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RANDOMIZED SHARKOVSKY-TYPE RESULTS AND RANDOM SUBHARMONIC SOLUTIONS OF DIFFERENTIAL INCLUSIONS

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Abstract. Two multivalued deterministic versions of the celebrated Sharkovsky cycle coexistence theorem are randomized in terms of very general random periodic orbits. It is also shown that nontrivial subharmonics of scalar random upper-Carathéodory differential inclusions imply the coexistence of random subharmonics of all orders.

1. Introduction

The aim of the present paper is to randomize: (i) two multivalued deterministic Sharkovsky-type theorems (cf. [20]) obtained by the authors in [3, 7] and (ii) the deterministic theorem for scalar differential equations and inclusions, saying that the existence of a pure (nontrivial) subharmonic solution implies the coexistence of subharmonic solutions of all orders, whose various proofs can be found in [2, 4, 18, 19, 21].

The first goal is related to the existence of random periodic orbits rather than random periodic points (cf. [9, 10]) and there are quite rare results in this field (see [1, 6, 8, 16, 23]). The main technique, Proposition 3.2 below, will be developed for a very general class of multivalued random operators in a Suslin space. In this way, all our earlier results in [1, 6, 8] will be generalized both as a method in Section 3 as well as its application to Sharkovsky-type theorems in Section 4.

As concerns the second goal, the forcing property for the subharmonic solutions of random differential inclusions was already studied by the authors in [4] and [6], but the proofs were either only indicated in [4] or incomplete in [6]. Here, full proofs of the related Theorem 5.6 below will be given in Section 5. Although random periodic solutions of differential equations have been investigated as random fixed points of the associated random operators (see e.g. [11, 12, 22]), as far as we know, there are no further results for random subharmonic solutions considered as random periodic orbits of the associated random operators.

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Although the proofs of the statements might seem to be rather technical (some preliminaries are therefore recalled in Section 2), the main randomized theorems are formulated in an extremely simple and transparent way. Moreover, since the developed randomization technique is quite universal and powerful, it can be easily applied elsewhere.

2. Preliminaries

By $\mathbb{N}$, $\mathbb{R}$, $\mathbb{I}$, we denote the set of natural numbers, the set of real numbers, and the interval $[0,1]$, respectively. For finite $I \subset \mathbb{N}$, the symbol $\mathrm{LCM}\, I$ denotes the lowest common multiple of elements of I. We denote $D(n) = \{p < n \colon p|n\}$ and $d(n) = |D(n)|$, for $n \in \mathbb{N}$. We also fix the following notation: (Ω, Σ) is a measurable space, $\mathcal{I}$ is a proper σ-ideal on Ω (i.e. $\Omega \notin \mathcal{I}$), and μ (if it exists) is a complete σ-finite nontrivial measure on (Ω, Σ). We say that Σ is *complete* if such a measure exists. We say that Σ is *nonatomic* with respect to $\mathcal{I}$ if for every $A \in \Sigma$ such that $A \notin \mathcal{I}$, there is a splitting $A = B \cup C$ such that $B, C \in \Sigma$ and $B, C \notin \mathcal{I}$.

We recall the notion of a Suslin family. Let S be the set of all finite sequences of natural numbers. For a family $\{A_s \colon s \in S\}$ we put $\mathcal{A}\{A_s \colon s \in S\} = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=0}^{\infty} A_{\sigma \restriction n}$. We say that a family $\mathcal{F}$ is a *Suslin family* if $\mathcal{A}\mathcal{F} = \mathcal{F}$, where $\mathcal{A}\mathcal{F} = \{\mathcal{A}\{A_s \colon s \in S\} \colon A_s \in \mathcal{F} \text{ for } s \in S\}$.

By (X, d), we always understand a metric space. A metric space which is a continuous image of a Polish space is called a *Suslin space*. Each Suslin space is separable. For $Y \subset X$, by $\overline{Y}$ we mean the closure of Y in X. For $X = \mathbb{R}^k$ or $X = \mathbb{I}^k$, the symbol d denotes the Euclidean metric. For a closed set $B \subset X$, we use the standard notation: $d(x, B) = \inf\{d(x, y) \colon y \in B\}$.

For σ-algebras Σ_1 and Σ_2, by $\Sigma_1 \otimes \Sigma_2$ we mean their product σ-algebra. The symbol $\mathcal{B}(X)$ denotes the σ-algebra of Borel sets in X. If X_1, X_2 are separable, then $\mathcal{B}(X_1 \times X_2) = \mathcal{B}(X_1) \otimes \mathcal{B}(X_2)$. For a complete measure space (Ω, Σ, μ), we denote $\mathcal{N}(\Omega) = \{A \subset \Omega \colon \mu(A) = 0\}$. For $\Omega = \mathbb{R}^k$ or $\Omega = \mathbb{I}^k$, the symbol $\mathcal{L}(\Omega)$ denotes the σ-algebra of Lebesgue measurable sets on Ω.

We naturally identify a relation $\varphi \subset A \times B$ with a map $\varphi \colon A \to \mathcal{P}(B)$, where $\mathcal{P}(B)$ stands for all subsets of B. If we want to emphasize the properties of φ as a subset of $A \times B$, we use the notion of a graph Γ_φ, where $\Gamma_\varphi = \{(a, b) \in A \times B \colon b \in \varphi(a)\}$. Furthermore, if $\Gamma_\varphi \subset (A \times B) \times C$, i.e. $\varphi \colon A \times B \to \mathcal{P}(C)$, then for any $x \in A$, we define the relation $\varphi_x = \varphi(x, \cdot)$. A multivalued map $\varphi \colon A \multimap B$ is a relation with nonempty values, i.e., $\varphi \colon A \to \mathcal{P}(B) \setminus \{\emptyset\}$. For single valued maps, we identify a map $f \colon A \to B$ with the multivalued map with one-element values fulfilling the condition $\varphi(x) = \{f(x)\}$, for every $x \in A$.

By a superposition of a map $f \colon B \to C$ with a relation $\varphi \subset A \times B$, we mean the relation $f \circ \varphi \subset A \times C$, defined by $f \circ \varphi(a) = f(\varphi(a))$, for $a \in A$. By a product of relations $F_i \subset A \times B$ for $i \in I$, we mean the relation $\prod_{i \in I} F_i$ defined by $(\prod_{i \in I} F_i)(a) = \prod_{i \in I} F_i(a)$, for $a \in A$.

We denote by

$$\varphi^-(B) = \{\omega \in \Omega \colon \varphi(\omega) \cap B \neq \emptyset\}$$

the large preimage of the set $B \subset X$ under the relation $\varphi \subset \Omega \times X$. A relation $\varphi \subset \Omega \times X$ is called *measurable* if $\varphi^-(F) \in \Sigma$, for every closed $F \subset X$. It is called *weakly measurable* if $\varphi^-(G) \in \Sigma$, for every open $G \subset X$. The paper [14] contains a thorough analysis of the notion of a measurable relation. We shall state several facts from that paper here. Every measurable relation is weakly measurable (see [14, Proposition 2.1]). If Σ is complete and X is a Suslin space, the notions of measurability, weak measurability and a measurable graph coincide for a relation with closed values (see [14, Theorem 3.5]). The countable sum of measurable relations is measurable (see [14, Proposition 2.3(i)]). If a codomain is separable, then the product of at most countably many weakly measurable relations is weakly measurable (see [14, Proposition 2.3(ii)]). For a weakly measurable multivalued map $\varphi\colon \Omega \multimap \mathbb{R}$, the map defined by $f(\omega) = \inf \varphi(\omega)$, for $\omega \in \Omega$, is measurable (see [14, Theorem 5.8]). The superposition of a continuous map with a weakly measurable relation is weakly measurable (which is clear from the definition).

Assume that X and Y are metric spaces. A relation $\varphi \subset X \times Y$ is called *upper semicontinuous* (u.s.c.) if $\varphi^{-}(F)$ is closed in X, for every closed $F \subset Y$. It is called *lower semicontinuous* (l.s.c.) if $\varphi^{-}(G)$ is open in X, for every open $G \subset Y$. If φ is both l.s.c. and u.s.c., then it is called *continuous*.

We call a map $f\colon \Omega \to X$ a *selection* of a multivalued map $\varphi\colon \Omega \multimap X$ and write $f \subset \varphi$ if $f(\omega) \in \varphi(\omega)$, for each $\omega \in \Omega$. We shall use a generalization of the Aumann-von Neumann selection theorem which is a simple consequence of Corollary to Theorem 7 in the article [17] by Leese. We state it as follows:

Lemma 2.1. *Assume that Σ is a Suslin family and X is a Suslin space. Let $\varphi\colon \Omega \multimap X$ have a measurable graph, i.e. $\Gamma_\varphi \in \Sigma \otimes \mathcal{B}(X)$. Then it has a measurable selection $f \subset \varphi$.*

Another fact is a simple consequence of [17, Lemma 3].

Lemma 2.2. *Assume that Σ is a Suslin family and X is a Suslin space. Then every $\varphi\colon \Omega \multimap X$ with a measurable graph is measurable.*

A sequence $(x_i)_{i=0}^{k-1} \in A^k$ is called a *k-orbit* of the multivalued map $\varphi\colon A \multimap A$ if $x_{i+1} \in \varphi(x_i)$, for $i < k-1$, $x_0 \in \varphi(x_{k-1})$, and there is no $m < k$ such that $m|k$ and $x_{sm+i} = x_i$, for $i < m$ and $s < \frac{k}{m}$.

We recall the Sharkovsky ordering of natural numbers [20]:

$$\begin{array}{ccccccccc}
3 & \rhd & 5 & \rhd & 7 & \rhd & 9 & \rhd & \dots \\
2\cdot 3 & \rhd & 2\cdot 5 & \rhd & 2\cdot 7 & \rhd & 2\cdot 9 & \rhd & \dots \\
2^2\cdot 3 & \rhd & 2^2\cdot 5 & \rhd & 2^2\cdot 7 & \rhd & 2^2\cdot 9 & \rhd & \dots \\
 & & & & \vdots & & & & \\
\dots & \rhd & 2^3 & \rhd & 2^2 & \rhd & 2 & \rhd & 1.
\end{array}$$

We state here a simple fact about this ordering (cf. [8, Lemma 4.1]):

Lemma 2.3. *If $\mathrm{LCM}\{i_j : j < l\} = n$ and $k \lhd n$, then there is $j' < l$ such that $k \lhd i_{j'}$.*

We also recall here some generalizations of Sharkovsky's Theorem which are simple corollaries of Theorem 6 from [3] and Theorem 2 form [7], respectively.

Proposition 2.4. *Let $\varphi\colon \mathbb{R} \multimap \mathbb{R}$ be an l.s.c. multivalued map with compact and convex values. If φ has an n-orbit, then φ has a k-orbit, for every $k \lhd n$.*

Proposition 2.5. *Let $\varphi\colon \mathbb{R} \multimap \mathbb{R}$ be a u.s.c. multivalued map with compact and convex values. Suppose that φ has an n-orbit ($n = 2^m q$, where q is odd), and n is the maximal number in the Sharkovsky ordering with that property.*

(1) *If $q > 3$, then φ has a k-orbit for every $k \lhd n$, except possibly for $k = 2^{m+2}$.*

(2) *If $q = 3$, then at least one of the following two cases occurs:*
  (i) *φ has a k-orbit for every $k \lhd n$, except possibly for $k = 2^{m+1} \cdot 3, 2^{m+2}$,*
  (ii) *φ has a k-orbit for every $k \lhd n$, except possibly for $k = 2^{m+1}$.*

(3) *If $q = 1$, then φ has a k-orbit for every $k \lhd n$.*

We recall here the definition of a random operator after [1].

Definition 2.6. Let $\varphi\colon \Omega \times X \multimap X$ be a multivalued map with closed values. We say that φ is a *random operator* if it is weakly measurable with respect to σ-algebra $\Sigma \otimes \mathcal{B}(X)$. We call it an *l.s.c. (u.s.c.) random operator* if φ_ω, i.e., $\varphi(\omega, \cdot)$, is l.s.c. (u.s.c., respectively), for almost all $\omega \in \Omega$.

Remark 2.7. For the definition of a random operator, it is usually still required that φ be compact-valued (cf. [13]), and $\varphi(\omega, \cdot)\colon X \multimap X$ to be u.s.c. (cf. again [13]) or Hausdorff-continuous (cf. [15, Chapter 5.6]), for almost all $\omega \in \Omega$. We omit these additional assumptions and introduce here a more general approach.

For a random operator $\varphi\colon \Omega \times X \multimap X$ and $k, m \in \mathbb{N}$, we define the function $d_{\varphi,k,m}\colon \Omega \times X^k \to [0, +\infty)$ as follows (cf. [1]):

$$d_{\varphi,k,m}\left(\omega, (x_i)_{i=0}^{k-1}\right) = d\Big((x_i)_{i=0}^{k-1}, \varphi(\omega, x_{m-1}) \times \varphi(\omega, x_0) \times \ldots \times \varphi(\omega, x_{m-2})$$
$$\times \left(\{x_0\} \times \{x_1\} \times \ldots \times \{x_{m-1}\}\right)^{\frac{k}{m}-1}\Big),$$

and the function $D_{\varphi,k,m}\colon \Omega \times X^k \to [0, +\infty)^{1+d(m)}$ as follows:

$$D_{\varphi,k,m} = \left(d_{\varphi,k,m}, (d_{\varphi,k,p})_{p \in D(m)}\right).$$

Moreover, for $\omega \in \Omega$, we denote $d_{\varphi,k,m,\omega} = d_{\varphi,k,m}(\omega, \cdot)$ and

$$D_{\varphi,k,m,\omega} = D_{\varphi,k,m}(\omega, \cdot) = \left(d_{\varphi,k,m,\omega}, (d_{\varphi,k,p,\omega})_{p \in D(m)}\right).$$

We also define the relation $\mathcal{O}_{\varphi,k,m} \subset \Omega \times X^k$ as follows:

$$\mathcal{O}_{\varphi,k,m}(\omega) = \left\{(x_i)_{i=0}^{k-1} \in X^k \colon (x_i)_{i=0}^{m-1} \text{ is an } m\text{-orbit of } \varphi_\omega \wedge \forall_{\substack{s < \frac{k}{m} \\ i < m}} x_{sm+i} = x_i\right\},$$

and set $\Delta_{\varphi,k} = \{\omega \in \Omega \colon \varphi_\omega \text{ has a } k\text{-orbit}\}$.

The following simple facts from the paper (cf. [8, Lemmas 3.1 and 3.2]) will be useful.

Lemma 2.8. *Assume that X is separable. Let $\varphi\colon \Omega \times X \multimap X$ be a random operator. Then, for every $k, m \in \mathbb{N}$ such that $m|k$, the function $d_{\varphi,k,m}$ (and as a result $D_{\varphi,k,m}$) is measurable with respect to σ-algebra $\Sigma \otimes \mathcal{B}(X^k)$.*

Lemma 2.9. *Let $\varphi\colon \Omega \times X \multimap X$ be a random operator and let $k, m \in \mathbb{N}$ be such that $m|k$. Then*

$$\mathcal{O}_{\varphi,k,m}(\omega) = \left\{(x_i)_{i=0}^{k-1} \in X^k \colon d_{\varphi,k,m,\omega}\left((x_i)_{i=0}^{k-1}\right) = 0 \wedge \forall_{\substack{p<m \\ p|m}} d_{\varphi,k,p,\omega}\left((x_i)_{i=0}^{k-1}\right) > 0\right\},$$

for all $\omega \in \Omega$.

Remark 2.10. Equivalently we can write:

$$\mathcal{O}_{\varphi,k,m}(\omega) = D^{-1}_{\varphi,k,m,\omega}\left(\{0\} \times (0, +\infty)^{d(m)}\right),$$

or simply:

$$\Gamma_{\mathcal{O}_{\varphi,k,m}} = D^{-1}_{\varphi,k,m}\left(\{0\} \times (0, +\infty)^{d(m)}\right).$$

As a result:

$$\Gamma_{\mathcal{O}_{\varphi,k,m}} \in \Sigma \otimes \mathcal{B}(X^k).$$

Thus, if Σ is a Suslin family and X is a Suslin space, then, by Lemma 2.2, $\mathcal{O}_{\varphi,k,m}$ is measurable, and its domain $\mathcal{O}^{-}_{\varphi,k,m}(\mathbb{R}^k) \in \Sigma$. In particular, $\Delta_{\varphi,k} = \mathcal{O}^{-}_{\varphi,k,k}(\mathbb{R}^k) \in \Sigma$.

Now, we can state the crucial definition of a random orbit.

Definition 2.11. Let $\varphi\colon \Omega \times X \multimap X$ be a random operator. A sequence of measurable functions $(\xi_i)_{i=0}^{k-1}$, where $\xi_i\colon \Omega \to X$, for $i = 0, \dots, k-1$, is called a *random k-orbit* of the operator φ if

(a) $\Omega \setminus \{\omega \in \Omega\colon \forall_{i<k-1}\, \xi_{i+1}(\omega) \in \varphi(\omega, \xi_i(\omega)) \wedge \xi_0(\omega) \in \varphi(\omega, \xi_{k-1}(\omega))\} \in \mathcal{I}$,

(b) there is no $m < k$ such that $m|k$ and

$$\Omega \setminus \{\omega \in \Omega\colon \forall_{s<\frac{k}{m}} \forall_{i<m}\, \xi_{sm+i}(\omega) = \xi_i(\omega)\} \in \mathcal{I}.$$

Remark 2.12. If we assume that (Ω, Σ, μ) is a complete measure space, $\mathcal{I} = \mathcal{N}(\Omega)$, and (X, d) is a Polish space, then the definition of a random orbit coincides with the definition given in [1], i.e.:

Let $\varphi\colon \Omega \times X \multimap X$ be a random operator. A sequence of measurable functions $(\xi_i)_{i=0}^{k-1}$, where $\xi_i\colon \Omega \to X$, for $i = 0, \dots, k-1$, is called a *random k-orbit* of the operator φ if

(a) $\xi_{i+1}(\omega) \in \varphi(\omega, \xi_i(\omega))$, for $i = 0, \dots, k-2$ and $\xi_0(\omega) \in \varphi(\omega, \xi_{k-1}(\omega))$, for almost all $\omega \in \Omega$,

(b) the sequence $(\xi_i)_{i=0}^{k-1}$ is not formed by going p-times around a shorter subsequence of m consecutive elements (i.e. it is not a concatenation), where $mp = k$ (for almost all $\omega \in \Omega$).

Equivalently:

(a) $\mu\left(\Omega \setminus \{\omega \in \Omega\colon \forall_{i<k-1}\, \xi_{i+1}(\omega) \in \varphi(\omega, \xi_i(\omega)) \wedge \xi_0(\omega) \in \varphi(\omega, \xi_{k-1}(\omega))\}\right) = 0$,

(b) there is no $m < k$ such that $m|k$ and

$$\mu\Big(\Omega \setminus \{\omega \in \Omega\colon \forall_{s<\frac{k}{m}} \forall_{i<m}\, \xi_{sm+i}(\omega) = \xi_i(\omega)\}\Big) = 0.$$

3. Characterization of operators with random orbits

Lemma 3.1. *Assume that X is separable. Let $\varphi\colon \Omega \times X \multimap X$ be a random operator. If φ has a random k-orbit, then there exists a partition of Ω such that*

$$\Omega = \Omega_0 \cup \bigcup_{j=0}^{l-1} \Omega_{i_j},$$

where:

- $\Omega_m \in \Sigma$ *for* $m = 0, i_0, i_1, \dots, i_{l-1}$,
- $\Omega_0 \in \mathcal{I}$ *and* $\Omega_{i_j} \notin \mathcal{I}$ *for* $j = 0, \dots, l-1$,
- $\mathrm{LCM}\{i_j\colon j = 0, \dots, l-1\} = k$,
- φ_ω *has an i_j-orbit for each* $\omega \in \Omega_{i_j}$ *for* $j < l$.

Proof. Let:

$$\Omega_m = \{\omega \in \Omega\colon (\xi_i(\omega))_{i=0}^{k-1} \in \mathcal{O}_{\varphi,k,m}(\omega)\},$$

for any $m|k$. Let $\{i_j\colon j < l\} = \{m|k\colon \Omega_m \notin \mathcal{I}\}$, and:

$$\Omega_0 = \Omega \setminus \bigcup_{j=0}^{l-1} \Omega_{i_j}.$$

The sets Ω_m for $m = 0, i_0, i_1, \dots, i_{l-1}$ are disjoint.

For all $m = i_0, i_1, \dots, i_{l-1}$, we have:

$$\Omega_m = \left\{ \omega \in \Omega \colon d_{\varphi,k,m}\left(\omega, (\xi_i(\omega))_{i=0}^{k-1}\right) = 0 \wedge \forall_{\substack{p<m\\ p|m}} d_{\varphi,k,p}\left(\omega, (\xi_i(\omega))_{i=0}^{k-1}\right) > 0 \right\}.$$

Let $\xi' = \left(id_\Omega, (\xi_i)_{i=0}^{k-1}\right)$. Then:

$$\Omega_m = (D_{\varphi,k,m} \circ \xi')^{-1} \left(\{0\} \times (0,+\infty)^{d(m)} \right).$$

The function $D_{\varphi,k,m}$ is measurable with respect to σ-algebra $\Sigma \otimes \mathcal{B}(X^k)$. Hence, $D^{-1}_{\varphi,k,m}(\{0\} \times (0,+\infty)^{d(m)}) \in \Sigma \otimes \mathcal{B}(X^k)$. Since X is separable, the σ-algebra $\Sigma \otimes \mathcal{B}(X^k)$ is generated by the sets $M \times \prod_{i=0}^{k-1} B_i$, where $M \in \Sigma$ and $B_i \in \mathcal{B}(X)$ for $i < k$. Fix such a set $M \times \prod_{i=0}^{k-1} B_i$. Then:

$$\xi'^{-1}\left(M \times \prod_{i=0}^{k-1} B_i \right) = M \cap \bigcap_{i=0}^{k-1} \xi_i^{-1}(B_i) \in \Sigma,$$

because functions ξ_i, for $i < k$, are measurable. Thus, $\xi'^{-1}(B) \in \Sigma$, for any $B \in \Sigma \otimes \mathcal{B}(X^k)$, and subsequently:

$$\Omega_m = \xi'^{-1}\left(D^{-1}_{\varphi,k,m}(\{0\} \times (0,+\infty)^{d(m)}) \right) \in \Sigma,$$

for any $m = i_0, i_1, \dots, i_{l-1}$. Therefore, $\Omega_0 \in \Sigma$. Clearly, $\Omega_{i_j} \notin \mathcal{I}$, for any $j < l$. By the condition (a), we have $\Omega \setminus \bigcup_{m|k} \Omega_m \in \mathcal{I}$, and since

$$\Omega_0 = \left(\Omega \setminus \bigcup_{m|k} \Omega_m \right) \cup \bigcup \{\Omega_m \colon \Omega_m \in \mathcal{I} \wedge m|k\},$$

we get $\Omega_0 \in \mathcal{I}$.

Suppose that $k' = \mathrm{LCM}\{i_j \colon j < l\} < k$. Obviously, $k'|k$. Then $\xi_{sm+i}(\omega) = \xi_i(\omega)$, for any $s < \frac{k}{k'}$ and $i < k'$, and $\omega \in \Omega \setminus \Omega_0$, which contradicts the condition (b). Hence, $\mathrm{LCM}\{i_j \colon j < l\} = k$. $\square$

Proposition 3.2. *Assume that Σ is a Suslin family and X is a Suslin space. Let $\varphi\colon \Omega \times X \multimap X$ be a random operator. Then φ has a random k-orbit iff there exists a partition of Ω such that*

$$\Omega = \Omega_0 \cup \bigcup_{j=0}^{l-1} \Omega_{i_j},$$

where:

- $\Omega_m \in \Sigma$ *for* $m = 0, i_0, i_1, \dots, i_{l-1}$,
- $\Omega_0 \in \mathcal{I}$ *and* $\Omega_{i_j} \notin \mathcal{I}$ *for* $j = 0, \dots, l-1$,
- $\mathrm{LCM}\{i_j \colon j = 0, \dots, l-1\} = k$,
- φ_ω *has an* i_j*-orbit for each* $\omega \in \Omega_{i_j}$ *for* $j < l$.

Proof. By Lemma 3.1, it is enough to show the implication to the left. We define the multivalued map $\mathcal{O}_{\varphi,k}\colon \Omega \to X^k$ as follows:

$$\mathcal{O}_{\varphi,k}(\omega) = \begin{cases} X^k & \text{for } \omega \in \Omega_0, \\ \mathcal{O}_{\varphi,k,i_j}(\omega) & \text{for } \omega \in \Omega_{i_j} \text{ for } j < l \text{ and } t \in \mathbb{N}. \end{cases}$$

By Remark 2.10, we get $\Gamma_{\mathcal{O}_{\varphi,k}} \in \Sigma \otimes \mathcal{B}(X^k)$. By Lemma 2.1, we receive a measurable function $\xi\colon \Omega \to X^k$ such that $\xi \subset \mathcal{O}_{\varphi,k}$.

The sequence $(\xi_i)_{i=0}^{k-1}$ is a random k-orbit of φ. To see this, first observe that each ξ_i is measurable, because ξ is measurable and $\mathcal{B}(X)^k = \mathcal{B}(X^k)$. For each $\omega \in \Omega_{i_j}$, where $j < l$, we have $\xi_{i+1}(\omega) \in \varphi(\omega, \xi_i(\omega))$, for $i < k-1$, and $\xi_0(\omega) \in \varphi(\omega, \xi_{k-1}(\omega))$. Thus, the condition (a) is fulfilled. Suppose that there is $m < k$ such that $m|k$ and

$$\Omega \setminus \{\omega \in \Omega\colon \forall_{s<\frac{k}{m}} \forall_{i<m}\, \xi_{sm+i}(\omega) = \xi_i(\omega)\} \in \mathcal{I}.$$

Fix any $j < l$. Then $\Omega_{i_j} \cap \{\omega \in \Omega\colon \forall_{s<\frac{k}{m}} \forall_{i<m}\, \xi_{sm+i}(\omega) = \xi_i(\omega)\} \neq \emptyset$, because $\Omega_{i_j} \notin \mathcal{I}$. Thus, $i_j|m$, and $\mathrm{LCM}\{i_j : j < l\} \leq m < k$, a contradiction. Consequently, the condition (b) is fulfilled. □

Remark 3.3. The above proposition is a generalization of Proposition 2 in [1].

4. Randomized Sharkovsky-type results

Now we can state two randomized versions of the Sharkovsky Theorem, whose proofs are based on the above Proposition 3.2.

Theorem 4.1. *Assume that Σ is a Suslin family, $\mathcal{I}$ is a proper σ-ideal, and $\varphi\colon \Omega \times \mathbb{R} \multimap \mathbb{R}$ is an l.s.c. random operator with compact and connected values. If φ has a random n-orbit, then it has a random k-orbit, for each $k \lhd n$.*

Proof. Suppose that φ has a random n-orbit. There is a splitting of Ω as in Proposition 3.2. Fix $k \lhd n$. By Lemma 2.3, there is $j' < l$ such that $k \lhd i_{j'}$. Moreover, $1 \lhd i_j$ for each $j < l$ and $j \neq j'$. Since φ_ω has an i_j-orbit, for each $\omega \in \Omega_{i_j}$ and every $j < l$, by Proposition 2.4, φ_ω has a k-orbit, for each $\omega \in \Omega_{i_{j'}}$ and a 1-orbit, for each $\omega \in \Omega_{i_j}$ and every $j < l$ and $j \neq j'$. Put $\Omega'_k = \Omega_{i_{j'}}$, $\Omega'_1 = \bigcup\{\Omega_{i_j} : j \leq l \wedge j \neq j'\}$, and $\Omega'_0 = \Omega_0$. As a consequence, we get the following splitting of Ω: $\Omega = \Omega'_0 \cup \Omega'_k$ if $l = 1$, or $\Omega = \Omega'_0 \cup \Omega'_k \cup \Omega'_1$ if $l > 1$. By Proposition 3.2, φ has a random k-orbit. □

As a special single-valued case of Theorem 4.1, we obtain the following randomization of the classical Sharkovsky theorem in [20].

Corollary 4.2. *Assume that Σ is a Suslin family, $\mathcal{I}$ is a proper σ-ideal, and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous random operator. If f has a random n-orbit, then it has a random k-orbit, for each $k \lhd n$.*

Theorem 4.3. *Assume that Σ is a Suslin family which is nonatomic with respect to a proper σ-ideal $\mathcal{I}$, and $\varphi\colon \Omega \times \mathbb{R} \multimap \mathbb{R}$ is a u.s.c. random operator with compact and connected values. Suppose that φ has a random n-orbit ($n = 2^m q$, where q is odd), and n is the maximal number in the Sharkovsky ordering with that property.*

(1) *If $q > 3$, then φ has a random k-orbit for every $k \lhd n$, except possibly for $k = 2^{m+2}$.*

(2) *If $q = 3$, then at least one of the following two cases occurs:*
  (i) *φ has a random k-orbit for every $k \lhd n$, except possibly for $k = 2^{m+2}$,*
  (ii) *φ has a random k-orbit for every $k \lhd n$, except possibly for $k = 2^{m+1}$.*

(3) *If $q = 1$, then φ has a k-orbit for every $k \lhd n$.*

Proof. There is a splitting of Ω as in Proposition 3.2.

Recall that $\Delta_{\varphi,k} \in \Sigma$, for any k.

First, fix $k \rhd n$, $k \neq n$. Suppose that $\Delta_{\varphi,k} \notin \mathcal{I}$. Then we can take $\Omega'_0 = \Omega_0$, $\Omega'_k = \Delta_{\varphi,k} \setminus \Omega_0$, and $\Omega'_1 = \Omega \setminus (\Omega'_k \cup \Omega'_0)$, and by Proposition 3.2, we get a random k-orbit for φ which contradicts the maximality of n. Thus, $\Delta_{\varphi,k} \in \mathcal{I}$.

Let $\Omega''_0 = \bigcup_{k \rhd n} \Delta_{\varphi,k} \cup \Omega_0$. Then $\Omega''_0 \in \Sigma$ and $\Omega''_0 \in \mathcal{I}$. Moreover, φ_ω has no k-orbit with $k \rhd n$ and $k \neq n$, for every $\omega \in \Omega \setminus \Omega''_0$. Fix $j = 0, \ldots, l-1$. Since $\Omega_{i_j} \setminus \Omega''_0 \neq \emptyset$, $i_j \lhd n$. Furthermore, $i_j | n$. Hence, $i_j = n$ or $i_j = 2^{m_j}$ (where $m_j \leq m$), and there is $j' < l$ such that $i_{j'} = n$. Let $\Omega''_n = \Omega_n \setminus \Omega''_0$. Thus, $\Omega''_n \in \Sigma$, $\Omega''_n \notin \mathcal{I}$, and for every $\omega \in \Omega''_n$, φ_ω has n-orbit, and n is the maximal number in the Sharkovsky ordering with that property.

Suppose that $q > 3$. Fix $k \lhd n$, $k \neq 2^{m+2}$. By Proposition 2.5, for every $\omega \in \Omega_n$, φ_ω has a k-orbit. Now, take $\Omega'_k = \Omega''_n$, $\Omega'_0 = \Omega''_0$, $\Omega'_1 = \Omega \setminus (\Omega'_k \cup \Omega'_0)$. Then by Proposition 3.2, φ has a random k-orbit.

Suppose that $q = 3$. For $k \lhd n$, $k \neq 2^{m+1} \cdot 3$, $k \neq 2^{m+2}$, $k \neq 2^{m+1}$, we proceed analogously to the previous paragraph. By Proposition 2.5, $\Omega''_n \subset \Delta_{\varphi,2^{m+1}} \cup \big(\Delta_{\varphi,2^{m+1}\cdot 3} \cap \Delta_{\varphi,2^{m+2}}\big)$. Then $\Delta_{\varphi,2^{m+1}} \cap \Omega''_n \notin \mathcal{I}$ or $\big(\Delta_{\varphi,2^{m+1}\cdot 3} \cap \Delta_{\varphi,2^{m+2}}\big) \cap \Omega''_n \notin \mathcal{I}$.

Suppose that $\Delta_{\varphi,2^{m+1}} \cap \Omega''_n \notin \mathcal{I}$. Let $k = 2^{m+1}$. Now, take $\Omega'_0 = \Omega''_0$, $\Omega'_k = \big(\Delta_{\varphi,2^{m+1}} \cap \Omega''_n\big) \setminus \Omega'_0$, $\Omega'_1 = \Omega \setminus (\Omega'_k \cup \Omega'_0)$. Then, by Proposition 3.2, φ has a random k-orbit. Let $k = 2^{m+1} \cdot 3$. For every $\omega \in \Omega''_n$, φ_ω has an n-orbit, i.e., $2^m \cdot 3$ -orbit. There is a splitting $\Delta_{2^{m+1}} \cap \Omega''_n = \Omega' \cup \Omega''$ such that $\Omega', \Omega'' \in \Sigma$, $\Omega', \Omega'' \notin \mathcal{I}$. Take $\Omega'_0 = \Omega_0$, $\Omega'_{2^{m+1}} = \Omega'$, $\Omega'_{2^m \cdot 3} = \Omega''$, and $\Omega'_1 = \Omega \setminus (\Omega'_{2^{m+1}} \cup \Omega'_{2^m \cdot 3} \cup \Omega'_0)$. Since $\mathrm{LCM}\{2^{m+1}, 2^m \cdot 3, 1\} = 2^{m+1} \cdot 3 = k$, by Proposition 3.2, φ has a random k-orbit.

Suppose that $\big(\Delta_{\varphi,2^{m+1}\cdot 3} \cap \Delta_{\varphi,2^{m+2}}\big) \cap \Omega''_n \notin \mathcal{I}$. Then $\Delta_{\varphi,2^{m+1}\cdot 3} \cap \Omega''_n \notin \mathcal{I}$, and $\Delta_{\varphi,2^{m+2}} \cap \Omega''_n \notin \mathcal{I}$. For $k = 2^{m+1} \cdot 3$ and $k = 2^{m+2}$, we proceed quite analogously as for $k = 2^{m+1}$ in the previous paragraph.

The case when $q = 1$ is analogous to the case, when $q > 3$. □

Corollary 4.4. *Assume that Σ is a Suslin family which is nonatomic with respect to a proper σ-ideal $\mathcal{I}$, and $\varphi\colon \Omega \times \mathbb{R} \multimap \mathbb{R}$ is a u.s.c. random operator with compact and connected values. If φ has a random n-orbit, then it has a random k-orbit for each $k \lhd n$ with at most one exception.*

Remark 4.5. Observe that in the case (2)(i), unlike in the deterministic Proposition 2.5, we have a larger area for manipulation in the randomized Theorem 4.3. We can namely divide Ω into two parts (by nonatomicity) and take the period 2^{m+1} on one side and $2^m \cdot 3$ on the other one. Thus, the period $2^{m+1} \cdot 3$ occurs by their combination via Proposition 3.2, and subsequently $2^{m+1} \cdot 3$ is no longer an exception. In particular, the maximal number of exceptional cases reduces to one, as stated in Corollary 4.4.

5. Subharmonics of random differential inclusions

Assume that (Ω, Σ, μ) is a complete measure space and $\mathcal{I}$ is a proper σ-ideal on Ω. On $\mathbb{R}$ and $[0,1]$, we use the σ-algebra of Lebesgue measurable sets and the σ-ideal of null sets. Assume that $\varphi\colon \Omega \times [0,1] \times \mathbb{R} \multimap \mathbb{R}$ is a *random u-Carathéodory map*, i.e.:

- $\varphi(\cdot,\cdot,x)\colon \Omega \times [0,1] \multimap \mathbb{R}$ is measurable, for all $x \in \mathbb{R}$,
- $\varphi(\omega,t,\cdot)\colon \mathbb{R} \multimap \mathbb{R}$ is u.s.c., for almost all $(\omega,t) \in \Omega \times [0,1]$,
- there exists $a, b > 0$ such that $\sup\{|y|\colon y \in \varphi(\omega,t,x)\} \leq a + b|x|$, for almost all $(\omega,t) \in \Omega \times [0,1]$ and all $x \in \mathbb{R}$.

We extend φ to $\Omega \times \mathbb{R} \times \mathbb{R}$ in the following manner: $\varphi(\omega, t+k, x) = \varphi(\omega, t, x)$, for $\omega \in \Omega$, $t \in [0,1]$, $k \in \mathbb{Z}$, and $x \in \mathbb{R}$. Let $\varphi_\omega(t,x) = \varphi(\omega, t, x)$. We shall consider the random differential inclusion:

$$(I_\varphi) \qquad x'(\omega,t) \in \varphi(\omega,t,x(\omega,t)) \quad [\equiv \varphi(\omega,t+1,x(\omega,t))]$$

and a one-parameter family of deterministic differential inclusions:

$$(I_{\varphi_\omega}) \qquad x'(t) \in \varphi_\omega(t,x(t)) \quad [\equiv \varphi_\omega(t+1,x(t))]\,.$$

We say that $x\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a *random solution* of (I_φ) if $x(\omega,\cdot)$ is absolutely continuous, for almost all $\omega \in \Omega$, $x(\cdot,t)$ is measurable, for each $t \in \mathbb{R}$, and the condition (I_φ) is fulfilled (where the differentiation is over t), for almost all $(\omega,t) \in \Omega \times \mathbb{R}$. For $k \in \mathbb{N}$, we say that a solution is a *random k-periodic subharmonic solution* if $x(\omega,t) = x(\omega,t+k)$, for almost all $(\omega,t) \in \Omega \times \mathbb{R}$ and there is no $m \in \mathbb{N}$, $m < k$ such that $x(\omega,t) = x(\omega,t+m)$, for almost all $(\omega,t) \in \Omega \times \mathbb{R}$.

We say that $x\colon \mathbb{R} \to \mathbb{R}$ is a *solution* of (I_{φ_ω}) if, for a fixed $\omega \in \Omega$, it is absolutely continuous and the condition (I_{φ_ω}) is fulfilled, for almost all $t \in \mathbb{R}$. For $k \in \mathbb{N}$, we say that a solution is a *k-periodic subharmonic solution* if $x(t) = x(t+k)$, for almost all $t \in \mathbb{R}$, and there is no $m \in \mathbb{N}$, $m < k$ such that $x(t) = x(t+m)$, for almost all $t \in \mathbb{R}$. For $k = 1$, we also speak about (*random*) *harmonic* solutions. Thus, for $k > 1$, we can speak about *pure* or *nontrivial* (*random*) *subharmonics.*

We define the (*random*, cf. [6, Proposition 4.3]) *Poincaré operator* $P_k\colon \Omega \times \mathbb{R} \times \mathbb{R} \multimap \mathbb{R}$ associated with (I_φ) as follows:

$$P_k(\omega,t_0,x_0) = \{x(t_0+k)\colon x \text{ is a solution of } (I_{\varphi_\omega}), x(t_0) = x_0\}.$$

It is known that $P_k = P_1^k$. For more details and properties of the random Poincaré operators, see [5, Chapter III,4,E] and [6]. We shall also use the following notation: $P_{k,\omega,t_0}(x_0) = P_k(\omega,t_0,x_0)$ and $P_{k,t_0}(\omega,x_0) = P_k(\omega,t_0,x_0)$.

We can now draw a diagram of implications between certain sentences. In that diagram, we assume that $n,k > 1$. By "$\exists \bigcup_i^m \Omega_i = \Omega\ \forall_\omega \Phi$", we mean that there exists a partition of Ω such that

$$\Omega = \Omega_0 \cup \bigcup_{j=0}^{l-1} \Omega_{i_j},$$

where:

- $\Omega_m \in \Sigma$ for $m = 0, i_0, i_1, \ldots, i_{l-1}$,
- $\mu(\Omega_0) = 0$ and $\mu(\Omega_{i_j}) > 0$ for $j = 0, \ldots, l-1$,
- $\mathrm{LCM}\{i_j\colon j = 0, \ldots, l-1\} = m$,
- for $i = i_j$ and for all $\omega \in \Omega_i$, Φ.

The implications in the diagram are proved or cited below. Theorem 2 from [2] will be stated here in the form of the following proposition.

Proposition 5.1. *Let $\varphi\colon \mathbb{R} \multimap \mathbb{R}$ be a multivalued mapping with nonempty connected values whose margins, i.e., $\varphi^*(x) = \sup\{y\colon y \in \varphi(x)\}$ and $\varphi_*(x) = \inf\{y\colon y \in \varphi(x)\}$, are nondecreasing. If φ has an n-orbit with $n > 1$, then it also has a k-orbit, for every $k \in \mathbb{N}$.*

Remark 5.2. We know (see [2] and [4]) that the Poincaré operators associated with (I_{φ_ω}) are u.s.c. maps with compact, connected values and their margins are nondecreasing. In particular, they satisfy the assumptions of Proposition 5.1.

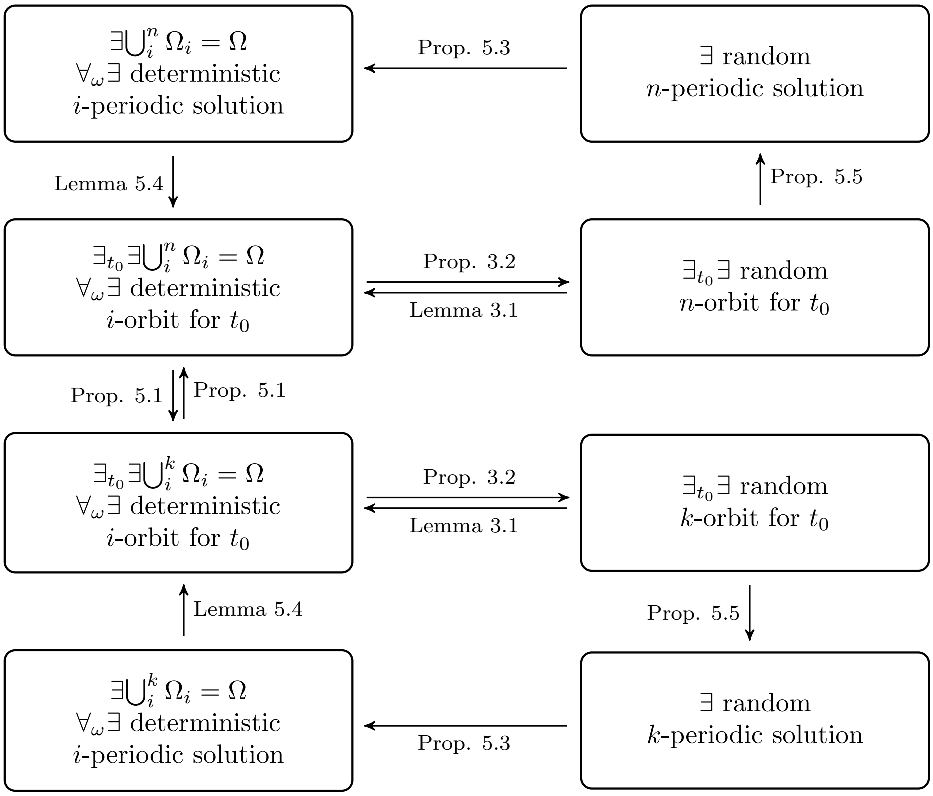

FIGURE 1. Diagram of implications among various sentences about the given inclusion (I_φ)

Proposition 5.3. *If the inclusion (I_φ) has a random n-periodic subharmonic solution (where $n \in \mathbb{N}$, $n > 1$), then there is a partition of Ω such that*

$$\Omega = \Omega_0 \cup \bigcup_{j=0}^{l-1} \Omega_{i_j},$$

where:

- $\Omega_m \in \Sigma$ *for* $m = 0, i_0, i_1, \ldots, i_{l-1}$,
- $\mu(\Omega_0) = 0$ *and* $\mu(\Omega_{i_j}) > 0$ *for* $j < l$,
- $\mathrm{LCM}\{i_j : j < l\} = n$,
- (I_{φ_ω}) *has an* i_j*-periodic solution for each* $\omega \in \Omega_{i_j}$, $j < l$.

Proof. Let x be a random n-periodic subharmonic solution of (I_φ). Let

$$\Omega_i = \{\omega \in \Omega \colon x_\omega \text{ is } i\text{-periodic subharmonic solution of } (I_{\varphi_\omega})\},$$

for $i|n$, $i \in \mathbb{N}$. Then

$$\begin{aligned}\Omega_i &= \{\omega \in \Omega\colon \forall_t x(\omega,t) = x(\omega,t+i)\} \setminus \bigcup_{\substack{j|n\\ j<i}} \{\omega \in \Omega\colon \forall_t x(\omega,t) = x(\omega,t+j)\}\\ &= \{\omega \in \Omega\colon \forall_{q\in\mathbb{Q}} x(\omega,q) = x(\omega,q+i)\} \setminus \bigcup_{\substack{j|n\\ j<i}} \{\omega \in \Omega\colon \forall_{q\in\mathbb{Q}} x(\omega,q) = x(\omega,q+j)\}\\ &= \bigcap_{q\in\mathbb{Q}} \{\omega \in \Omega\colon x(\omega,q) = x(\omega,q+i)\} \setminus \bigcup_{\substack{j|n\\ j<i}} \bigcap_{q\in\mathbb{Q}} \{\omega \in \Omega\colon x(\omega,q) = x(\omega,q+j)\}.\end{aligned}$$

Hence, $\Omega_i \in \Sigma$, for any $i|n$, $i \in \mathbb{N}$. Furthermore, $\mu(\Omega \setminus \bigcup_{i|n} \Omega_i) = 0$. Let $\{i_j\colon j < l\} = \{i|n\colon \mu(\Omega_i) > 0\}$ and $\Omega_0 = \Omega \setminus \bigcup_{j<l} \Omega_{i_j}$. Clearly, $\Omega_0 \in \Sigma$ and $\mu(\Omega_0) = 0$. At the same time, $\mathrm{LCM}\{i_j\colon j < l\} = n$, because otherwise $\mathrm{LCM}\{i_j\colon j < l\} = k < n$, and for almost all $\omega \in \Omega$, x_ω has some period $i|k$, by which x has some period $i|k$, a contradiction. □

Lemma 5.4. *If there exists a partition of Ω such that*

$$\Omega = \Omega_0 \cup \bigcup_{j=0}^{l-1} \Omega_{i_j},$$

where:

- $\Omega_m \in \Sigma$ *for* $m = 0, i_0, i_1, \ldots, i_{l-1}$,
- $\mu(\Omega_0) = 0$ *and* $\mu(\Omega_{i_j}) > 0$ *for* $j < l$,
- $\mathrm{LCM}\{i_j\colon j < l\} = n$,
- (I_{φ_ω}) *has an i_j-periodic solution for each* $\omega \in \Omega_{i_j}$, $j < l$,

then there is $t_0 \in [0,n)$ and a partition of Ω such that

$$\Omega = \Omega'_0 \cup \Omega'_1 \cup \Omega'_n,$$

where:

- $\Omega'_m \in \Sigma$ *for* $m = 0, 1, n$,
- $\mu(\Omega'_0) = 0$ *and* $\mu(\Omega'_n) > 0$,
- P_{1,ω,t_0} *has an n-orbit for each* $\omega \in \Omega'_n$.

Proof. Since $n > 1$, there is $j_0 < l$ such that $i_{j_0} > 1$. Fix $\omega \in \Omega_{j_{i_0}}$. (I_{φ_ω}) has a j_{i_0}-periodic solution $x_\omega\colon \mathbb{R} \to \mathbb{R}$. The set $A_\omega = \{t\colon x_\omega(t) \neq x_\omega(t+1)\}$ is open and nonempty, because otherwise $n = 1$. Take $q_\omega \in \mathbb{Q} \cap A_\omega$. Then $(x_\omega(q_\omega + s))_{s=0}^{i_0-1}$ is an orbit of P_{1,ω,q_ω} or a concatenation of identical orbits of period larger than 1. Thus, P_{1,ω,q_ω} has an l-orbit for some $l > 1$. Let

$$\Omega''_q = \{\omega \in \Omega\colon P_{1,\omega,q} \text{ has an } l\text{-orbit for some } l > 1\},$$

for $q \in \mathbb{Q} \cap [0,n)$.

$$\Omega''_q = \bigcup_{p=2}^{\infty} \mathcal{O}^-_{P_{1,q},p,p}(\mathbb{R}^n).$$

The relation $\mathcal{O}_{P_{1,q},p,p}$ is measurable. Hence, $\Omega''_q \in \Sigma$. Since $\bigcup_{q\in\mathbb{Q}\cap[0,n)} \Omega''_q \supset \Omega_{i_{j'}}$, there is $q_0 \in \mathbb{Q}\cap[0,n)$ such that $\mu(\Omega''_{q_0}) > 0$. Now, take $t_0 = q_0$, $\Omega'_0 = \Omega_0$, $\Omega'_n = \Omega''_{q_0}$, $\Omega'_1 = \Omega \setminus (\Omega'_0 \cup \Omega'_n)$. Then P_{1,ω,t_0} has an n-orbit, for each $\omega \in \Omega'_n$, by Proposition 5.1. □

In Proposition 5.6 in [4], we can find the proof of the following fact.

Proposition 5.5. *(I_φ) has a random n-periodic subharmonic solution, provided P_{1,t_0} has a random n-orbit, for some $t_0 \in [0,1]$.*

As a result of commutativity of the diagram, we can state the following main theorem of this section.

Theorem 5.6. *If (I_φ) has a random n-periodic subharmonic solution, for some $n > 1$, then it has random k-periodic subharmonic solutions, for all $k \in \mathbb{N}$.*

Remark 5.7. Although the diagram in Figure 1 does not explicitly contain all implications indicated there by the arrows, it is completely commutative. In other words, the missing arrows can be completed.

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