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# CONDUCTORS OF $\ell$-ADIC REPRESENTATIONS

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Abstract. We give a new formula for the Artin conductor of an $\ell$-adic representation of the Weil group of a local field of residue characteristic $p \neq \ell$.

## 1. Introduction

Our aim in this note is to give a new formula for the Artin conductor of an $\ell$-adic representation of the Galois group of a non-archimedean local field of residue characteristic $p \neq \ell$; see Theorem 1 in Section 9. The proof that this formula is equivalent to the standard one is a simple unwinding of the definitions, but the new formula has the virtue that it does not require any auxiliary constructions such as semi-simplification or Weil-Deligne representations, and it works equally well for representations with finite or infinite inertial image. We also take the occasion to correct a typographical error in a canonical reference.

We would have imagined that formula (9.1) was well known, and make no claim of priority, but a proof that it is equivalent to other definitions does not seem to appear in the literature.[1] We hope having a complete treatment will be useful to the community.

Here is a brief explanation of the main issue: Let $F$ be a local field with Weil group $W_F$. The conductor of a representation $\rho$ of $W_F$ depends only on the restriction of $\rho$ to the inertia group $I_F$, and it is defined in the first instance in [Art31] only for representations which factor through finite quotients of $W_F$. Since the image of an $\ell$-adic representation restricted to inertia need not be finite, further discussion is required. This problem was solved by Serre, who in [Ser70] gave a definition of the conductor of an $\ell$-adic representation using the key fact that there is a subgroup of inertia of finite index on which the representation is unipotent. This last fact was conjectured by Serre and Tate and proven by Grothendieck; see [ST68, Appendix]. Later, Deligne gave a definition using what is now known as a Weil-Deligne representation, a technical device with several uses in the Langlands program; see [Del73] and [Tat79].

For the convenience of the reader, we discuss the previous definitions of the conductor (Artin's definition in Sections 2 through 4, Serre's in Sections 5 and 6, and Deligne's in Sections 7 and 8) with full definitions, but minimal proofs. In Section 9 we give the new formula, and in Section 10 we give two applications, one of which motivated this work.

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[1]The formula appears as a definition in [DDT97, §2.1] and in the notes [Wie12], but without proof that it is equivalent to the standard definition.

## 2. Galois groups and representations

Throughout the paper, $F$ will be a non-archimedean local field with residue field of characteristic $p$ and cardinality $q$. In other words, $F$ is a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. Let $\overline{F}$ be a separable closure of $F$, $G_F$ the Galois group $\mathrm{Gal}(\overline{F}/F)$, and $\Phi$ a geometric Frobenius element, i.e., an element of $G_F$ which induces the inverse of the $q$-power Frobenius automorphism of the residue field of $\overline{F}$. Let $W_F$ be the Weil group of $F$, the subgroup of $G_F$ inducing integral powers of the $q$-power Frobenius on the residue field, and let $I_F$ be the inertia subgroup of $G_F$, the subgroup acting as the identity on the residue field. We give $W_F$ the topology characterized by the requirement that $I_F$ be an open subgroup.

Let $\ell$ be a prime number distinct from $p$ and let $E$ be a finite extension of $\mathbb{Q}_\ell$, the $\ell$-adic numbers. An $\ell$-adic representation with values in $E$ is a continuous homomorphism

$$\rho_\ell : W_F \to \mathrm{GL}_n(E)$$

where $\mathrm{GL}_n(E)$ is given the topology induced by the metric ($\ell$-adic) topology on $E$. (When $E \neq \mathbb{Q}_\ell$, some authors refer to $\rho_\ell$ as a $\lambda$-adic representation.)

We are concerned with (the exponent of) the Artin conductor of $\rho_\ell$, which we denote $a(\rho_\ell)$ and call simply the conductor. It will be defined in Section 6 below.

## 3. Ramification groups

In this section, we review the lower and upper ramification filtrations on Galois groups. See [Ser79, Ch. IV] for more details.

Let $K$ be a finite Galois extension of $F$ with group $G = \mathrm{Gal}(K/F)$. We write $\mathcal{O}_K$ for the ring of integers of $K$, $\pi_K$ for a generator of the maximal ideal of $\mathcal{O}_K$, and $v_K$ for the valuation of $K$ with $v_K(\pi_K) = 1$.

The ramification filtration on $G$ in the lower numbering is defined by the requirement that

$$\sigma \in G_i \quad \Longleftrightarrow \quad v_K(\sigma(x) - x) \geq i + 1 \quad \forall x \in \mathcal{O}_K$$

for $i$ an integer $\geq -1$. Clearly $G_{-1} = G$, $G_0$ is the inertia subgroup of $G$, and $G_i = 0$ for all sufficiently large $i$. By convention, if $r \geq -1$ is a real number, we set $G_r = G_i$ where $i$ is the smallest integer $\geq r$.

Let $\varphi : [-1, \infty) \to [-1, \infty)$ be the continuous, piecewise linear function with $\varphi(-1) = -1$, slope 1 on $[-1, 0)$, and slope $1/[G_0 : G_i]$ on $(i-1, i)$. Let $\psi = \varphi^{-1}$, the inverse function. The upper numbering of the ramification filtration on $G$ is given by

$$G^s = G_{\psi(s)} \qquad \text{and} \qquad G^{\varphi(r)} = G_r.$$

Note that the breaks in the upper numbering (i.e., the values $s$ so that $G^{s+\epsilon} \neq G^s$ for all $\epsilon > 0$) are in general rational numbers, not necessarily integers.

The upper numbering is adapted to quotients in the following sense: if $L/F$ is a Galois extension with $L \subset K$ and $H = \mathrm{Gal}(L/F)$, so that $H$ is a quotient of $G$, then the upper numbering satisfies

$$H^s = \mathrm{Im}(G^s \to H).$$

This property allows us to define a ramification filtration on $G_F = \mathrm{Gal}(\overline{F}/F)$ by declaring that

$$G_F^s = \left\{\sigma \in G_F \;\middle|\; \sigma_{|K} \in \mathrm{Gal}(K/F)^s \;\forall K\right\}$$

where $K$ runs through all finite Galois extensions of $F$. Clearly we have $G_F^{-1} = G_F$ and $G_F^0 = I_F$.

We define

$$G_F^{>0} = \bigcup_{\epsilon>0} G_F^{\epsilon}$$

where the union is over all positive real numbers $\epsilon$. We also write $P_F$ for $G_F^{>0}$ and call this the wild inertia group of $F$. It is known to be a pro-$p$ group, and the quotient $I_F/P_F$ is isomorphic as a profinite group to $\prod_{\ell\neq p}\mathbb{Z}_\ell$.

## 4. Artin conductor

In this section we review the definition of the Artin conductor of a representation of $G = \mathrm{Gal}(K/F)$ where $K/F$ is a finite Galois extension. The standard reference for this material is [Ser79, Ch. VI].

Let $\rho : G \to \mathrm{GL}_n(E)$ be a representation where $E$ is a field of characteristic zero. We write $V$ for the space where $\rho$ acts, namely $E^n$, and for a subgroup $H$ of $G$ we write $V^H$ for the invariants under $H$:

$$V^H = \{v \in V \,|\rho(h)(v) = v \ \forall h \in H\}.$$

Recall the ramification subgroups $G_i$ of the previous section. For a subspace $W$ of $V$, we write $\operatorname{codim} W$ for the codimension of $W$ in $V$, i.e., $\dim V - \dim W$. Following Artin [Art31], we define the *Artin conductor* of $\rho$ as

$$a(\rho) := \sum_{i=0}^{\infty} \frac{\operatorname{codim} V^{G_i}}{[G_0 : G_i]}. \tag{4.1}$$

Note that this is in fact a finite sum and that it depends only on the restriction of $\rho$ to $G_0$, the inertia subgroup of $G$. It is true but not at all obvious that $a(\rho)$ is an integer; see [Ser79, Ch. VI, §2, Thm. 1′].

Because the definition of $a(\rho)$ depends only on $\rho$ restricted to inertia, we may extend it to representations $\rho$ which are only assumed to have finite image after restriction to inertia.

We give two alternate expressions for $a(\rho)$ which will be useful in what follows. First, we have

$$a(\rho) = \int_{-1}^{\infty} \frac{\operatorname{codim} V^{G_r}}{[G_0 : G_r]}\, dr$$

because the integrand is constant on intervals $(i-1, i)$ and the corresponding Riemann sum for the integral is exactly the sum defining $a(\rho)$. Second,

$$a(\rho) = \int_{-1}^{\infty} \operatorname{codim} V^{G^s}\, ds.$$

This follows from the previous expression and the definition of the function $\varphi$ relating the upper and lower numberings. Indeed, if $s = \varphi(r)$, then $ds = \varphi'\, dr$ and $\varphi'(r) = 1/[G_0 : G_i]$ for $r \in (i-1, i)$.

This last formula for $a(\rho)$ turns out to be useful as it generalizes without change to $\ell$-adic representations. It also makes evident the fact that if $\rho$ factors through $H = \mathrm{Gal}(L/F)$ for some subextension $L \subset K$, then the conductor of $\rho$ as a representation of $G$ is the same as its conductor as a representation of $H$.

For use later we note that the first term in the sum for $a(\rho)$ and the first part of the integrals for it are all equal:

$$\epsilon(\rho) := \int_{-1}^{0} \operatorname{codim} V^{G^s}\,ds = \int_{-1}^{0} \frac{\operatorname{codim} V^{G_r}}{[G_0 : G_r]}\,dr = \operatorname{codim} V^{G_0} = \operatorname{codim} V^{I_F}.$$

It is convenient to break $a(\rho)$ into two parts, $\epsilon(\rho)$ as above, and

$$\delta(\rho) := \int_{0}^{\infty} \operatorname{codim} V^{G^s}\,ds = \int_{0}^{\infty} \frac{\operatorname{codim} V^{G_r}}{[G_0 : G_r]}\,dr = \sum_{i=1}^{\infty} \frac{\operatorname{codim} V^{G_i}}{[G_0 : G_i]},$$

so that $a(\rho) = \epsilon(\rho) + \delta(\rho)$. One calls $\epsilon(\rho)$ the *tame conductor* of $\rho$ and $\delta(\rho)$ the *wild conductor* or *Swan conductor* of $\rho$.

The Artin and Swan conductors may also be realized as the inner products of the character of $\rho$ with those of certain representations, known as the *Artin representation* and the *Swan representation* respectively. The existence of representations with this property is essentially equivalent to the fact that the Artin and Swan conductors are integers; see [Ser79, VI, §2].

## 5. $\ell$-adic representations

Recall that an $\ell$-adic representation is a continuous homomorphism

$$\rho_\ell : W_F \to \mathrm{GL}_n(E)$$

where $E$ is a finite extension of $\mathbb{Q}_\ell$ and $\mathrm{GL}_n(E)$ is given the $\ell$-adic topology. A primary source of such representations is $\ell$-adic cohomology. More precisely, if $X$ is a variety over $F$, then the $\ell$-adic étale cohomology groups $H^i(X \times \overline{F}, \mathbb{Q}_\ell)$ (and variants) are equipped with continuous actions of $G_F$ and we may restrict to $W_F$ to obtain $\ell$-adic representations as defined above.

Let $\mathcal{O}_E$ be the ring of integers of $E$ and let $\mathfrak{m} \subset \mathcal{O}_E$ be the maximal ideal. It is well known (see, e.g., [Ser06, p. 104]) that every compact subgroup of $\mathrm{GL}_n(E)$ is conjugate to a subgroup of $\mathrm{GL}_n(\mathcal{O}_E)$. Since $I_F$ is a closed subgroup of $G_F$, it is compact. Thus replacing $\rho_\ell$ by a conjugate representation if necessary, we may assume that the image of $I_F$ under $\rho_\ell$ is contained in $\mathrm{GL}_n(\mathcal{O}_E)$.

Now the $\mathrm{GL}_n(\mathcal{O}_E)$ has a finite index subgroup which is a pro-$\ell$ group, namely, the kernel of reduction $\mathrm{GL}_n(\mathcal{O}_E) \to \mathrm{GL}_n(\mathcal{O}_E/\mathfrak{m})$. Since $P_F \subset I_F$ is a pro-$p$ group and $\ell \neq p$, it follows that the image of $P_F$ under $\rho_\ell$ is finite.

On the other hand, it is not in general the case that the image of all of $I_F$ under $\rho_\ell$ is finite. For example, if $\rho_\ell$ is the representation of $G_F$ on the Tate module of an elliptic curve over $F$ with split multiplicative reduction, it is known that the restriction of $\rho_\ell$ to $P_F$ is trivial, and in a suitable basis the restriction of $\rho_\ell$ to $I_F$ has the form

$$\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$$

where $\tau : I_F \to \mathbb{Z}_\ell$ is the projection $I_F \to I_F/P_F \cong \prod_{r \neq q} \mathbb{Z}_r \to \mathbb{Z}_\ell$. See [Roh94, §15] for more details.

Since the restriction of $\rho_F$ to inertia need not have finite image, the definition of Artin does not apply directly because the denominators $[G_0 : G_i]$ in (4.1) need not be finite, so further discussion is needed.

The crucial observation that makes it possible to define a conductor for $\rho_\ell$ is the following, which was conjectured by Serre and Tate and proven by Grothendieck.

*Key Fact.* If $\rho_\ell : W_F \to \mathrm{GL}_n(E)$ is a continuous representation, then there is a subgroup $I' \subset I_F$ of finite index such that $\rho_\ell(g)$ is unipotent for every $g \in I'$.

The proof is given in [ST68, Appendix]. It applies more generally to discrete valuation fields $F$ whose residue field has the property that no finite extension contains all roots of unity of $\ell$-power order. Note that we may assume that $I'$ is also normal in $W_F$.

## 6. Semi-simplification

Fix an $\ell$-adic representation $\rho_\ell : W_F \to \mathrm{GL}_n(E)$. We write $V_\ell$ for $E^n$ with its action of $W_F$ via $\rho_\ell$.

Let $V_{ss}$ be the semi-simplification of $V_\ell$, defined as the direct sum of the Jordan-Hölder factors of $V_\ell$ as a $W_F$ module, and let $\rho_{ss} : W_F \to \mathrm{GL}_n(E)$ be the corresponding homomorphism.

It follows from the key fact of the previous section that $\rho_{ss}$ is trivial on a finite index subgroup of $I_F$. Indeed, by Clifford's theorem [Cli37, Thm. 1], $\rho_{ss}$ is semi-simple when restricted to any normal subgroup of $W_F$, and by Kolchin's theorem [Ser06, V.3*, p. 35], a semi-simple and unipotent representation is trivial. In particular, $\rho_{ss}(I_F)$ is a finite group. Therefore, the wild conductor $\delta(\rho_{ss})$ and the Artin conductor $a(\rho_{ss})$ are well defined.

Following Serre [Ser70], we define

$$a(\rho_\ell) := \epsilon(\rho_\ell) + \delta(\rho_{ss})$$

where as usual $\epsilon(\rho_\ell) = \dim V_\ell - \dim V_\ell^{I_F}$.

Note that if $\rho_\ell$ restricted to inertia has finite image, then $\rho_\ell$ and $\rho_{ss}$ have the same restriction to inertia, so this definition agrees with that in Section 4 when they both apply.

Note also that

$$\epsilon(\rho_\ell) - \epsilon(\rho_{ss}) = \dim V_{ss}^{I_F} - \dim V_\ell^{I_F}$$

so we have

$$a(\rho_\ell) = \dim V_{ss}^{I_F} - \dim V_\ell^{I_F} + a(\rho_{ss}).$$

This last formula appears in [Tat79, 4.2.4], but is missing the exponent $I_F$ on $V_\ell$.

The wild conductor $\delta(\rho_{ss})$ has two other useful descriptions. We noted above that $\rho_\ell|_{P_F}$ (the restriction to wild inertia) has finite image, so is already semi-simple, and $\rho_{ss}|_{P_F}$ is also semi-simple by the same reasoning (or by Clifford's theorem). Since $\rho_\ell|_{P_F}$ and $\rho_{ss}|_{P_F}$ have the same character, they are isomorphic and have the same wild conductor. For another description, choose a basis of $V_\ell$ so that the image of $\rho_\ell$ lies in $\mathrm{GL}_n(\mathcal{O}_E)$ and let $\overline{\rho}_\ell$ be the reduction modulo $\mathfrak{m}_E$. Then $\overline{\rho}_\ell$ restricted to $P_F$ is isomorphic to $\rho_\ell$ restricted to $P_F$ and also gives the same wild conductor. Summarizing:

$$\delta(\rho_{ss}) = \delta(\rho_\ell|_{P_F}) = \delta(\overline{\rho}_\ell).$$

## 7. Weil-Deligne representations

In this section we review (with the minimum of details) the notion of a Weil-Deligne representation. See [Del73, §8] or [Tat79, 4.1] for more details, and [Roh94] for a motivated introduction aimed at arithmetic geometers.

We write $||\cdot||$ for the homomorphism $W_F \to \mathbb{Q}$ which sends $\Phi$ to $q^{-1}$ and which is trivial on $I_F$.

Let $V$ be a vector space over a field of characteristic zero. We define a Weil-Deligne representation of $W_F$ on $V$ as a pair $(\rho, N)$ where $\rho : W_F \to \mathrm{GL}(V)$ is a homomorphism continuous with respect to the discrete topology on $V$ and $N : V \to V$ is an endomorphism satisfying

$$\rho(w)N\rho(w)^{-1} = ||w||\, N.$$

Continuity of $\rho$ implies that it has finite image restricted to inertia, and the displayed formula implies that $N$ is nilpotent (because its eigenvalues are stable under multiplication by $q$).

Because $\rho$ has finite image when restricted to $I_F$, its conductor is defined by the formulas of Section 4. We define the Artin conductor of a Weil-Deligne representation $(\rho, N)$ as

$$a(\rho, N) := a(\rho) + \dim V^{I_F} - \dim V_N^{I_F}.$$

Here $V_N$ is the kernel of $N$ on $V$, so that

$$V_N^{I_F} = \{v \in V\, |N(v) = 0, \rho(w)(v) = v\ \forall w \in I_F\}.$$

## 8. $\ell$-adic representations and Weil-Deligne representations

Fix an $\ell$-adic representation $\rho_\ell$. The key fact stated at the end of Section 5 leads to a description of the behavior of $\rho_\ell$ restricted to $I_F$ in terms of Weil-Deligne representations.

First, note that because $P_F$ is a pro-$p$ group and $I_F/P_F \cong \prod_{\ell \neq p} \mathbb{Z}_l$, there is a non-zero homomorphism $t_\ell : I_F \to \mathbb{Q}_\ell$ which is unique up to a scalar. It satisfies $t_\ell(w\sigma w^{-1}) = ||w||\, t_\ell(\sigma)$ for all $w \in W_F$.

The key fact implies that there is a unique nilpotent linear transformation $N : E^n \to E^n$ such that for all $\sigma$ in some finite index subgroup of $I_F$

$$\rho_\ell(\sigma) = \exp(t_\ell(\sigma)N)$$

as automorphisms of $E^n$. Here exp is defined by the usual series $1 + x + x^2/2! + \cdots$ and $\exp(t_l(\sigma)N)$ is in fact a finite sum because $N$ is nilpotent.

It follows from this (see [Del73, §8]) that there exists a unique Weil-Deligne representation $(\rho, N)$ on $V = E^n$ such that for all $m \in \mathbb{Z}$ and all $\sigma \in I_F$

$$\rho_\ell(\Phi^m\sigma) = \rho(\Phi^m\sigma)\exp\left(t_\ell(\sigma)N\right). \tag{8.1}$$

Conversely, given a Weil-Deligne representation $(\rho, N)$ on $V$, the displayed formula defines an $\ell$-adic representation. This correspondence gives a bijection on isomorphism classes. (The correspondence $\rho_\ell \leftrightarrow (\rho, N)$ depends on the choices of $t_\ell$ and $\Phi$, but after passing to isomorphism classes it is independent of these choices; see [Del73].)

The point of introducing Weil-Deligne representations is that their definition uses only the discrete topology on $V$, so is convenient for shifting between different ground fields (such as $\mathbb{Q}_\ell$ for varying $\ell$ and $\mathbb{C}$).

Following Deligne [Del73], we define

$$a_D(\rho_\ell) := a(\rho, N) = a(\rho) + \dim V^{I_F} - \dim V_N^{I_F}.$$

We note that $\rho_\ell(I_F)$ is finite if and only if the corresponding $N = 0$, and in this case the definition above reduces to that of Section 4.

Since $\epsilon(\rho) = \dim V - \dim V^{I_F}$, we also have

$$a_D(\rho_\ell) = \dim V - \dim V_N^{I_F} + \delta(\rho).$$

We note also that $t_\ell$ is trivial on the wild inertia group $P_F = G_F^{>0}$, so $\rho_\ell$ and $\rho$ are equal on $P_F$. It follows that $\delta(\rho) = \delta(\rho_\ell|_{P_F}) = \delta(\rho_{ss})$. On the other hand, equation (8.1) (with $m = 0$) implies that $V_N^{I_F} = V_\ell^{I_F}$. Therefore

$$a_D(\rho_\ell) = \dim V - \dim V_\ell^{I_F} + \delta(\rho_\ell) = a(\rho_\ell),$$

in other words, the definitions of Deligne and Serre agree.

## 9. Another formula for $a(\rho_\ell)$

We can now state the main result of this note. The left hand side of (9.1) appears as a definition in [Wie12, Def. 3.1.27]. This reference seems to include everything needed to prove that Definition 3.1.27 agrees with the definitions of Serre and Deligne, but the proof is not given there.

**Theorem 1.** *Let $\rho_\ell$ be an $\ell$-adic representation of $W_F$ on $V_\ell$ with corresponding Weil-Deligne representation $(\rho, N)$ on $V$ and semi-simplification $\rho_{ss}$ on $V_{ss}$. Let $a(\rho_\ell)$ be the Artin conductor of $\rho_\ell$, defined as in Section 6, and let $a_D(\rho_\ell)$ be defined as in Section 8. Then*

$$\int_{-1}^{\infty} \operatorname{codim} V_\ell^{G^s}\, ds = a(\rho_\ell) = a_D(\rho_\ell). \tag{9.1}$$

*Proof.* We saw at the end of Section 8 that $a(\rho_\ell) = a_D(\rho_\ell)$, so we need only check that the integral is equal to $a(\rho_\ell)$. For $-1 < s < 0$, $G^s = G_s = I_F$, so

$$\int_{-1}^{0} \operatorname{codim} V_\ell^{G^s}\, ds = \operatorname{codim} V_\ell^{I_F} = \epsilon(\rho_\ell).$$

On the other hand, for $s > 0$, $G^s \subset P_F$ and $\rho_\ell$ restricted to $G^s$ is isomorphic to $\rho_{ss}$ restricted to $G^s$. Therefore

$$\int_{0}^{\infty} \operatorname{codim} V_\ell^{G^s}\, ds = \int_{0}^{\infty} \operatorname{codim} V_{ss}^{G^s}\, ds = \delta(\rho_{ss}).$$

It follows that

$$\int_{-1}^{\infty} \operatorname{codim} V_\ell^{G^s}\, ds = \epsilon(\rho_\ell) + \delta(\rho_{ss}) = a(\rho_\ell),$$

as desired. □

## 10. An application to twisting

We give an easy application of the theorem which is the motivation for this work.

Let $\rho_\ell : W_F \to \mathrm{GL}_n(E)$ be an $\ell$-adic representation and let $\chi : W_F \to E^\times$ be a character. We say "$\chi$ is more deeply ramified than $\rho_\ell$" if there exists a non-negative real number $s$ such that $\rho_\ell(G_F^s) = \{id\}$ and $\chi(G_F^s) \neq \{id\}$. In other words, $\chi$ is non-trivial further into the ramification filtration than $\rho_\ell$ is. Let $m$ be the supremum of the set of $s$ such that $\chi$ is non-trivial on $G_F^s$. It follows from Section 4 that $a(\chi) = m + 1$.

**Proposition 1.** *If $\chi$ is more deeply ramified than $\rho_\ell$, then*

$$a(\rho_\ell \otimes \chi) = \deg(\rho_\ell) a(\chi).$$

*Proof.* Let $V_\ell$ be the space where $W_F$ acts via $\rho_\ell$ and let $V_{\ell,\chi}$ be the same space where $W_F$ acts via $\rho_\ell \otimes \chi$. By the theorem we have

$$a(\rho_\ell \otimes \chi) = \int_{-1}^{\infty} \operatorname{codim} V_{\ell,\chi}^{G_F^s}\, ds.$$

If $s \leq m$, then $V_{\ell,\chi}^{G_F^s} \subset V_{\ell,\chi}^{G_F^m}$ and the latter is zero because $\rho_\ell(G_F^m) = \{id\}$ and $\chi(G_F^m) \neq \{id\}$. Thus in this range the integrand is $\dim V_\ell = \deg(\rho_\ell)$. On the other hand, if $s > m$, then $\rho_\ell \otimes \chi(G_F^s) = \{id\}$ and the integrand is zero. Thus

$$\int_{-1}^{\infty} \operatorname{codim} V_{\ell,\chi}^{G_F^s}\, ds = \deg(\rho_\ell)(m+1) = \deg(\rho_\ell) a(\chi),$$

as desired. □

A particularly useful case of the proposition occurs when $\rho_\ell$ is tamely ramified and $\chi$ is wildly ramified, e.g., when $\chi$ is an Artin-Schreier character.

A variant of the proposition where $\chi$ and $\rho_\ell$ are both assumed to be irreducible, but $\chi$ may be of dimension $> 1$, is stated as Lemma 9.2(3) of [DD13]

We end with another application in the same spirit, namely a very simple solution to Exercise 2 in [Ser79, p. 103].

**Proposition 2.** *Suppose that $\rho_\ell$ is an irreducible $\ell$-adic representation of $W_F$ on $V_\ell$, and let $m$ be the supremum of the set of numbers $s$ such that $\rho_\ell(G^s) \neq 0$. Then*

$$a(\rho_\ell) = (\dim V_\ell)(m+1).$$

*Proof.* Since $G^s$ is a normal subgroup of $G_F$, the subspace of invariants $V_\ell^{G^s}$ is preserved by $G_F$. Since $V_\ell$ is irreducible, we have that $\operatorname{codim} V_\ell^{G^s}$ is 0 if $s > m$ and $\dim V_\ell$ if $s < m$. Therefore

$$a(\rho_\ell) = \int_{-1}^{\infty} \operatorname{codim} V_\ell^{G_F^s}\, ds = (\dim V_\ell)(m+1),$$

as desired. □

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