{C}[G])$, then $\mathcal{X}(\hat{K}x^{-1})$ is a scalar matrix. Assuming (a) or (b), therefore, we deduce that $\omega\mathcal{X}(x)$ or $\omega\mathcal{X}(x^{-1})$ is a scalar matrix. If $\chi(x) \neq 0$, then $\omega \neq 0$, and thus $\mathcal{X}(x)$ or $\mathcal{X}(x^{-1})$ is a scalar matrix, and we have $|\chi(x)| = \chi(1)$. Conversely, suppose for each character $\chi \in \mathrm{Irr}(G)$, that either $\chi(x) = 0$ or $|\chi(x)| = \chi(1)$. Then for every irreducible representation $\mathcal{X}$ of G, we see that $\omega\mathcal{X}(x)$ and $\omega\mathcal{X}(x^{-1})$ are (possibly zero) scalar matrices, and thus $\mathcal{X}(\hat{K}x)$ and $\mathcal{X}(\hat{K}x^{-1})$ are scalar matrices. When $\mathbb{C}[G]$ is written

as a direct sum of matrix algebras, therefore, the component of $\hat{K}x$ and of $\hat{K}x^{-1}$ in each summand is a scalar matrix, and it follows that $\hat{K}x$ and $\hat{K}x^{-1}$ are central in $\mathbb{C}[G]$. □

Theorem 2.2. *Let K be a conjugacy class of G, where G is a finite group. Then the following are equivalent:*

(a) K^2 is a conjugacy class of G.
(b) If $x \in K$, then $\mathbf{C}_G(x) = \mathbf{C}_G(x^2)$, and $K = Nx$, where $N = [x, G]$.
(c) If $x \in K$, then $\mathbf{C}_G(x) = \mathbf{C}_G(x^2)$, and for all $\chi \in \mathrm{Irr}(G)$, either $\chi(x) = 0$ or $|\chi(x)| = \chi(1)$.

Proof. First suppose that K^2 is a class, and let $x \in K$. Then $xK \subseteq K^2$, so $|K| \leq |K^2|$. Also, K^2 is the class of x^2, and since $\mathbf{C}_G(x) \subseteq \mathbf{C}_G(x^2)$, we have $|K| \geq |K^2|$. Thus equality holds, so $\mathbf{C}_G(x) = \mathbf{C}_G(x^2)$, and $xK = K^2$. Similarly, $yK = K^2$ for $y \in K$, and thus $x^{-1}yK = K$, and we see that $[x, g]K = K$ for all $g \in G$. Since $N = [x, G]$ is generated by the elements $[x, g]$, it follows that $NK = K$, and thus $Nx \subseteq K$. Also $K = x\{[x, g] \,|\, g \in G\} \subseteq xN$, and hence $|K| \leq |N|$. It follows that $Nx = K$, proving (b).

Now assume (b), and let $x \in K$. Then $K = Nx$, so $Kx^{-1} = N$ and thus $\hat{K}x^{-1} = \hat{N}$. Also, $\hat{N} \in \mathbf{Z}(\mathbb{C}[G])$ since $N \lhd G$. Now Lemma 2.1 shows that (c) holds.

Assuming (c) now, Lemma 2.1 guarantees that $\hat{K}x$ is central in $\mathbb{C}[G]$, and thus the set Kx is closed under conjugation. This set, therefore, contains the full conjugacy class L of x^2. By hypothesis, $\mathbf{C}_G(x) = \mathbf{C}_G(x^2)$, and so $|L| = |K| = |xK|$, and we have that $Kx = L$. Then $Kx^g = (Kx)^g = L$ for all $g \in G$, and it follows that $K^2 = L$, proving (a). □

3. Special cosets

We start with an elementary lemma.

Lemma 3.1. *Let G be a finite group, $N \lhd G$, $x \in G$, and assume that all the elements in Nx are G-conjugate. If $\chi \in \mathrm{Irr}(G)$ does not contain N in its kernel, then $\chi(x) = 0$.*

Proof. Let $\mathcal{X}$ be a representation of G affording χ. Since $N \lhd G$, we have that $\hat{N}$ is central in $\mathbb{C}[G]$, and by Schur's Lemma it follows that $\mathcal{X}(\hat{N})$ is a scalar matrix. The trace of $\mathcal{X}(\hat{N})$ is

$$\sum_{n \in N} \chi(n) = |N|[\chi_N, 1_N] = 0\,,$$

and we conclude that $\mathcal{X}(\hat{N}) = 0$. Now,

$$\mathcal{X}(\widehat{xN}) = \mathcal{X}(x)\mathcal{X}(\hat{N}) = 0\,.$$

Since all the elements of xN are G-conjugate by hypothesis, it follows that the trace of $\mathcal{X}(\widehat{xN})$ is a multiple of $\chi(x)$, and the result follows. □

We next complete the proof of Theorem A and the first two parts of Theorem B. We use the Classification of Finite Simple Groups in order to ensure that every non-abelian minimal normal subgroup of a finite group possesses a non-trivial irreducible character that extends to G. (See for instance [BCLP].) The following generalizes some of the results in [L].

Theorem 3.2. *Let G be a finite group and let N be a normal subgroup of G. Let $x \in G$ be such that all elements of xN are conjugate in G. Then:*

(a) N is solvable.
(b) If π is the set of prime divisors of $o(x)$, then x normalizes some Hall π-complement H of N on which it acts fixed point freely.
(c) If x is a p-element for some prime p, then N has a normal p-complement.

Proof. We first prove (a). We argue by induction on $|N|$. If E is a minimal normal subgroup of G contained in N, it then follows by induction that N/E is solvable. In particular, we may assume that E is the direct product of non-abelian simple groups.

Note that all elements of xE are conjugate in G and so we may assume by induction that $N = E$.

By Theorems 2, 3, 4 and Lemma 5 of [BCLP], there exists $1 \neq \theta \in \mathrm{Irr}(E)$ that extends to $\chi \in \mathrm{Irr}(G)$. Now, by Lemma 3.1, we conclude that $\chi(xn) = 0$ for all $n \in N$. Now, let $\tau = \chi_{\langle N,x\rangle} \in \mathrm{Irr}(\langle N, x\rangle)$. Since τ_N is irreducible, by Lemma (8.14) of [I2], there exists $m \in N$ such that $\tau(xm) = \chi(xm) \neq 0$, and this is a contradiction.

We next prove (b). Let H be a Hall π-complement of N. Since all Hall π-complements are conjugate in N (since N is solvable), the Frattini argument gives $x = mn$, for some $m \in \mathbf{N}_G(H)$ and $n \in N$. Therefore xn^{-1} normalizes H, and therefore some G-conjugate of x normalizes H, using the hypothesis. If $H_1 = H^g \subseteq N$ is normalized by x, notice that all the elements of xH_1 have the same order (a π-number), and therefore x cannot centralize any element of H_1.

Finally, we give two different proofs of (c). Our first proof uses character theory. If $D/N = \mathbf{C}_{G/N}(Nx)$, working in D and by induction, we may assume that Nx is central in G/N. Hence, $Nx = x^G$.

Write $C = \mathbf{C}_G(x)$. We have $|G : C| = |N|$, and it follows easily that $|G : NC| = |C \cap N| = |\mathbf{C}_N(x)|$. Since all the elements of Nx are p-elements, then $\mathbf{C}_N(x)$ is a p-group, and we deduce that $|G : NC|$ is a prime power. Therefore $G = QNC$, for every $Q \in \mathrm{Syl}_p(G)$. In particular, all the elements of Nx are NQ-conjugate, and by working in NQ and by induction, we may assume that $NQ = G$.

Now let $\chi \in \mathrm{Irr}(G)$ be non-linear. If $N \subseteq \ker(\chi)$, then $\chi \in \mathrm{Irr}(G/N)$ has degree a non-trivial power of p. If N is not contained in the kernel of χ, then $\chi(x) = 0$ by Lemma 3.1. Since

$$\chi(x) \equiv \chi(1) \bmod p$$

(because p-power roots of unity are congruent with 1 modulo p in the ring of algebraic integers), then we deduce that p divides $\chi(1)$. Now we apply a result of Thompson (Corollary 12.2 of [I2]), to conclude that G (and therefore N) has a normal p-complement. (We notice that this proof does not use the Classification of Finite Simple Groups.)

Our second proof is group-theoretical. Note that G acts via conjugation on the set X of p-complements in N and that N acts transitively on X. Suppose that $|X| > 1$. Then $|X|$ is a non-trivial power of p. By (b), every element in the coset xN has a fixed point on X, whence by a minor extension of Burnside's Lemma (see [FGS, Lemma 13.1]), every element in xN has a unique fixed point in X. On the other hand, since x is a p-element and and p divides $|X|$, the number of fixed points of x on X is a multiple of p. This contradiction completes the proof. $\square$

Contrary to the case of Theorem 3.2 when a p-complement of N is normal in G, if the set of prime divisors π of $o(x)$ involves more than one prime, there are examples where the Hall π-complement of N need not be normal.

We now complete the proof of Theorem B. It remains only to prove part (c).

Lemma 3.3. *Suppose that N is a normal subgroup of a finite group G. Let $x \in G$ and assume that all elements in xN have odd order. Then N is solvable.*

Proof. We induct on the order of N. If $\mathbf{F}(N) \neq 1$, we can pass to $G/\mathbf{F}(N)$. So we may assume that N contains a normal subgroup E of G with E a direct product of non-abelian simple groups. Then all elements of xE still have odd order and so by minimality $E = N$. Write $E = L \times \ldots \times L$ with L simple. By [FGS, 12.1], there is a class of involutions z^L that is $\mathrm{Aut}(L)$ invariant. Thus, the class of $w := (z, \ldots, z)$ is $\mathrm{Aut}(E)$ invariant. Thus $G = E\mathbf{C}_G(w)$. In particular $xE \cap \mathbf{C}_G(w)$ is non-empty. Thus, we may assume that x centralizes w. Then x has odd order and so xw has order twice that of x, a contradiction. □

Recall that a pair (G, N) is a *Camina pair* if for every $x \in G - N$, all the elements of xN consist of G-conjugates. Our hypothesis in our theorem resembles Camina pairs, but one coset at a time.

Finally, we make some remarks on fixed-point-free automorphisms of finite groups. If x is a fixed-point-free automorphism of N, then xN is a single G-conjugacy class in $G = \langle N, x\rangle$. (This is elementary and follows because every element of N can be uniquely written in the form $n^{-1}n^{x^{-1}}$. In particular, $xn^{-1}n^{x^{-1}} = x^{nx^{-1}}$.)

Lemma 3.4. *Let X be a group of automorphisms of a finite group N. Assume that either X is cyclic or that $|X|$ and $|N|$ are coprime. The following are equivalent:*

(a) X acts fixed point freely on N;
(b) X fixes no non-trivial conjugacy class of N.
(c) X fixes no non-trivial irreducible character of N.

Proof. Clearly, (b) implies (a). So assume that X acts fixed point freely. Let G be the semidirect product NX.

First suppose that $X = \langle x\rangle$. Since x acts fixed point freely on N, all elements in xN are conjugate via an element of N. Suppose the $C^x = C$ for some $C = z^N$ for some $1 \neq z \in N$. Then xy centralizes z for some $y \in N$, whence xy does not act fixed point freely on N and so neither does x.

Finally assume that $|X|$ and $|N|$ are coprime. Note that X is a Hall π-subgroup of G for π the set of prime divisors of $|X|$. By the Schur-Zassenhaus theorem, all Hall π-subgroups of G are conjugate. If X fixes the class z^N, then $G = \mathbf{C}_G(z)N$ and so $\mathbf{C}_G(z)/\mathbf{C}_N(z) \cong X$, whence $\mathbf{C}_G(z)$ contains a Hall π-subgroup. Thus, X centralizes a conjugate of z.

If X is cyclic, (b) is also equivalent to (c) by Brauer's Lemma on character tables [I2, Theorem (6.32)].

If $|X|$ is coprime to $|N|$, (b) is equivalent to (c) by the Glauberman correspondence [G] when N is solvable and by [I1] when X is solvable (and by Feit-Thompson, either N or X is solvable). □

Note that if X is cyclic generated by x, then x acting fixed point freely on N is equivalent to the fact that all elements in xN (in the semidirect product NX) are

conjugate. Note that this condition obviously descends to X-invariant subgroups of N and to quotients of N by X-invariant normal subgroups. If X has coprime order, then the same is true by general results about coprime actions.

This gives a quick proof of the fact that non-solvable groups do not admit fixed-point-free automorphisms or fixed-point-free coprime order groups of automorphisms (using the classification of finite simple groups). The following result is essentially [FGS, 12.1]. See also Rowley [R] for a different proof.

Theorem 3.5. *Let N be a non-solvable finite group. Let X be a group of automorphisms of N. Assume that either X has order coprime to $|N|$ or is cyclic. Then there exists a non-central conjugacy class C of N that is X-invariant. In particular, X does not act fixed point freely on N.*

Proof. By the remarks preceding the theorem and Lemma 3.4, we see that the invariance of a non-trivial class C is equivalent to the action being fixed-point-free and that these properties descend to quotients. Thus, we may assume that $\mathbf{F}(N) = 1$. Let A be a minimal characteristic subgroup of N. Then $A = L \times \ldots \times L$ with L non-abelian simple. By [FGS, 12.1], L contains a class of involutions z^L that is $\mathrm{Aut}(L)$-invariant and so the class of $w := (z, \ldots, z)$ is $\mathrm{Aut}(N)$ invariant. $\square$

As we have noted, by [BCLP] there is always a character of a finite non-abelian simple group invariant under the full automorphism group. One might ask whether the same is true for Brauer characters and for p'-classes. If $p \neq 2$, this is true for p'-classes by [FGS, 12.1]. However for $p = 2$, this fails. Indeed, taking $G = M_{10}$ (a point stabilizer in M_{11}), we see that no conjugacy class of elements of odd order in $\mathrm{Alt}(6)$ is invariant under G (the two classes of elements of order 3 and the two classes of elements of order 5 are interchanged by the outer automorphism). By Brauer's Lemma on character tables, no non-trivial irreducible Brauer character of $\mathrm{Alt}(6)$ (for $p = 2$) is invariant under G (and also therefore not invariant under the full automorphism group).

If N is non-solvable, one might also ask if N necessarily contains a non-trivial conjugacy class that is invariant under $\mathrm{Aut}(N)$.

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