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A NOTE ON THE SPECTRAL AREA OF TOEPLITZ OPERATORS

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Abstract. In this note, we show that for hyponormal Toeplitz operators, there exists a lower bound for the area of the spectrum. This extends the known estimate for the spectral area of Toeplitz operators with an analytic symbol.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane. Let L^2 denote the Lebesgue space of square integrable functions on the unit circle $\partial\mathbb{D}$. The Hardy space H^2 is the subspace of L^2 of analytic functions on $\mathbb{D}$. Let P be the orthogonal projection from L^2 to H^2. For $\varphi \in L^\infty$, the space of bounded Lebesgue measurable functions on $\partial\mathbb{D}$, the Toeplitz operator T_φ and the Hankel operator H_φ with symbol φ are defined on H^2 by

$$T_\varphi h = P(\varphi h),$$

and

$$H_\varphi h = U(I-P)(\varphi h), \tag{1.1}$$

for $h \in H^2$. Here U is the unitary operator on L^2 defined by

$$Uh(z) = \bar{z}h(\bar{z}).$$

Recall that the spectrum of a linear operator T, denoted as $sp(T)$, is the set of complex numbers λ such that $T - \lambda I$ is not invertible; here I denotes the identity operator. Let $[T^*, T]$ denote the operator $T^*T - TT^*$, called the self-commutator of T. An operator T is called hyponormal if $[T^*, T]$ is positive. Hyponormal operators satisfy the celebrated Putnam inequality [11]

Theorem 1.1. *If T is a hyponormal operator, then*

$$\|[T^*, T]\| \leqslant \frac{Area(sp(T))}{\pi}.$$

Notice that a Toeplitz operator with analytic symbol f is hyponormal, and it is well known that $sp(T_f) = \overline{f(\mathbb{D})}$. The lower bounds of the area of $sp(T_f)$ were obtained in [9] (see [2], [1], [13], and [14] for generalizations to uniform algebras and further discussions). Together with Putnam's inequality, such lower bounds were used to prove the isoperimetric inequality (see [4], [5], and the references there). Recently, there has been revived interest in the topic in the context of analytic Toeplitz operators on the Bergman space (cf. [3], [10], and [7]). Together with

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Putnam's inequality, the latter lower bounds have provided an alternative proof of the celebrated St. Venant's inequality for torsional rigidity.

In the general case, Harold Widom [15] proved the following theorem for arbitrary symbols.

Theorem 1.2. *Every Toeplitz operator has a connected spectrum.*

The main purpose of this note is to show that for a rather large class of Toeplitz operators on H^2, hyponormal operators with a harmonic symbol, there is still a lower bound for the area of the spectrum, similar to the lower bound obtained in [9] in the context of uniform algebras.

We shall use the following characterization of the hyponormal Toeplitz operators given by Cowen in [6]

Theorem 1.3. *Let $\varphi \in L^\infty$, where $\varphi = f + \bar{g}$ for f and g in H^2. Then T_φ is hyponormal if and only if*

$$g = c + T_{\bar{h}} f,$$

for some constant c and $h \in H^\infty$ with $\|h\|_\infty \leqslant 1$.

2. Main results

In this section, we obtain the lower bound for the area of the spectrum for hyponormal Toeplitz operators by estimating the self-commutators.

Theorem 2.1. *Suppose $\varphi \in L^\infty$ and*

$$\varphi = f + \overline{T_{\bar{h}} f},$$

for $f, h \in H^\infty$, $\|h\|_\infty \leqslant 1$, and $h(0) = 0$. Then

$$\|[T_\varphi^*, T_\varphi]\| \geqslant \int |f - f(0)|^2 \frac{d\theta}{2\pi} = ||P(\varphi) - \varphi(0)||_2^2.$$

Proof. Let

$$g = T_{\bar{h}} f. \tag{2.1}$$

For every p in H^2,

$$\begin{aligned}
\langle [T_\varphi^*, T_\varphi] p, p\rangle &= \langle T_\varphi p, T_\varphi p\rangle - \langle T_\varphi^* p, T_\varphi^* p\rangle \\
&= \langle fp + P(\bar{g}p), fp + P(\bar{g}p)\rangle - \langle gp + P(\bar{f}p), gp + P(\bar{f}p)\rangle \\
&= ||fp||^2 - ||P(\bar{f}p)||^2 - ||gp||^2 + ||P(\bar{g}p)||^2 \\
&= ||\bar{f}p||^2 - ||P(\bar{f}p)||^2 - ||\bar{g}p||^2 + ||P(\bar{g}p)||^2 \\
&= ||H_{\bar{f}} p||^2 - ||H_{\bar{g}} p||^2,
\end{aligned}$$

where $||\cdot||$ means the $||\cdot||_{L^2(\partial\mathbb{D})}$. The third equality holds because

$$\langle fp, P(\bar{g}p)\rangle = \langle fp, \bar{g}p\rangle = \langle gp, \bar{f}p\rangle = \langle gp, P(\bar{f}p)\rangle.$$

By the computation in [6, pp. 811], (2.1) implies that

$$H_{\bar{g}} = T_{\bar{k}} H_{\bar{f}},$$

where $k(z) = \overline{h(\bar{z})}$. Thus

$$\langle [T_\varphi^*, T_\varphi] p, p\rangle = ||H_{\bar{f}} p||^2 - ||T_{\bar{k}} H_{\bar{f}} p||^2, \tag{2.2}$$

for $k \in H^\infty$, $\|k\|_\infty \leqslant 1$, and $k(0) = 0$.

First, we assume k is a Blaschke product vanishing at 0. Then

$$|k| = 1 \text{ on } \partial\mathbb{D}.$$

Let $u = H_{\bar{f}}p \in H^2$. By (2.2) we have

$$\langle [T_\varphi^*, T_\varphi]p, p\rangle = ||u||^2 - ||T_{\bar{k}}u||^2 = ||u||^2 - ||\bar{k}u||^2 + ||H_{\bar{k}}u||^2 = ||H_{\bar{k}}u||^2. \tag{2.3}$$

Then

$$\begin{aligned} ||H_{\bar{k}}u|| &= ||(I-P)(\bar{k}u)|| = ||k\bar{u} - \overline{P(\bar{k}u)}|| \\ &\geqslant \sup_{\substack{m\in H^2 \\ m(0)=0}} \frac{|\langle k\bar{u} - \overline{P(\bar{k}u)}, m\rangle|}{||m||} \\ &= \sup_{\substack{m\in H^2 \\ m(0)=0}} \frac{1}{||m||}\left|\int k\bar{u}\bar{m}\,\frac{d\theta}{2\pi}\right|. \end{aligned}$$

The last equality holds because $m(0) = 0$ implies that $\bar{m}$ is orthogonal to H^2. Since $k(0) = 0$, taking $m = k$, we find

$$||H_{\bar{k}}u|| \geqslant \left|\int \bar{u}\,\frac{d\theta}{2\pi}\right| = |u(0)|. \tag{2.4}$$

Next, suppose k is a convex linear combination of Blaschke products vanishing at 0, i.e.,

$$k = \alpha_1 B_1 + \alpha_2 B_2 + \cdots + \alpha_l B_l,$$

where B_j's are Blaschke products with $B_j(0) = 0$, $\alpha_j \in [0,1]$, and $\sum_{j=1}^{l} \alpha_j = 1$.

By (2.3) and (2.4), for each j

$$\begin{aligned} &||u||^2 - ||T_{\bar{B}_j}u||^2 = ||H_{\bar{B}_j}u||^2 \geqslant |u(0)|^2 \\ \Longrightarrow\ &||T_{\bar{B}_j}u|| \leqslant \sqrt{||u||^2 - |u(0)|^2} = ||u - u(0)||. \end{aligned}$$

Then

$$\begin{aligned} ||u||^2 - ||T_{\bar{k}}u||^2 &= ||u||^2 - ||\alpha_1 T_{\bar{B}_1}u + \alpha_2 T_{\bar{B}_2}u + \cdots + \alpha_l T_{\bar{B}_l}u||^2 \\ &\geqslant ||u||^2 - \Big(\alpha_1||T_{\bar{B}_1}u|| + \alpha_2||T_{\bar{B}_2}u|| + \cdots + \alpha_l||T_{\bar{B}_l}u||\Big)^2 \\ &\geqslant ||u||^2 - ||u - u(0)||^2 = |u(0)|^2. \end{aligned} \tag{2.5}$$

In general, for k in the closed unit ball of H^∞, vanishing at 0, by Carathéodory's Theorem (cf. [8, p. 6]), there exists a sequence $\{B_n\}$ of finite Blaschke products such that

$$B_n \longrightarrow k \quad \text{pointwise on } \mathbb{D}.$$

Since B_n's are bounded by 1 in H^2, passing to a subsequence we may assume that

$$B_n \longrightarrow k \quad \text{weakly in } H^2.$$

Then by [12, Theorem 3.13], there is a sequence $\{k_n\}$ of convex linear combinations of Blaschke products such that

$$k_n \longrightarrow k \quad \text{in } H^2.$$

Since $k(0) = 0$, we can let those k_n's be convex linear combinations of Blaschke products vanishing at 0.

Then

$$||T_{\bar{k}_n}u - T_{\bar{k}}u|| = ||P(\bar{k}_n u - \bar{k}u)|| \leqslant ||k_n - k|| \cdot ||u|| \to 0.$$

Since (2.5) holds for every k_n, we have

$$\begin{aligned}\langle [T_\varphi^*, T_\varphi]p, p\rangle &= ||u||^2 - ||T_{\bar{k}}u||^2 = \lim_{n\to\infty}(||u||^2 - ||T_{\bar{k}_n}u||^2) \\ &\geqslant |u(0)|^2 = |(H_{\bar{f}}p)(0)|^2.\end{aligned}$$

By the definition of the Hankel operator (1.1),

$$|(H_{\bar{f}}p)(0)| = |\langle p\bar{f}, \bar{z}\rangle| = \left|\int \bar{f}zp\,\frac{d\theta}{2\pi}\right|.$$

From the standard duality argument (cf. [8, Chapter IV]), we have

$$\begin{aligned}\sup_{\substack{||p||=1\\ p\in H^2}}\left|\int \bar{f}zp\,\frac{d\theta}{2\pi}\right| &= \sup\left\{\left|\int \bar{f}p\,\frac{d\theta}{2\pi}\right| : p\in H^2, ||p||=1, p(0)=0\right\} \\ &= \operatorname{dist}(\bar{f}, H^2) = ||f - f(0)||.\end{aligned}$$

Hence

$$||[T_\varphi^*, T_\varphi]|| = \sup_{\substack{||p||=1\\ p\in H^2}} |\langle [T_\varphi^*, T_\varphi]p, p\rangle| \geqslant ||f - f(0)||^2.$$

□

Remark 2.1. For arbitrary h in the closed unit ball of H^∞, it follows directly from (2.2) that T_φ is normal if and only if h is a unimodular constant. So we made the assumption that $h(0) = 0$ to avoid these trivial cases. Of course, Theorem 2.1 implies right away that T_φ is normal if and only if $f = f(0)$, i.e., when φ is a constant, but under more restrictive hypothesis that $h(0) = 0$.

Applying Theorem 1.1 and 1.3, we have

Corollary 2.1. *Suppose $\varphi \in L^\infty$ and*

$$\varphi = f + \overline{T_{\bar{h}}f},$$

for $f, h \in H^\infty$, $\|h\|_\infty \leqslant 1$, and $h(0) = 0$. Then

$$Area(sp(T_\varphi)) \geqslant \pi||P(\varphi) - \varphi(0)||_2^2.$$

Remark 2.2. Thus the lower bound for the spectral area of a general hyponormal Toeplitz operator T_φ on $\partial\mathbb{D}$ still reduces to the H^2 norm of the analytic part of φ. For analytic symbols this is encoded in [9, Theorem 2] in the context of Banach algebras. In other words, allowing more general symbols does not reduce the area of the spectrum.

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