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GENERALIZED TORSION ELEMENTS IN THE KNOT GROUPS OF TWIST KNOTS

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Abstract. It is well known that any knot group is torsion-free, but it may admit a generalized torsion element. We show that the knot group of any negative twist knot admits a generalized torsion element. This is a generalization of the same claim for the knot 5_2, which is the (-2)-twist knot, by Naylor and Rolfsen.

1. Introduction

For a knot K in the 3-sphere S^3, the *knot group* of K is the fundamental group of the complement $S^3 - K$. It is a classical fact that any knot group is torsion-free. However, many knot groups can have generalized torsion elements. For a group, a non-trivial element is called a *generalized torsion element* if some non-empty finite product of its conjugates is the identity. Typical examples are the knot groups of torus knots (see [6]). It was open whether the knot group of a hyperbolic knot admits a generalized torsion element or not. However, Naylor and Rolfsen [6] first found such an element in the knot group of the hyperbolic knot 5_2. As they wrote, the element was found with the help of a computer, so a (topological) meaning of the element is not obvious.

The knot 5_2 is the (-2)-twist knot. See below for the convention of twist knots. In this article, we completely determine the existence of generalized torsion elements in the knot groups of twist knots as a generalization of Naylor-Rolfsen's result. Figure 1 shows the m-twist knot, where the rectangular box contains the right-handed (resp. left-handed) horizontal m-full twists if $m > 0$ (resp. $m < 0$). By this convention, the 1-twist knot is the figure-eight knot, and the (-1)-twist knot is the right-handed trefoil as shown in Figure 1. We may call the m-twist knot a positive or negative twist knot, according to the sign of m.

There is a well-known relation between the bi-orderability and the existence of generalized torsion elements in a group. We recall that a group is said to be *bi-orderable* if it admits a strict total ordering, which is invariant under multiplication on both sides. Then any bi-orderable group has no generalized torsion elements ([6]). The converse statement is not true, in general ([5]).

According to [2], the knot group of any positive twist knot is bi-orderable, but that of any negative twist knot is not bi-orderable. Therefore, the former does not

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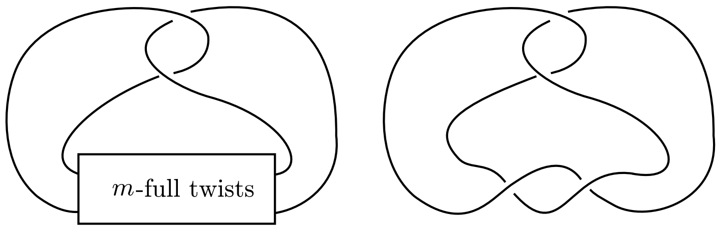

FIGURE 1. The m-twist knot and the (-1)-twist knot

admit a generalized torsion element, but we can expect that the latter would admit it. In fact, Naylor and Rolfsen [6] confirmed this claim for the (-2)-twist knot. The main result of the present article is to verify this expectation.

Theorem 1.1. *The knot group of any negative twist knot admits a generalized torsion element.*

Unfortunately, a topological meaning of the generalized torsion elements found here is unknown.

Throughout the paper, we use the convention $x^g = g^{-1}xg$ for the conjugate of x by g and $[x, y] = x^{-1}y^{-1}xy$ for the commutator.

2. Presentations of knot groups

For $n \geq 1$, let K be the $(-n)$-twist knot. We prepare a certain presentation of the knot group of K by using its Seifert surface. Figure 2 indicates an isotopy of the twist knot from the standard position as in Figure 1 into another which is helpful to identify a Seifert surface. Then Figure 3 shows a Seifert surface S of K, where $n = 2$. The fundamental group $\pi_1(S)$ is a free group of rank two, generated by x and y as illustrated there. Since the complement of the regular neighborhood $N(S)$ of S in S^3 is a genus two handlebody, $\pi_1(S^3 - N(S))$ is also a free group of rank two, generated by a and b.

Theorem 2.1. *The knot group G of the $(-n)$-twist knot K has a presentation*

$$\langle a, b, t \mid tat^{-1} = b^{-1}a, t(b^n a^{-1})t^{-1} = b^n \rangle,$$

where t is a meridian.

Proof. If we push x and y off from the Seifert surface to the front side, then we obtain $x^+ = a$ and $y^+ = b^n a^{-1}$. Similarly, if we push them off to the back side, then $x^- = b^{-1}a$ and $y^- = b^n$. As shown in [4, Lemma 2.1] (it is just an application of the van Kampen theorem), the knot group G is generated by a, b and a meridian t with two relations $tx^+t^{-1} = x^-$ and $ty^+t^{-1} = y^-$. □

From the second relation, we have

$$a = [t, b^n]. \tag{2.1}$$

Thus the knot group G is generated by b and t.

Remark 2.2. This presentation is distinct from the one used in [6] when $n = 2$, but it is equivalent, of course.

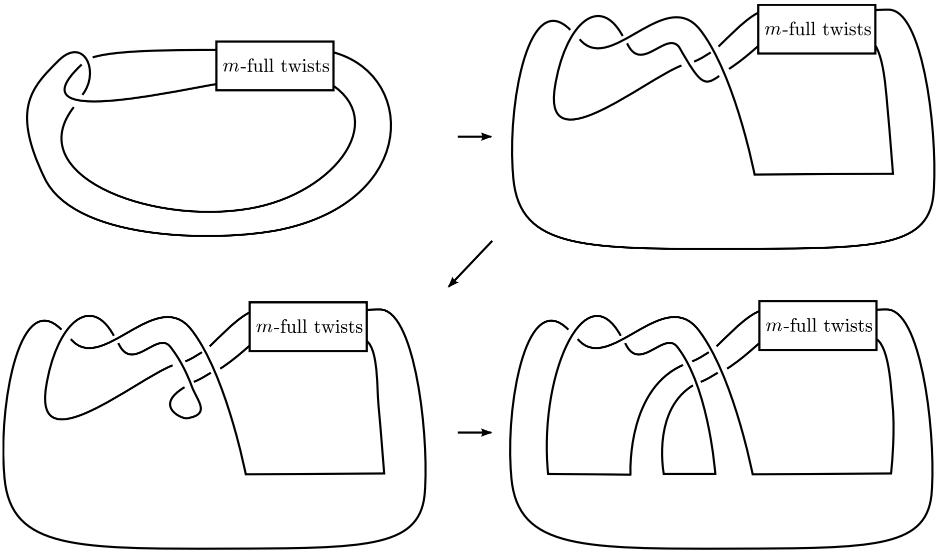

FIGURE 2. An isotopy of the m-twist knot

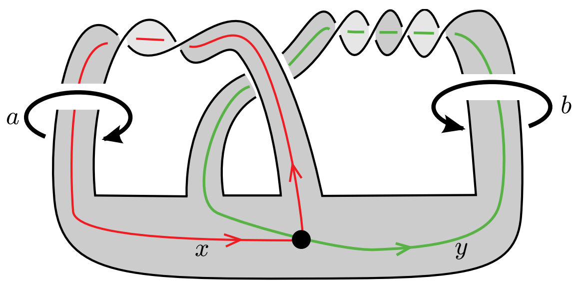

FIGURE 3. A Seifert surface ($n = 2$)

By their choices, the generators a and b belong to the commutator subgroup of G. Furthermore, we have $a = [t, b^n]$ and $b = [a^{-1}, t^{-1}]$. This implies that these elements belong to the intersection of all members of the lower central series of G.

Lemma 2.3. *Let $D = [t, b]$. Then $D \neq 1$.*

Proof. If $D = 1$, then the knot group G would be abelian, because G is generated by t and b. Since the knot is non-trivial, this is impossible. □

Lemma 2.4. $a = DD^b \cdots D^{b^{n-1}}$.

Proof. In general, $[x, y^m] = [x, y][x, y^{m-1}]^y$. The conclusion immediately follows from this and (2.1). □

3. Proof of Theorem 1.1

In this section, we prove that $D = [t, b]$ is a generalized torsion element of G. Let $\langle D \rangle$ be the semigroup consisting of all non-empty products of conjugates of D in G. That is,

$$\langle D \rangle = \{D^{g_1} D^{g_2} \cdots D^{g_m} \mid g_1, g_2, \ldots, g_m \in G, m \geq 1\}.$$

If the semigroup $\langle D \rangle$ is shown to contain the identity, then we can conclude that D is a generalized torsion element.

From the second relation of the presentation in Theorem 2.1, $(tb^n t^{-1})(ta^{-1}t^{-1}) = b^n$. Then $tb^n t^{-1} = b^n(tat^{-1})$. By the first relation of the presentation, we have $tb^n t^{-1} = b^{n-1}a$. Thus, by (2.1), we have

$$tb^n t^{-1} = b^{n-1} t^{-1} b^{-n} t b^n. \tag{3.1}$$

This can also be modified to

$$b^{-(n-1)} t b^n t^{-1} = [t, b^n]. \tag{3.2}$$

Proposition 3.1. *$b^n \in \langle D \rangle$.*

Proof. We show $tb^n t^{-1} \in \langle D \rangle$, which implies $b^n \in \langle D \rangle$. By (3.1), it is sufficient to show $b^{n-1} t^{-1} b^{-n} t b^n \in \langle D \rangle$.

Conjugate $b^{n-1} t^{-1} b^{-n} t b^n$ by $t^{-(n-1)}$. By applying (3.2) repeatedly, we have

$$\begin{aligned}
t^{n-1}(b^{n-1}t^{-1}b^{-n}tb^n)t^{-(n-1)} &= t^{n-1}b^{n-1}(t^{-1}b^{-1}) \cdot b^{-(n-1)}tb^n t^{-1} \cdot t^{-(n-2)} \\
&= t^{n-1}b^{n-1}(t^{-1}b^{-1}) \cdot [t, b^n] \cdot t^{-(n-2)} \\
&= t^{n-1}b^{n-1}(t^{-1}b^{-1})^2 \cdot b^{-(n-1)}tb^n t^{-1} \cdot t^{-(n-3)} \\
&= t^{n-1}b^{n-1}(t^{-1}b^{-1})^2 \cdot [t, b^n] \cdot t^{-(n-3)} \\
&\;\;\vdots \\
&= t^{n-1}b^{n-1}(t^{-1}b^{-1})^{n-1} \cdot [t, b^n].
\end{aligned}$$

Since $[t, b^n] = a \in \langle D \rangle$ by Lemma 2.4, we can stop here when $n = 1$. Hereafter, we suppose $n \geq 2$. Then, it suffices to show $t^{n-1}b^{n-1}(t^{-1}b^{-1})^{n-1} \in \langle D \rangle$.

Claim 3.2.

$$\begin{aligned}
t^{n-1}b^{n-1}(t^{-1}b^{-1})^{n-1} = D^{(bt)^{n-2}b^{-(n-1)}t^{-(n-1)}} \\
\cdot t^{n-1}b^{n-1}(t^{-1}b^{-1})^{n-2}b^{-1}t^{-1}.
\end{aligned}$$

Proof of Claim 3.2. This follows from

$$\begin{aligned}
D^{(bt)^{n-2}b^{-(n-1)}t^{-(n-1)}} &= t^{n-1}b^{n-1}(t^{-1}b^{-1})^{n-2} D (bt)^{n-2} b^{-(n-1)} t^{-(n-1)} \\
&= t^{n-1}b^{n-1}(t^{-1}b^{-1})^{n-2} \cdot t^{-1}b^{-1}tb \cdot (bt)^{n-2} b^{-(n-1)} t^{-(n-1)} \\
&= t^{n-1}b^{n-1}(t^{-1}b^{-1})^{n-1} \cdot tb(bt)^{n-2} b^{-(n-1)} t^{-(n-1)}.
\end{aligned}$$

□

If $n = 2$, then the right-hand side is $D^{b^{-1}t^{-1}}$, so we are done.

Claim 3.3. *If $n \geq 3$, then*

$$t^{n-1}b^{n-1}(t^{-1}b^{-1})^{n-2}b^{-1}t^{-1} = [t^{-1}, b^{-(n-1)}]^{t^{-(n-2)}} \cdots [t^{-1}, b^{-3}]^{t^{-2}} [t^{-1}, b^{-2}]^{t^{-1}}.$$

Proof of Claim 3.3. We use an induction on n. When $n = 3$, the left-hand side is $t^2b^2(t^{-1}b^{-1})b^{-1}t^{-1}$, which is equal to $[t^{-1}, b^{-2}]^{t^{-1}}$.

Assume that

$$t^{n-2}b^{n-2}(t^{-1}b^{-1})^{n-3}b^{-1}t^{-1} = [t^{-1}, b^{-(n-2)}]^{t^{-(n-3)}} \cdots [t^{-1}, b^{-3}]^{t^{-2}}[t^{-1}, b^{-2}]^{t^{-1}}.$$

Multiplying the left-hand side $[t^{-1}, b^{-(n-1)}]^{t^{-(n-2)}}$ from the left gives

$$[t^{-1}, b^{-(n-1)}]^{t^{-(n-2)}} t^{n-2}b^{n-2}(t^{-1}b^{-1})^{n-3}b^{-1}t^{-1} = t^{n-1}b^{n-1}(t^{-1}b^{-1})^{n-2}b^{-1}t^{-1}.$$

□

Recall that $[t, b^m] \in \langle D\rangle$ for $m \geq 1$, as in the proof of Lemma 2.4. Also, $[t^{-1}, b^{-m}] = [t, b^m]^{t^{-1}b^{-m}} \in \langle D\rangle$. By Claims 3.2 and 3.3, $t^{n-1}b^{n-1}(t^{-1}b^{-1})^{n-1} \in \langle D\rangle$. This completes the proof. □

Remark 3.4. We remark that the proofs of Claims 3.2 and 3.3 do not use the relation (3.2). In other words, the two equations stated in these claims hold in the free group generated by t and b.

Proposition 3.5. *$b^{-n} \in \langle D\rangle$.*

Proof. The argument is similar to that of the proof of Proposition 3.1. The relation (3.1) can be changed into

$$t^{-1}b^{-n}t = b^{-(n-1)}tb^nt^{-1}b^{-n}. \tag{3.3}$$

Also, this is modified into

$$b^{n-1}t^{-1}b^{-n}t = [t^{-1}, b^{-n}]. \tag{3.4}$$

This means that the relations (3.1) and (3.2) are invariant under the transformation $t \leftrightarrow t^{-1}$, $b \leftrightarrow b^{-1}$. Thus, conjugating (3.3) by t^{n-1} yields the relation

$$t^{-n}b^{-n}t^n = t^{-(n-1)}b^{-(n-1)}(tb)^{n-1} \cdot [t^{-1}, b^{-n}]$$

by using (3.4), instead of (3.2) in the proof of Proposition 3.1.

When $n = 1$, the right-hand side is $[t^{-1}, b^{-1}] = D^{b^{-1}t^{-1}} \in \langle D\rangle$. So, assume $n \geq 2$. As above, $[t^{-1}, b^{-n}] \in \langle D\rangle$.

Claim 3.6. *$t^{-(n-1)}b^{-(n-1)}(tb)^{n-1} = D^{(b^{-1}t^{-1})^{n-1}b^{n-1}t^{n-1}} \cdot t^{-(n-1)}b^{-(n-1)}(tb)^{n-2}bt$.*

Proof of Claim 3.6. By the transformation $t \leftrightarrow t^{-1}$, $b \leftrightarrow b^{-1}$, the relation of Claim 3.2 changes into

$$t^{-(n-1)}b^{-(n-1)}(tb)^{n-1} = [t^{-1}, b^{-1}]^{(b^{-1}t^{-1})^{n-2}b^{n-1}t^{n-1}} \cdot t^{-(n-1)}b^{-(n-1)}(tb)^{n-2}bt.$$

Since $[t^{-1}, b^{-1}] = D^{b^{-1}t^{-1}}$, we have the conclusion. □

Again, if $n = 2$, then the right-hand side is $D^{b^{-1}t^{-1}bt} = D$.

Claim 3.7. *If $n \geq 3$, then*

$$t^{-(n-1)}b^{-(n-1)}(tb)^{n-2}bt = [t, b^{n-1}]^{t^{n-2}} \cdots [t, b^3]^{t^2}[t, b^2]^t.$$

Proof. This can be proved by an induction on n as in the proof of Claim 3.3. We omit this. □

Since $[t, b^m] \in \langle D\rangle$ for $m \geq 1$, we obtain $b^{-n} \in \langle D\rangle$. □

Proof of Theorem 1.1. By Propositions 3.1 and 3.5, both b^n and b^{-n} belong to the semigroup $\langle D \rangle$. Hence $\langle D \rangle$ contains the identity. Since $D \neq 1$ by Lemma 2.3, D is a generalized torsion element. □

Remark 3.8. For the case $n = 2$, Naylor and Rolfsen [6] showed that $[b^{-1}, t]$ in our notation is a generalized torsion element. This equals $D^{b^{-1}}$, which is a conjugate of our D.

4. Comments

A group without generalized torsion elements is called an R^*-group in the literature ([5]). It was an open question whether the class of bi-orderable groups coincided with the class of R^*-groups [5, p.79]. Unfortunately, the answer is known to be negative. However, there might be a possibility that two classes coincide among knot groups. Thus we expect that any non-bi-orderable knot group would admit a generalized torsion element. Many knot groups are now known to be non-bi-orderable by [1–3, 7]. It would be a future problem to examine such knot groups.

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References

[1] I. M. Chiswell, A. M. W. Glass, and John S. Wilson, *Residual nilpotence and ordering in one-relator groups and knot groups*, Math. Proc. Cambridge Philos. Soc. **158** (2015), no. 2, 275–288, DOI 10.1017/S0305004114000644. MR3310246

[2] Adam Clay, C. Desmarais, and P. Naylor, *Testing bi-orderability of knot groups*, preprint.

[3] Adam Clay and Dale Rolfsen, *Ordered groups, eigenvalues, knots, surgery and L-spaces*, Math. Proc. Cambridge Philos. Soc. **152** (2012), no. 1, 115–129, DOI 10.1017/S0305004111000557. MR2860419

[4] Xiao Song Lin, *Representations of knot groups and twisted Alexander polynomials*, Acta Math. Sin. (Engl. Ser.) **17** (2001), no. 3, 361–380, DOI 10.1007/s101140100122. MR1852950 (2003f:57018)

[5] Roberta Botto Mura and Akbar Rhemtulla, *Orderable groups*, Lecture Notes in Pure and Applied Mathematics, Vol. 27, Marcel Dekker, Inc., New York-Basel, 1977. MR0491396 (58 #10652)

[6] G. Naylor and D. Rolfsen, *Generalized torsion in knot groups*, to appear in Canad. Math. Bull.

[7] Bernard Perron and Dale Rolfsen, *On orderability of fibred knot groups*, Math. Proc. Cambridge Philos. Soc. **135** (2003), no. 1, 147–153, DOI 10.1017/S0305004103006674. MR1990838 (2004f:57015)

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