

STRONG UNIQUE CONTINUATION FOR A RESIDUAL STRESS SYSTEM WITH GEVREY COEFFICIENTS

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ABSTRACT. We consider the problem of the strong unique continuation for an elasticity system with general residual stress. Due to the known counterexamples, we assume the coefficients of the elasticity system are in the Gevrey class of appropriate indices. The main tools are Carleman estimates for product of two second order elliptic operators.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In this paper, we prove the strong unique continuation property (SUCP) for the isotropic elasticity system with residual stress under appropriate conditions. We formulate the mathematical problem in the following.

Let Ω be a connected open domain in \mathbb{R}^3 and consider the time-harmonic elasticity system

$$(1) \quad \nabla \cdot \sigma + \kappa^2 \rho u = 0 \text{ in } \Omega,$$

where $\sigma = (\sigma_{ij})_{i,j=1}^3$ is the stress tensor field, $\kappa \in \mathbb{C}$ is the frequency and $\rho = \rho(x) > 0$ denotes the density of the medium. The vector field $u(x) = (u_i(x))_{i=1}^3$ is the displacement vector. Suppose that the stress tensor is given by

$$\sigma(x) = T(x) + (\nabla u)T(x) + \lambda(x)(\text{tr}E)I + 2\mu(x)E,$$

where $E(x) = \frac{\nabla u + \nabla u^t}{2}$ is the infinitesimal strain and $\lambda(x), \mu(x)$ are the Lamé parameters. The second-rank tensor $T(x) = (t_{ij}(x))_{i,j=1}^3$ is the residual stress and satisfies

$$t_{ij}(x) = t_{ji}(x), \quad \forall i, j = 1, 2, 3 \text{ and } x \in \Omega,$$

and

$$\nabla \cdot T = \sum_j \partial_j t_{ij} = 0 \text{ in } \Omega, \quad \forall i = 1, 2, 3.$$

If we define the elastic tensor $C = (C_{ijkl})_{i,j,k,l=1}^3$ with

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{jk} \delta_{jl} + \delta_{jk} \delta_{il}) + t_{jl} \delta_{ik},$$

then (1) is equivalent to

$$\nabla \cdot (C \nabla u) + \kappa^2 \rho u = 0 \text{ in } \Omega.$$

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We consider the SUCP for (1); i.e., if $u \in H_{loc}^2(\Omega)$ satisfies (1) and $u(x)$ vanishes to infinite order at a point $x_0 \in \Omega$, then u must vanish identically in Ω . Without loss of generality, we assume $x_0 = 0$. A brief history of the results on the (strong) unique continuation for (1) is the following. In [13], Nakamura and Wang proved the unique continuation property for (1) under the condition $\max_{i,j} \|t_{ij}\|_\infty$ is small and $T(x), \lambda(x), \mu(x) \in W^{2,\infty}$ and $\rho(x) \in W^{1,\infty}$. In [12], Lin proved the SUCP for (1) under the assumptions that $T(0) = 0$, $\max_{i,j} \|t_{ij}\|_\infty$ is small, $\lambda(x), \mu(x)$ and $\rho(x)$ are in C^2 . In addition, in [15], Uhlmann and Wang proved the unique continuation principle for (1) under the conditions $T(x), \lambda(x), \mu(x) \in W^{2,\infty}$, $\rho(x) \in W^{1,\infty}$ and general residual stress.

Motivated by [15], we want to prove the SUCP for (1) with arbitrary residual stress. In this paper, we will give a reduction algorithm to transform (1) into a special fourth order elliptic system. The main difficulty is that when $T(0) \neq 0$, the leading terms of (1) will not be the Laplacian at zero, so we cannot use a perturbation argument to derive suitable Carleman estimates in order to obtain the SUCP. In [2], Alinhac and Baouendi proved the SUCP for any fourth order operator with smooth coefficients verifying $P = Q_2 Q_1 + R$, where Q_i 's are second order elliptic operators with $Q_i(0, D) = -\Delta$ for $i = 1, 2$. Moreover, in [10], Le Borgne proved the SUCP for fourth order differential inequality with Q_i 's are Lipschitz continuous and $Q_i(0, D) = -\Delta$ for $i = 1, 2$. In [12], Lin introduced $v = \nabla \cdot u$ and $w = \nabla \times u$ to transform (1) into a second order differential system, but the system is weakly-coupled; i.e., the principal part of the second order derivatives are not diagonal. Moreover, Lin also introduced a fourth order elliptic system $P = \Delta Q_i$ where Q_i 's are second order elliptic operators with $Q_i(0, D) = \Delta$ for $i = 1, 2$ and gave another approach to derive the SUCP. For more details, we refer the reader to [12].

In this note, our transformation will reduce (1) to a fourth order principally diagonal elliptic system with the same leading coefficients. The key observation is that the leading terms of the fourth order elliptic system are the same. Notice that principally diagonal strongly elliptic systems allow the application of Carleman estimates for scalar operators since these estimates are flexible with respect to perturbations by lower order terms. Therefore, it is possible to derive suitable Carleman estimates for the fourth order elliptic system.

In general, the SUCP does not hold even if the coefficients are smooth; Alinhac gave a counterexample in [1]. Thus, we consider all the coefficients in the Gevrey class and we will use the Carleman estimates proved in [6] for the scalar higher order elliptic equations in order to prove the SUCP for the new fourth order strongly elliptic system.

Definition 1.1. We say that $f \in C^\infty(\Omega)$ belongs to the Gevrey class of order s —denote it as $G^s(\Omega)$ —if there exist constants c, A and multiindices β such that

$$|\partial^\beta f| \leq cA^{|\beta|} |\beta|!^s \text{ in } \Omega.$$

To simplify the notation, from now on, we use G^s to denote $G^s(\Omega)$. In this paper, we assume all the coefficients $T(x), \lambda(x), \mu(x)$ and $\rho(x)$ lie in the Gevrey class G^s . We are interested in the SUCP for (1) with Gevrey coefficients, which means if u satisfies (1) and u is flat at the origin in the sense that

$$(2) \quad \sup_{r \leq \delta} r^{-N} \|u\|_{L^2(B(0,r))} < \infty$$

for all N , then u vanishes near the origin. If u is smooth, the condition (2) is equivalent to all partial derivatives of u vanishing at 0.

The SUCP for the second order elliptic equations in the Gevrey class were studied in many works [4–6, 11]. In 1981, Lerner [11] considered a second order elliptic operator L in \mathbb{R}^2 with simple characteristics and the coefficients in the Gevrey class of order s . Lerner proved that if s is smaller than a quantity depending on the principal symbol of $l_2(0, \mathbb{R}^2)$, then L has the SUCP near 0. In [5], the authors extended Lerner’s result to \mathbb{R}^N , which means the SUCP holds for a second order elliptic operator L in \mathbb{R}^N with the Gevrey order s smaller than a quantity depending on the principal symbol of $l_2(0, \mathbb{R}^N)$.

Recall that the strongly elliptic condition is given as: there exists $c_0 > 0$ such that for all vectors $\xi = (\xi_i)_{i=1}^3$,

$$\sum_{ij} a_{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2, \quad \forall x \in \Omega.$$

In this paper, we assume P_1 and P_2 are two strongly elliptic operators, where

$$(3) \quad P_1(x, D) := \sum_{jk} a_{jk}^1(x)\partial_{x_jx_k}^2 := \sum_{jk} (\mu\delta_{jk} + t_{jk})\partial_{x_jx_k}^2,$$

$$(4) \quad P_2(x, D) := \sum_{jk} a_{jk}^2(x)\partial_{x_jx_k}^2 := \sum_{jk} ((\lambda + 2\mu)\delta_{jk} + t_{jk})\partial_{x_jx_k}^2$$

with $a_{jk}^1(x) = \mu(x)\delta_{jk} + t_{jk}(x)$ and $a_{jk}^2(x) = (\lambda(x) + 2\mu(x))\delta_{jk} + t_{jk}(x)$. Further, there exists $c_0 > 0$ such that for any $\xi = (\xi_i)_{i=1}^3 \in \mathbb{R}^3$,

$$(5) \quad \sum_{jk} a_{jk}^1(x)\xi_j\xi_k = \sum_{jk} t_{jk}\xi_j\xi_k + \mu|\xi|^2 \geq c_0|\xi|^2$$

$$(6) \quad \sum_{jk} a_{jk}^2(x)\xi_j\xi_k = \sum_{jk} t_{jk}\xi_j\xi_k + (\lambda + 2\mu)|\xi|^2 \geq c_0|\xi|^2$$

for all $x \in \Omega$. Note that $(a_{jk}^\ell(x))_{j,k=1}^3$ is a symmetric matrix for $\ell = 1, 2$.

We also assume that there exists a constant $\alpha > 0$ such that the eigenvalues $\lambda_1^\ell \leq \lambda_2^\ell \leq \lambda_3^\ell$ of $(a_{jk}^\ell(0))$ satisfy

$$(7) \quad \alpha > \frac{\lambda_3^\ell - \lambda_1^\ell}{\lambda_1^\ell}$$

and

$$(8) \quad s < 1 + \frac{1}{\alpha}$$

for $\ell = 1, 2$.

The following theorem derives the SUCP for (1) when all the coefficients lie in the Gevrey class G^s .

Theorem 1.2. *Let the residual stress $(t_{ij}(x))_{i,j=1}^3$, the Lamé parameters $\lambda(x)$, $\mu(x)$ and the density of the medium $\rho(x)$ be in the Gevrey class $G^s(\Omega)$ with s satisfying (8). Then, for all $u \in H_{loc}^2(\Omega; \mathbb{R}^3)$ solving (1) and for all $N > 0$,*

$$\int_{R \leq |x| \leq 2R} |u|^2 dx = O(R^N) \text{ as } R \rightarrow 0;$$

thus u is identically zero in Ω .

This paper is organized as follows. In section 2, we will reduce (1) to a fourth order principally diagonal elliptic system. We use the ideas in [12] and give more detailed transformations. In section 3, we will use the property of the strongly elliptic system in the Gevrey class, so we can get the asymptotic behavior of u near 0. In section 4, we state the SUCP for the fourth order elliptic system and prove the theorem by using the Carleman estimates.

2. REDUCTION TO A FOURTH ORDER STRONGLY ELLIPTIC SYSTEM

In this section, we want to transform (1) into a principally diagonal fourth order strongly elliptic system, as the calculation in [12]. Let

$$(9) \quad Ru = \nabla \cdot (\nabla u T)$$

with $Ru = ((Ru)_1, (Ru)_2, (Ru)_3)$, where $(Ru)_i = \sum_{jk} t_{jk} \partial_{jk}^2 u_i, i = 1, 2, 3$.

As in section 2, we set $U = (u, v, w)^t$, where $v = \nabla \cdot u, w = \nabla \times u$ and u satisfies (1). From (1), (9), let P_1 and P_2 be two elliptic operators:

$$\begin{aligned} P_1(x, D) &= R + \mu \Delta, \\ P_2(x, D) &= R + (\lambda + 2\mu) \Delta; \end{aligned}$$

then (u, v, w) satisfies

$$(10) \quad P_1(x, D)u = A_{1,1}(u, v) + A_{1,0}(u, v),$$

$$(11) \quad \begin{aligned} P_2(x, D)v &= - \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u \\ &\quad + A_{2,1}(u, v, w) + A_{2,0}(u, v, w), \end{aligned}$$

$$(12) \quad \begin{aligned} P_1(x, D)w &= - \sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u \\ &\quad + A_{3,1}(u, v, w) + A_{3,2}(u, v, w), \end{aligned}$$

where $A_{\ell,m}$ are m -th order differential operators. For more details, we refer the reader to [12].

Notice that $u \in H_{loc}^2(\Omega; \mathbb{R}^3)$ satisfies (10) and $v = \nabla \cdot u \in H_{loc}^1(\Omega)$ and $\nabla v \in L_{loc}^2(\Omega)$; then the right hand side of (10) lies in $L_{loc}^2(\Omega)$. Therefore, we use the standard elliptic higher order regularity theory for (10) (see Theorem 2.2 in [7]) and the strongly elliptic property. Then we have $u \in H_{loc}^3(\Omega; \mathbb{R}^3)$. Iterating the procedures, we obtain $u \in H_{loc}^k(\Omega; \mathbb{R}^3) \forall k \in \mathbb{N}$ (which implies $v, w \in H_{loc}^k(\Omega) \forall k \in \mathbb{N}$).

Let $P(x, D)$ be the principal part of the system to get

$$P(x, D)U = (P_1(x, D)u, P_2(x, D)v, P_1(x, D)w)^t,$$

where $U := (u, v, w)^t : \Omega \rightarrow \mathbb{R}^7$. Component-wise, we have

$$(P(x, D)U)_i = \mu \Delta u_i + \sum_{jk} t_{jk} \partial_{jk}^2 u_i, \quad i = 1, 2, 3$$

$$(P(x, D)U)_i = (\lambda + 2\mu) \Delta v + \sum_{jk} t_{jk} \partial_{jk}^2 v, \quad i = 4$$

$$(P(x, D)U)_i = \mu \Delta w_{i-4} + \sum_{jk} t_{jk} \partial_{jk}^2 w_{i-4}, \quad i = 5, 6, 7.$$

Now, let us take the second order elliptic operator $P_2(x, D)$ on (10). We get

$$(13) \quad \begin{aligned} P_2 P_1(x, D)u &= P_2(x, D)[A_{1,1}(u, v) + A_{1,0}(u, v)] \\ &:= \sum_{m=0}^3 B_{1,m}(u, v), \end{aligned}$$

where $B_{1,m}$ is an m -th order differential operator. Similarly, we can take $P_1(x, D)$ on (11) and $P_2(x, D)$ on (12) to obtain

$$(14) \quad \begin{aligned} P_1 P_2(x, D)v &= P_1(x, D)\left(-\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u\right) + P_1(x, D)(A_{2,1}(u, v, w) + A_{2,0}(u, v, w)) \\ &= -P_1(x, D)\left(\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u\right) + \sum_{m=0}^3 B_{2,m}(u, v, w) \end{aligned}$$

and

$$(15) \quad \begin{aligned} P_2 P_1(x, D)w &= P_2(x, D)\left(-\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u\right) + P_2(x, D)(A(u, v, w) + A_{3,2}(u, v, w)) \\ &= -P_2(x, D)\left(\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u\right) + \sum_{m=0}^3 B_{3,m}(u, v, w). \end{aligned}$$

Now, if we interchange P_1, P_2 on (14) and use

$$P_2 P_1 = P_1 P_2 - [P_1, P_2],$$

where $[P_1, P_2]$ is the commutator of two second order elliptic operators, then $[P_1, P_2]$ is a third order differential operator. Thus, (14) becomes

$$(16) \quad \begin{aligned} P_2 P_1(x, D)v &= -P_1(x, D)\left[\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u\right] \\ &\quad + \sum_{m=0}^3 \widetilde{B}_{2,m}(u, v, w), \end{aligned}$$

where $\widetilde{B}_{2,m}$ is an m -th order differential operator and

$$\sum_{m=0}^3 \widetilde{B}_{2,m}(u, v, w) = \sum_{m=0}^3 B_{2,m}(u, v, w) - [P_1, P_2](x, D)v.$$

Now, combining (13), (15) and (16), we get

$$(17) \quad P_2 P_1(x, D) \begin{pmatrix} u \\ v \\ w \end{pmatrix} = - \begin{pmatrix} 0 \\ P_1(x, D)\left[\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u\right] \\ P_2(x, D)\left[\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u\right] \end{pmatrix} + \sum_{m=0}^3 \begin{pmatrix} B_{1,m}(u, v) \\ \widetilde{B}_{2,m}(u, v, w) \\ B_{3,m}(u, v, w) \end{pmatrix}.$$

Now, for $P_1\left(\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u\right)$ in (17), recall that $P_1(x, D) = R + \mu\Delta$ and $Ru = \nabla \cdot (\nabla u T)$. Then we have

$$(18) \quad P_1(x, D)\left[\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u\right] = R\left(\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u\right) + \mu\Delta\left(\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u\right).$$

For the second term of (18), by using the vector identity $\Delta u = \nabla(\nabla \cdot u) - \nabla \times \nabla \times u = \nabla v - \nabla \times w$, it is easy to see that

$$\begin{aligned} & \Delta\left(\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u\right) \\ &= \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2(\Delta u) + \widetilde{A}_{2,3}(u) + \widetilde{A}_{2,2}(u) \\ &= \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2(\nabla v - \nabla \times w) + \widetilde{A}_{2,3}(u) + \widetilde{A}_{2,2}(u) \\ &= \widetilde{B}_{2,3}(u, v, w) + \widetilde{A}_{2,2}(u), \end{aligned}$$

where $\widetilde{A}_{2,m}$ and $\widetilde{B}_{2,m}$ are m -th order differential operators and

$$\widetilde{B}_{2,3}(u, v, w) = \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2(\nabla v - \nabla \times w) + \widetilde{A}_{2,3}(u).$$

For the first term of (18), we have

$$\begin{aligned} & R\left(\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u\right) \\ &= \sum_{\ell m} t_{\ell m} \partial_{\ell m}^2 \left(\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u\right) \\ &= \sum_{jk} \nabla(t_{jk}) \cdot \left[\sum_{\ell m} t_{\ell m} \partial_{\ell m}^2 \partial_{jk}^2 u\right] + \widetilde{C}_{2,3}(u) + \widetilde{C}_{2,2}(u) \\ &= \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 \left(\sum_{\ell m} t_{\ell m} \partial_{\ell m}^2 u\right) + \widetilde{D}_{2,3}(u) + \widetilde{D}_{2,2}(u) \\ &= \sum_{jk} \nabla(t_{jk}) \cdot Ru + \widetilde{D}_{2,3}(u) + \widetilde{D}_{2,2}(u), \end{aligned}$$

and using (10), we have $Ru = -\mu\Delta u + A_{1,1}(u, v) + A_{1,0}(u, v)$. So we have

$$\begin{aligned} & R\left(\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u\right) \\ &= \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2(-\mu\Delta u + A_{1,1}(u, v) + A_{1,0}(u, v)) + \widetilde{D}_{2,3}(u) + \widetilde{D}_{2,2}(u) \\ &= \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2(-\mu(\nabla v - \nabla \times w)) + \sum_{m=0}^3 \widetilde{E}_{2,3}(u, v) \\ &= \sum_{m=0}^3 \widetilde{F}_{2,m}(u, v, w) \end{aligned}$$

where $\widetilde{C}_{2,m}, \widetilde{D}_{2,m}, \widetilde{E}_{2,m}$ and $\widetilde{F}_{2,m}$ are m -th order differential operators. From the above calculation and (17), we have

$$(19) \quad P_2 P_1(x, D)v = \sum_{m=0}^3 \widehat{E}_{2,m}(u, v, w),$$

where $\widehat{E}_{2,m}$ are m -th order differential operators. Similarly, for $P_2 \left(\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u \right)$, it is easy too see that

$$\Delta \left(\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u \right) = \widetilde{A}_{3,3}(u, v, w) + \widetilde{A}_{3,2}(u),$$

where $\widetilde{A}_{3,m}$ is an m -th order differential operator. Similarly, for $R \left(\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u \right)$, component-wise, we have

$$\begin{aligned} & \left[R \left(\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u \right) \right]_i \\ &= \sum_{\ell m} t_{\ell m} \partial_{\ell m}^2 \left(\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u \right)_i \\ &= \sum_{\ell m} \sum_{jk} (\nabla(t_{jk}) \times t_{\ell m} \partial_{\ell m}^2 \partial_{jk}^2 u)_i + \widetilde{B}_{3,3}(u) + \widetilde{B}_{3,2}(u) \\ &= \sum_{jk} (\nabla(t_{jk}) \times \partial_{jk}^2 \left(\sum_{\ell m} t_{\ell m} \partial_{\ell m}^2 u \right))_i + \widetilde{C}_{3,3}(u) + \widetilde{C}_{3,2}(u) \\ &= \sum_{jk} (\nabla(t_{jk}) \times \partial_{jk}^2 R u)_i + \widetilde{D}_{3,3}(u) + \widetilde{D}_{3,2}(u) \end{aligned}$$

and using (10) again, we obtain

$$\begin{aligned} & \left[R \left(\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u \right) \right]_i \\ &= \sum_{jk} (\nabla(t_{jk}) \times \partial_{jk}^2 [-\mu(\nabla v - \nabla \times w)])_i + \sum_{m=0}^3 \widetilde{E}_{3,m}(u, v) \\ &= \sum_{m=0}^3 \widetilde{F}_{3,m}(u, v, w), \end{aligned}$$

where $\widetilde{B}_{3,m}, \widetilde{C}_{3,m}, \widetilde{D}_{3,m}, \widetilde{E}_{3,m}$ and $\widetilde{F}_{3,m}$ are m -th order differential operators.

Therefore, we transform the equation (15) into

$$(20) \quad P_2 P_1(x, D)w = \sum_{m=0}^3 \widehat{E}_{3,m}(u, v, w),$$

where $\widehat{E_{3,m}}$ are m -th order differential operators. From (19), (20) and (17), we can obtain

$$P_2P_1(x, D) \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \sum_{m=0}^3 \begin{pmatrix} \widehat{E_{1,m}}(u, v, w) \\ \widehat{E_{2,m}}(u, v, w) \\ \widehat{E_{3,m}}(u, v, w) \end{pmatrix},$$

with $\widehat{E_{\ell,m}}$ are m -th order differential operators or, equivalently,

$$(21) \quad P_2P_1U = \sum_{m=0}^3 \widehat{E_m}(U),$$

where $\widehat{E_m} = (\widehat{E_{3,m}}, \widehat{E_{3,m}}, \widehat{E_{3,m}})^t$ is an m -th order differential operator and $U = (u, v, w)^t$, which means this fourth order differential equation has the same leading term P_2P_1 and all coefficients of (21) lie in G^s . Moreover, if we use the elliptic regularity for (21) with Gevrey coefficients, then $U \in G^s$ by Proposition 2.13 in [3].

3. THE ASYMPTOTIC BEHAVIOR OF u NEAR 0

As in section 2, we set $U = (u, v, w)^t$, where $v = \nabla \cdot u$ and $w = \nabla \times u$. If we can prove that U solves (21) and satisfies the SUCP, then u solves (1) and fulfills the SUCP. In the following theorem, we describe the asymptotic behavior of u near 0. Recall that if $u \in H_{loc}^2(\Omega; \mathbb{R}^3)$, then $u \in C^\infty(\Omega)$ by the standard elliptic regularity. Thus, $\forall k \in \mathbb{N}$, we can consider $u \in H_{loc}^k(\Omega; \mathbb{R}^3)$ for arbitrary $k \in \mathbb{N}$ in the following results.

Theorem 3.1 ([12]). *Let u be a solution to (1) and for all $N > 0$,*

$$\int_{R \leq |x| \leq 2R} |u|^2 dx = O(R^N) \text{ as } R \rightarrow 0.$$

Then for $|\beta| \leq 2$, we have

$$\int_{R \leq |x| \leq 2R} |R^{|\beta|} D^\beta u|^2 dx = O(R^N) \text{ as } R \rightarrow 0.$$

Proof. The theorem was proved by Corollary 17.1.4 in Hörmander [9]. By using Theorem 3.1, we will get the following corollary. □

Corollary 3.2. *Let $U = (u, v, w)^t$ with $v = \nabla \cdot u$ and $w = \nabla \times u$. Then for $|\beta| \leq 1$, $\forall N > 0$, we have*

$$(22) \quad \int_{R \leq |x| \leq 2R} |D^\beta U|^2 dx = O(R^N) \text{ as } R \rightarrow 0.$$

In fact, we can get higher derivatives for $|\beta| \geq 2$ in Corollary 3.2.

Lemma 3.3 ([9]). *If U satisfies a fourth order strongly elliptic system (21)*

$$PU = \sum_{m=0}^3 \widehat{E_m}(U)$$

and U satisfies $\forall N > 0$,

$$\int_{R \leq |x| \leq 2R} |U|^2 dx = O(R^N) \text{ as } R \rightarrow 0,$$

then it follows that if $|\beta| \leq 4$ that

$$(23) \quad \int_{R \leq |x| \leq 2R} |R^{|\beta|} D^\beta U|^2 dx = O(R^N) \text{ as } R \rightarrow 0.$$

Proof. Since U satisfies (21), a fourth order strongly elliptic system, by using Corollary 17.1.4 in [9], we obtain (23). \square

Remark 3.4. In section 3 of [12], the author proved that (23) holds for $|\beta| \leq 2$. From Lemma 3.3 and the coefficients of P in the Gevrey class G^s , we have $U \in G^s$ and

$$\int_{|x| \leq R} |D^\beta U|^2 dx = O(R^N) \text{ as } R \rightarrow 0,$$

for $|\beta| \leq 4$ and $\forall N > 0$.

4. PROOF OF THE MAIN THEOREM

In this section, we want to prove Theorem 1.2. If $U = (u, v, w)^t$ satisfies (21) and the SUCP, then the SUCP holds for u , where u fulfills (1).

4.1. SUCP for U . In the following theorem, we will prove the SUCP for U .

Theorem 4.1. *Suppose that the second order elliptic operators P_ℓ satisfies (3), (4), (5) and (6) for $\ell = 1, 2$; $\alpha > 0$ satisfies (7) at $x = 0$; and s satisfies (8). Let $P = P_2 P_1$ be a fourth order elliptic operator. Then the SUCP holds for the elliptic system*

$$(24) \quad PU = \sum_{|\beta| \leq 3} a_\beta \partial^\beta U$$

provided the coefficients of P_ℓ are in the Gevrey class G^s .

Proof. The proof follows from [6] and section 1. To prove Theorem 4.1, there are two steps. First, Gevrey regularity of the elliptic system implies the solution U of (24) is in the Gevrey class G^s (see Proposition 2.13 in [3]). Using the vanishing order assumption and $U \in G^s$, we have

$$(25) \quad |U| \lesssim e^{-|x|^{-\gamma}},$$

near $x = 0$ and for some constant $\gamma > 0$. Second, we can show that (25) implies U vanishes near 0 by using appropriate Carleman estimates. In addition, since U vanishes near 0, by the results in [15], we have $U \equiv 0$ in Ω . \square

Remark 4.2. We give the Carleman estimates which were used in the proof of Theorem 4.1 in the following section.

4.2. Carleman estimates. We are going to derive the Carleman estimates with the weight $e^{\tau|x|^{-\alpha}}$ for the fourth order elliptic operator $P = P_2 P_1$ in this section. The following Carleman estimates for the scalar case have been proven in [5] and [6]. Similar to the scalar elliptic equation, we can derive the following Carleman estimate for the special elliptic system.

Proposition 4.3. *Let $P_\ell(x, D) = \sum_{j,k} a_{jk}^\ell(x) \partial_{jk}^2$ be a principally diagonal second order elliptic operator where $a_{jk}^\ell(x) \in G^s$ satisfies (3), (4), (5) and (6) for $\ell = 1, 2$; $\alpha > 0$ satisfies (7) at $x = 0$; and s satisfies (8). Then there exist $\tau_0 > 0$ and $r_0 > 0$*

such that for $\tau > \tau_0$ and for all $V \in C^\infty((B_{r_0} \setminus \{0\}); \mathbb{R}^7)$, $\ell = 1, 2$, the following inequality holds:

$$\begin{aligned} & \tau \int |D^2(|x|^{\alpha/2} e^{\tau|x|^{-\alpha}} V)|^2 dx + \tau^3 \int |x|^{-4-3\alpha} e^{2\tau|x|^{-\alpha}} V|^2 dx \\ & \lesssim \int |e^{2\tau|x|^{-\alpha}} (P_\ell V)|^2 dx. \end{aligned}$$

Proof. Since P_ℓ is the principally diagonal second order elliptic operator for $\ell = 1, 2$, we can directly follow the consequences in [6] and use the proof in [5]. For more details and classical results, we refer the reader to [8, 14]. \square

By using the integration by parts, we can get a stronger inequality in the following. For more details, we refer the reader to [6] and section 3. We have

$$\begin{aligned} & \sum_{j=0}^2 \tau^{3-2j} \int e^{2\tau|x|^{-\alpha}} |x|^\alpha |x|^{2(j-2)(1+\alpha)} |D^j V|^2 dx \\ & \lesssim \int e^{2\tau|x|^{-\alpha}} |P_\ell V|^2 dx, \end{aligned}$$

with P_ℓ satisfying all the assumptions in Proposition 4.3 for $\ell = 1, 2$. Note that the right hand side of (11) and (12) involve second order derivatives of u , so we cannot apply the Carleman estimates for the second order differential systems directly to get the SUCP for U . Since we have transformed (1) into a special fourth order elliptic system with the same leading operator (see (21)), we can derive the Carleman estimates for the operator $P = P_2 P_1$.

Corollary 4.4 ([6]). *Let*

$$A = \sum_{jk} a_{jk}(x) \partial_{x_j x_k}^2$$

be a second order strongly elliptic operator with a_{ij} in the Gevrey class G^s . Suppose $\alpha > 0$ satisfies (7) at $x = 0$. Then there exists τ_0 such that for all $|s|$, $k \leq \nu$ and $\tau \geq \tau_0$,

$$\begin{aligned} (26) \quad & \sum_{j=0}^{k+2} \tau^{3-2j} \int |x|^{\alpha+2j(1+\alpha)} |x|^{2s} e^{2\tau|x|^{-\alpha}} |D^j V|^2 dx \\ & \lesssim \sum_{j=0}^k \tau^{-2j} \int |x|^{2(2+j)(1+\alpha)} |x|^{2s} e^{2\tau|x|^{-\alpha}} |D^j(AV)|^2 dx. \end{aligned}$$

Proof. See [6] and section 3. We can use the induction hypothesis to prove Corollary 4.4. \square

For the fourth order elliptic operator, $PU = P_2 P_1 U$ is the product of two second order elliptic operators which satisfy (21), where $U = (u, v, w)^t$ and $P_\ell(x, D)U = \sum_{jk} a_{jk}^\ell(x) \partial_{jk}^2 U$. Recall that $a_{jk}^\ell \in G^s$ and $\alpha > 0$ satisfy (7) uniformly in x and for $\ell = 1, 2$; then we have the following key estimates.

Proposition 4.5. *We have the following Carleman estimate:*

$$\sum_{j=0}^4 \tau^{6-2j} \int |x|^{-8-6\alpha} |x|^{2j(1+\alpha)} e^{2\tau|x|^{-\alpha}} |D^j V|^2 dx \leq C \int e^{2\tau|x|^{-\alpha}} |PV|^2 dx.$$

Proof. Apply Corollary 4.4 iteratively. Then we have

$$\begin{aligned}
 (27) \quad & \sum_{j=0}^4 \tau^{6-2j} \int |x|^{-8-6\alpha} |x|^{2j(1+\alpha)} e^{2\tau|x|^{-\alpha}} |D^j V|^2 dx \\
 & \lesssim \sum_{j=9}^2 \tau^{3-2j} \int |x|^{-4-3\alpha} |x|^{j(1+\alpha)} e^{2\tau|x|^{-\alpha}} |D^j(P_1 V)|^2 dx \\
 & \lesssim \int e^{2\tau|x|^{-\alpha}} |(P_2 P_1 V)|^2 dx = \int e^{2\tau|x|^{-\alpha}} |PV|^2 dx,
 \end{aligned}$$

where the first inequality is obtained by (26) with $k = 2$, $s = -4 - \frac{7}{2}\alpha$ and the second inequality is obtained by (26) with $k = 0$, $s = -2(1 + \alpha)$. For more details, we refer the reader to [6]. \square

Now, we want to prove the SUCP for (1). Here we prove Theorem 1.2.

Proof of Theorem 4.1. The operator $P = P_2 P_1$ is strongly elliptic in the Gevrey class G^s ; then U is also in the Gevrey class G^s . Therefore, we have that the vanishing of infinite order implies that

$$|u| \lesssim e^{-|x|^{-\gamma}}$$

for some $\gamma > \alpha$. Let $\chi \in C_0^\infty(\mathbb{R}^3)$ be such that $\chi \equiv 1$ for $|x| \leq R$ and $\chi \equiv 0$ for $|x| \geq 2R$ ($R > 0$ is small enough). Then we can apply (27) to the function χU , which means

$$\begin{aligned}
 (28) \quad & C \sum_{|\beta|=0}^4 \tau^{6-2|\beta|} \int_{|x|<R} |x|^{(2|\beta|-6)(1+\alpha)-2} e^{2\tau|x|^{-\alpha}} |D^\beta U|^2 dx \\
 & \leq \int e^{2\tau|x|^{-\alpha}} |PU|^2 dx \\
 & \leq \int_{|x|<R} e^{2\tau|x|^{-\alpha}} |PU|^2 dx + \int_{|x|>R} e^{2\tau|x|^{-\alpha}} |P(\chi U)|^2 \\
 & \leq \int_{|x|<R} e^{2\tau|x|^{-\alpha}} \left| \sum_{m=0}^3 \widehat{E}_m(U) \right|^2 dx + \int_{|x|>R} e^{2\tau|x|^{-\alpha}} |P(\chi U)|^2,
 \end{aligned}$$

by using the reduction elliptic system (21).

If τ is large and R is sufficiently small, then (28) implies that

$$\begin{aligned}
 (29) \quad & C \sum_{|\beta|=0}^4 \tau^{6-2|\beta|} \int_{|x|<R} |x|^{(2|\beta|-6)(1+\alpha)-2} e^{2\tau|x|^{-\alpha}} |D^\beta U|^2 dx \\
 & \leq \int_{|x|>R} e^{2\tau|x|^{-\alpha}} |P(\chi U)|^2,
 \end{aligned}$$

for some constant $C > 0$. Notice that $e^{\tau|x|^{-\alpha}} \geq e^{\tau R^{-\alpha}}$ for $|x| < R$ and $e^{\tau|x|^{-\alpha}} \leq e^{\tau R^{-\alpha}}$ for $|x| > R$. Therefore, we can use (29) to obtain

$$\begin{aligned} & C \sum_{|\beta|=0}^4 \tau^{6-2|\beta|} \int_{|x|<R} |x|^{(2|\beta|-6)(1+\alpha)-2} |D^\beta U|^2 dx \\ & \leq \int_{|x|>R} |P(\chi U)|^2. \end{aligned}$$

Letting $\tau \rightarrow \infty$, we get $U = 0$ in $\{|x| < R\}$ for R small, which implies $u = 0$ in $\{|x| < R\}$. Furthermore, by using the unique continuation principal in [15], we can obtain $u \equiv 0$ in Ω , and then we are done.

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