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ON THE STRUCTURE OF THE SECOND EIGENFUNCTIONS OF THE p-LAPLACIAN ON A BALL

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Abstract. In this paper, we prove that the second eigenfunctions of the p-Laplacian, $p > 1$, are not radial on the unit ball in $\mathbb{R}^N$, for any $N \geq 2$. Our proof relies on the variational characterization of the second eigenvalue and a variant of the deformation lemma. We also construct an infinite sequence of eigenpairs $\{\tau_n, \Psi_n\}$ such that Ψ_n is nonradial and has exactly $2n$ nodal domains. A few related open problems are also stated.

1. Introduction

Let $B_1 \subset \mathbb{R}^N$ be the open unit ball centred at the origin. We consider the following eigenvalue problem:

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2} u \quad \text{in } B_1, \\ u &= 0 \quad \text{on } \partial B_1, \end{aligned} \tag{1.1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplace operator with $p > 1$ and λ is the spectral parameter. A real number λ for which (1.1) admits a nonzero weak solution in $W_0^{1,p}(B_1)$ is called an eigenvalue of (1.1), and corresponding solutions are called the eigenfunctions associated with λ.

For $p = 2$, it is well known that the set of all eigenvalues of (1.1) can be arranged in a sequence

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \ldots \to \infty$$

and the corresponding normalized eigenfunctions form an orthonormal basis for the Sobolev space $W_0^{1,2}(B_1)$. Further, using the Courant-Weinstein variational

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principle (Theorem 7.8.14 of [4]), these eigenvalues can be expressed as follows:

$$\lambda_k := \inf_{\{u \perp \{u_1,\ldots,u_{k-1}\}, \|u\|_2=1\}} \int_{B_1} |\nabla u|^2 \,\mathrm{d}x,\ k = 1, 2, 3, \ldots,$$

where u_i is an eigenfunction corresponding to λ_i. For $p \neq 2$, using the Ljusternik-Schnirelman theorem, an infinite sequence $\{\mu_n\}$ of eigenvalues of (1.1) is provided in [8]. Possibly a different sequence $\{\lambda_n\}$ of variational eigenvalues of (1.1) is provided in [5]. We stress that a complete description of the set of all eigenvalues of (1.1) for $p \neq 2$ is a challenging open problem. Nevertheless, a complete description of the set of all radial eigenvalues $\{\gamma_n\}$ (an eigenvalue with a radial eigenfunction) of (1.1) is given in [3]. The authors of [3] showed that λ is a radial eigenvalue of (1.1) if and only if the following ODE has a nonzero solution:

$$\begin{aligned} -\left(r^{N-1}\,|u'(r)|^{p-2}u'(r)\right)' &= \lambda r^{N-1}\,|u(r)|^{p-2}u(r) \quad \text{in } (0,1), \\ u'(0) = 0, &\quad u(1) = 0. \end{aligned} \tag{1.2}$$

Regardless of the methods by which the eigenvalues are obtained, one can uniquely identify the first two eigenvalues of (1.1) as below:

$$\begin{aligned} \lambda_1 &= \min\{\lambda : \lambda \text{ is an eigenvalue of } (1.1)\}, \\ \lambda_2 &= \min\{\lambda > \lambda_1 : \lambda \text{ is an eigenvalue of } (1.1)\}. \end{aligned}$$

It is well known that the eigenfunctions corresponding to λ_1 are radial and keep the same sign on B_1. All other eigenfunctions change their sign on B_1. The structures of the second eigenfunctions are not well understood, except for $p = 2$. In this case, the Fourier method for the Laplacian in the polar coordinates gives the precise form of the second eigenfunctions. In particular, it is evident that the second eigenfunctions are not radial. One anticipates the same results as well for $p \neq 2$.

In [13], Parini proved that the second eigenfunctions are not radial in a special case, where B_1 is the disc ($B_1 \subset \mathbb{R}^2$) and p is close to 1. In [1], this result is extended for every $p \in (1, \infty)$ using a computer aided proof. Indeed, these methods are not readily extendable to dimensions greater than 2. Here, we give a simple analytic proof for their result which works in all dimensions ($N \geq 2$) and for every $p \in (1, \infty)$. Our proof relies on the variational characterization of λ_2 given in [5] and a variation of the deformation lemma given in [9]. We also use a result of [2] that gives the monoticity of the first eigenvalue with respect to annular domains in B_1. More precisely, for a fixed $r \in (0, 1)$, let $B_r(x) \subset B_1$ be the ball with centre x and radius r. Then Theorem 2 of [2] gives that $\lambda_1(B_1 \setminus \overline{B_r(x)})$ is monotonically decreasing as $|x| \to 1 - r$. It is worth mentioning that the strict monoticity of $\lambda_1(B_1 \setminus \overline{B_r(x)})$ is still an open problem for $p \neq 2$. In [7], the strict monoticity of $\lambda_1(B_1 \setminus \overline{B_r(x)})$ is obtained for certain weighted eigenvalue problems; unfortunately their results do not fit in our case.

Now we state our main result:

Theorem 1.1. *Let B_1 be the unit ball centred at the origin in $\mathbb{R}^N$ with $N \geq 2$ and let $1 < p < \infty$. Let λ_2 be the second eigenvalue of* (1.1). *Then the eigenfunctions corresponding to λ_2 are not radial.*

In this paper we also construct a sequence $\{\tau_n, \Psi_n\}$ of eigenpairs of (1.1) such that the eigenfunction Ψ_n is nonradial and has exactly $2n$ nodal domains. Furthermore, the sequence $\{\tau_n\}$ is strictly increasing and unbounded. In fact the nodal domains can be specified using the spherical coordinate system for $\mathbb{R}^N$ which consists of a radial coordinate r and angular coordinates $\theta_1, \ldots, \theta_{N-1}$ where $\theta_1, \ldots, \theta_{N-2} \in [0, \pi]$ and $\theta_{N-1} \in [0, 2\pi)$. By a sector of the ball B_1 we mean the set S given by $S = \{x \in B_1 : 0 < \theta_* < \theta_{N-1} < \theta^* < 2\pi\}$. Now we state our next result.

Theorem 1.2. *Let $B_1 \subset \mathbb{R}^N$. Then for each $n \in \mathbb{N}$ there exists an eigenpair $\{\tau_n, \Psi_n\}$ of* (1.1) *such that Ψ_n has exactly $2n$ nodal domains where each nodal domain is a sector with measure $\frac{|B_1|}{2n}$.*

The rest of this paper is organized as follows. In Section 2, we consider a Dirichlet eigenvalue for the p-Laplacian on a general domain and discuss the existence and the regularity properties of the eigenfunctions. We also discuss the variational characterizations of eigenvalues and state a version of the deformation lemma. In Section 3, we give a proof for Theorem 1.1. The last section consists of a proof of Theorem 1.2 and some important open problems related to eigenvalues of the p-Laplacian.

2. Preliminary

In this section we consider the eigenvalue problem on a bounded domain Ω in $\mathbb{R}^N$:

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2} u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

We discuss the existence and regularity properties of the eigenfunctions of (2.1). If λ is an eigenvalue of (2.1) and $u \in W_0^{1,p}(\Omega)$ is an associated eigenfunction, then we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, \mathrm{d}x = \lambda \int_\Omega |u|^{p-2} u v \, \mathrm{d}x, \quad \forall\, v \in W_0^{1,p}(\Omega). \tag{2.2}$$

Now we consider the following two functionals on $W_0^{1,p}(\Omega)$:

$$J(u) = \int_\Omega |\nabla u|^p \mathrm{d}x, \qquad G(u) = \int_\Omega |u|^p \mathrm{d}x.$$

Using the Lagrange multiplier theorem, it can be easily verified that the critical values and critical points of J on the manifold $\mathcal{S} = G^{-1}(1)$ satisfy (2.2). Indeed, the eigenvalues of (2.1) and the critical values of J on S are one and the same. The least critical value of J on $\mathcal{S}$ is given by

$$\lambda_1 = \inf_{u \in \mathcal{S}} J(u).$$

In the next proposition, we list some of the important properties of λ_1 and the corresponding eigenfunctions.

Proposition 2.1. *Let λ_1 be the first eigenvalue of* (2.1). *Then*

(i) *λ_1 is simple,*
(ii) *any eigenfunction corresponding to λ_1 keeps the same sign on Ω,*
(iii) *any eigenfunction corresponding to an eigenvalue $\lambda > \lambda_1$ changes its sign on Ω,*
(iv) *if $\Omega = B_r(0)$, then the eigenfunctions corresponding to λ_1 are radial.*

Proof. A proof of (i) (for nonsmooth Ω) can be found in Lemma 3.2 of [10] and (iii) in Lemma 3.1 of [10]. For a proof of (ii), see Lemma 2.4 of [12]. Finally (iv) is evident from (i) and (iii) by noting the existence of a radial positive eigenfunction for (2.1) when $\Omega = B_r(0)$. $\square$

An infinite set of critical values of J on S are obtained in [8] using the variational methods. Their approach relies on the notion of Krasnoselskii genus of a symmetric closed set. For a symmetric closed subset $\mathcal{A} \subset \mathcal{S}$, Krasnoselskii genus of $\mathcal{A}$ is defined as

$$\gamma(\mathcal{A}) := \inf\{n \in \mathbb{N} : \exists \text{ a continuous odd map from } \mathcal{A} \text{ into } \mathbb{R}^n \setminus \{0\}\}$$

with the convention $\inf\{\emptyset\} = \infty$. For each $n \in \mathbb{N}$, let

$$\begin{aligned} \mathcal{E}_n &:= \left\{\mathcal{A} \subset \mathcal{S} : \mathcal{A} = \overline{\mathcal{A}},\ \mathcal{A} = -\mathcal{A} \text{ and } \gamma(\mathcal{A}) \geq n\right\}, \\ \mu_n &:= \inf_{\mathcal{A} \in \mathcal{E}_n} \sup_{u \in \mathcal{A}} J(u). \end{aligned}$$

Then μ_n is a critical value of J on $\mathcal{S}$ (see Proposition 5.4 of [8]). Possibly another set of critical values is obtained in [5] by considering a special collection of sets with genus n in $\mathcal{S}$. Note that the unit sphere $\mathcal{S}^{n-1}$ in $\mathbb{R}^n$ has genus n, and hence its image under an odd continuous map has the same genus. For each $n \in \mathbb{N}$, let

$$\begin{aligned} \mathcal{F}_n &:= \left\{\mathcal{A} \subset \mathcal{S} :\ \mathcal{A} = h(\mathcal{S}^{n-1}),\ h \text{ is an odd continuous map from } \mathcal{S}^{n-1} \to \mathcal{S}\right\}, \\ \mu_n^* &:= \inf_{\mathcal{A} \in \mathcal{F}_n} \sup_{u \in \mathcal{A}} J(u). \end{aligned}$$

Then μ_n^* is a critical value of J on $\mathcal{S}$ (see Theorem 5 of [5]). Since $\mathcal{F}_n \subset \mathcal{E}_n$, we always have $\mu_n \leq \mu_n^*$. It is known that $\lambda_i = \mu_i = \mu_i^*$ for $i = 1, 2$. This result for $i = 1$ follows as the set $\{u, -u\}$ lies in both $\mathcal{E}_1$ and $\mathcal{F}_1$ for $u \in \mathcal{S}$. Let u be an eigenfunction corresponding to λ_2. Then by (ii) of Proposition 2.1 both u^+ and u^- are nonzero. Thus the set $\mathcal{A} := \left\{au^+ + bu^- : |a|^p \|u^+\|_p^p + |b|^p \|u^-\|_p^p = 1\right\}$ lies in both $\mathcal{E}_2$ and $\mathcal{F}_2$. Now as $J(au^+ + bu^-) = \lambda_2$, we get $\mu_2 \leq \lambda_2$ and $\mu_2^* \leq \lambda_2$. Since there is no eigenvalue between λ_1 and λ_2, it follows that $\lambda_2 = \mu_2 = \mu_2^*$. In particular, we have the following variational characterization of λ_2 that we use later:

$$\lambda_2 = \inf_{\mathcal{A} \in \mathcal{F}_2} \sup_{u \in \mathcal{A}} J(u). \tag{2.3}$$

The next proposition is a consequence of the deformation lemma (see Lemma 3.7 of [9]; see also Theorem 2.1 and Remark 2.3 of [6]). Note that $J \in \mathcal{C}^1(W_0^{1,p}(\Omega); \mathbb{R})$ and $\mathcal{S}$ is a $\mathcal{C}^1$-manifold. Further, $J(u) = J(-u)$ and $\mathcal{S} = -\mathcal{S}$.

Proposition 2.2. *Let $\mathcal{S}, J$ be as before. Let $\mathcal{K}$ be a compact subset of S. If $\|J'(u)\|_* \geq \varepsilon > 0$ for all $u \in \mathcal{K}$, then there exists a continuous one parameter family of homeomorphisms $\Psi : S \times [0,1] \to S$ such that*

(i) *$J(\Psi(u,t)) \leq J(u) - \varepsilon t$, for every $u \in \mathcal{K},\ t \in [0,1]$,*
(ii) *$\Psi(-u,t) = -\Psi(u,t)$, for all $u \in \mathcal{S},\ t \in [0,1]$.*

In particular, if $\mathcal{K} \in \mathcal{F}_n$ and J has no critical point on $\mathcal{K}$, then the set $\widetilde{\mathcal{K}} = \{\Psi(u,1) : u \in \mathcal{K}\}$ is in $\mathcal{F}_n$ and

$$\sup_{u \in \widetilde{\mathcal{K}}} J(u) < \sup_{u \in \mathcal{K}} J(u). \tag{2.4}$$

We also need the following result on the regularity of the eigenfunctions of (2.1) which is a consequence of Theorem 1 of [11].

Proposition 2.3. *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. Let ϕ be an eigenfunction of* (2.1)*. Then there exists $\alpha \in (0,1)$ such that $\phi \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$.*

3. Radial asymmetry of the second eigenfunctions

In this section we prove our main result. First we state a lemma that follows from Proposition 4.1 of [3].

Lemma 3.1. *Let γ_2 be the second radial eigenvalue of* (1.2)*. Then any radial eigenfunction corresponding to γ_2 has exactly two nodal domains — a ball and an annulus with centre at the origin. In particular, there exists $r \in (\frac{1}{2}, 1)$ such that $\lambda_1(B_r(0)) = \gamma_2 = \lambda_1(B_1 \setminus \overline{B_r(0)})$.*

Now using the 'r' given by the above lemma, we construct a special collection of sets in $\mathcal{F}_2$. For each $n \in \mathbb{N} \cup \{0\}$, we construct a special set $\mathcal{A}_n \in \mathcal{F}_2$ such that $\sup_{u \in \mathcal{A}_n} J(u) = \gamma_2$. Let $\{t_n\}$ be a sequence in $[0, 1-r)$ such that $t_0 = 0$ and $t_n \to 1-r$. For each $n \in \mathbb{N} \cup \{0\}$, let $\Omega_n = B_1 \setminus \overline{B_r(t_n e_1)}$, where e_1 is the unit vector in the direction of the first coordinate axis. Let u_n, v_n be the respective first eigenfunctions on $B_r(t_n e_1)$ and Ω_n satisfying $u_n > 0$ on B_n, $v_n > 0$ on Ω_n and $\|u_n\|_p = \|v_n\|_p = 1$. By translation invariance of the p-Laplacian, we have $\lambda_1(B_r(t_n e_1)) = \gamma_2$. Further, from Theorem 1 of [2], we also have $\lambda_1(\Omega_n) \leq \gamma_2$. Let $\widetilde{u}_n$ and $\widetilde{v}_n$ be the zero extensions to the entire B_1. For each $n \in \mathbb{N} \cup \{0\}$, we consider

$$\mathcal{A}_n := \{a\widetilde{u}_n + b\widetilde{v}_n : |a|^p + |b|^p = 1\}.$$

One can easily verify that $\mathcal{A}_n \in \mathcal{F}_2$ and $\sup_{u \in \mathcal{A}_n} J(u) = \gamma_2, \forall n \in \mathbb{N} \cup \{0\}$.

Now we ask the question of whether or not $\mathcal{A}_n$ contains a critical point of J on S. This leads to the following two alternatives:

(i) for every $n \in \mathbb{N}$, $\mathcal{A}_n$ contains at least one critical point of J on S,
(ii) there exists $n_0 \in \mathbb{N}$ such that $\mathcal{A}_{n_0}$ does not contain any critical point of J on S.

In the next lemma we show that alternative (i) does not hold.

Lemma 3.2. *Let $\mathcal{A}_n$ be as above. Then alternative* (i) *does not hold.*

Proof. Let u_n and $\widetilde{u}_n$ be as above. Then $u_n(x) = u_0(x - t_n e_1)$, and hence the sequence $\{\widetilde{u}_n(x)\}$ converges to $u^*(x) = \widetilde{u}_0(x - (1-r)e_1)$ both pointwise and in $W_0^{1,p}(B_1)$. On the other hand, the sequence $\{\widetilde{v}_n\}$ is bounded by γ_2 in $W_0^{1,p}(B_1)$. Thus, up to a subsequence, $\widetilde{v}_n$ converges to some v^* weakly in $W_0^{1,p}(B_1)$ and a.e. in B_1. If alternative (i) holds, then we get a sequence $\{\phi_n = a_n\widetilde{u}_n + b_n\widetilde{v}_n : |a_n|^p + |b_n|^p = 1\}$ of eigenfunctions of (1.1) with eigenvalues $J(\phi_n)$. By Proposition 2.3, the eigenfunctions are in $\mathcal{C}^1(\overline{B_1})$, and hence we must have $a_n b_n < 0$. Now we may assume that $a_n > 0$ and $b_n < 0$ for each n. Further, the sequences $\{J(\phi_n)\}, \{a_n\}$ and $\{b_n\}$ are bounded. Thus for a subsequence we get $J(\phi_n) \to \lambda^*$, $a_n \to a^*$ and $b_n \to b^*$ for some $\lambda^*, a^* \geq 0$ and $b^* \leq 0$. The sequence $\{\phi_n\}$ is bounded in $W_0^{1,p}(B_1)$ and hence up to a subsequence $\phi_n \rightharpoonup \phi^*$ in $W_0^{1,p}(B_1)$ and a.e. in B_1. Since $a_n\widetilde{u}_n + b_n\widetilde{v}_n \to a^*u^* + b^*v^*$ a.e. in B_1, we must have

$$\phi^* = a^*u^* + b^*v^*.$$

Since each ϕ_n is an eigenfunction of (1.1), it is easy to verify that ϕ^* is an eigenfunction corresponding to the eigenvalue λ^*. Thus by the regularity of ϕ^*, we must have $a^*b^* < 0$, and hence

$$a^* > 0, \quad b^* < 0.$$

Let $B^* = B_r((1-r)e_1)$ and $\Omega^* = B_1 \setminus B^*$. Clearly $u^* > 0$ on B^* and $u^* = 0$ on Ω^*. On the other hand, $v^* = 0$ a.e. in B^* and $v^* \geq 0$ a.e. on Ω^*. Thus from the continuity of the ϕ^* we get

$$\phi^*(x) > 0, \ \forall\, x \in B^*, \qquad \phi^*(x) \leq 0, \ \forall\, x \in \Omega^*.$$

Now we apply Theorem 5 of [16] (a Hopf's lemma type result for p-Laplacian) on $B^* \cup \{e_1\}$ to get

$$\frac{\partial \phi^*}{\partial x_1}(e_1) = c < 0.$$

Since $\phi^* \leq 0$ on Ω^* we also have

$$\frac{\partial \phi^*}{\partial \eta(x)}(x) \geq 0, \ \forall\, x \in \partial B_1 \setminus \{e_1\},$$

where $\eta(x)$ is the outward unit normal to B_1 at x. The above two inequalities contradict the fact that ϕ^* is in $\mathcal{C}^1(\overline{B_1})$. Thus we conclude that alternative (i) does not hold. □

Proof of Theorem 1.1. Let $\mathcal{A}_n$ be as before. Thus we have $\sup_{v \in \mathcal{A}_n} J(v) \leq \gamma_2$. By the above lemma, the alternative (ii) holds, i.e. there exists $n_0 \in \mathbb{N}$ such that $\mathcal{A}_{n_0}$ does not contain any critical points of J on $\mathcal{S}$. Thus by Proposition 2.2 and by (2.4), we get $\widetilde{\mathcal{A}}_{n_0} \in \mathcal{F}_2$ such that

$$\sup_{u \in \widetilde{\mathcal{A}}_{n_0}} J(u) < \sup_{v \in \mathcal{A}_{n_0}} J(v) \leq \gamma_2.$$

Now from (2.3) we get $\lambda_2 < \gamma_2$. □

4. Construction of nonradial eigenfunctions

In this section we construct an infinite sequence of nonradial eigenfunctions of (1.1). First we fix the following conventions. A vector x in $\mathbb{R}^N$ is always taken as a $1 \times N$ row vector, i.e $x = (x_1, x_2, \ldots, x_N)$. The transpose of x, denoted by x^T, is an $N \times 1$ column vector. We denote the scalar product in $\mathbb{R}^N$ by $x \cdot y \,(= xy^T)$. Let H be the hyperplane given by $H = \{x \in \mathbb{R}^N : x \cdot a = 0\}$ for some unit vector $a \in \mathbb{R}^N$. Let σ_H be the reflection about H. Then

$$\sigma_H(x) = x - 2(x \cdot a)a = x(I - 2a^T a).$$

Next we list some of the elementary properties of σ_H that we use in this article.

(i) σ_H is linear and $\sigma_H = (I - 2a^T a)$.
(ii) ${\sigma_H}^{-1} = \sigma_H$.
(iii) σ_H is symmetric and orthogonal.
(iv) $D\sigma_H(x) = \sigma_H$ and $\det D\sigma_H(x) = -1, \ \forall\, x \in \mathbb{R}^N$.

Let $\mathcal{O}$ be a bounded domain symmetric about H, i.e, $\sigma_H(\mathcal{O}) = \mathcal{O}$. Let $\mathcal{O}^+ := \{x \in \mathcal{O} : x \cdot a > 0\}$ and let $\mathcal{O}^- = \sigma_H(\mathcal{O}^+)$. Let $u \in W_0^{1,p}(\mathcal{O}^+)$ be a weak solution of (2.1) on $\Omega = \mathcal{O}^+$. Define u^* on $\mathcal{O}$ as

$$u^*(x) = \begin{cases} u(x), \quad x \in \mathcal{O}^+, \\ 0, \quad x \in \partial(\mathcal{O}^+) \cup \partial(\mathcal{O}^-), \\ -u(\sigma_H(x)), \quad x \in \mathcal{O}^-. \end{cases}$$

Clearly $u^* \in W_0^{1,p}(\mathcal{O})$, and we also have the following lemma:

Lemma 4.1. *Let u^* be defined as above. Then u^* is a weak solution of* (2.1) *on* $\Omega = \mathcal{O}$.

Proof. Let $\phi \in W_0^{1,p}(\mathcal{O})$ be a test function. We show that

$$\int_{\mathcal{O}} |\nabla u^*(x)|^{p-2}\nabla u^*(x) \cdot \nabla\phi(x)\mathrm{d}x = \lambda \int_{\mathcal{O}} |u^*(x)|^{p-2}u^*(x)\phi(x)\mathrm{d}x. \tag{4.1}$$

From the definition of u^*,

$$\begin{aligned}\int_{\mathcal{O}} &|\nabla u^*(x)|^{p-2}\nabla u^*(x) \cdot \nabla\phi(x)\mathrm{d}x \\ &= \int_{\mathcal{O}^+} |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla\phi(x)\mathrm{d}x \\ &+ \int_{\mathcal{O}^-} |\nabla(-u(\sigma_H(x)))|^{p-2}\nabla(-u(\sigma_H(x))) \cdot \nabla\phi(x)\mathrm{d}x.\end{aligned}$$

Now by noting that $D\sigma_H(x) = \sigma_H$ and σ_H is an isometry we get

$$\begin{aligned}\int_{\mathcal{O}^-} &|\nabla(-u(\sigma_H(x)))|^{p-2}\nabla(-u(\sigma_H(x))) \cdot \nabla\phi(x)\mathrm{d}x \\ &= -\int_{\mathcal{O}^-} |\nabla u(\sigma_H(x))\sigma_H|^{p-2}[\nabla u(\sigma_H(x))\sigma_H] \cdot \nabla\phi(x)\mathrm{d}x \\ &= -\int_{\mathcal{O}^-} |\nabla u(\sigma_H(x))|^{p-2}\nabla u(\sigma_H(x)) \cdot [\nabla\phi(x)\sigma_H]\mathrm{d}x,\end{aligned}$$

where the equality in the last step also uses the fact that σ_H is symmetric. Now the change of variable $y = \sigma_H(x)$ along with properties (ii) and (iv) of σ_H will give

$$\begin{aligned}\int_{\mathcal{O}^-} &|\nabla(-u(\sigma_H(x)))|^{p-2}\nabla(-u(\sigma_H(x))) \cdot \nabla\phi(x)\mathrm{d}x \\ &- \int_{\mathcal{O}^+} |\nabla u(y)|^{p-2}\nabla u(y) \cdot [\nabla\phi(\sigma_H(y))\sigma_H]\mathrm{d}y.\end{aligned}$$

Thus

$$\begin{aligned}\int_{\mathcal{O}} &|\nabla u^*(x)|^{p-2}\nabla u^*(x) \cdot \nabla\phi(x)\mathrm{d}x \\ &= \int_{\mathcal{O}^+} |\nabla u(x)|^{p-2}\nabla u(x) \cdot [\nabla\phi(x) - [\nabla\phi(\sigma_H(x))\sigma_H]]dx.\end{aligned}$$

Let $\psi(x) = \phi(x) - \phi(\sigma_H(x))$. Then we have

$$\int_{\mathcal{O}} |\nabla u^*(x)|^{p-2}\nabla u^*(x) \cdot \nabla\phi(x)\mathrm{d}x = \int_{\mathcal{O}^+} |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla\psi(x)dx. \tag{4.2}$$

Further,

$$\int_{\mathcal{O}} |u^*(x)|^{p-2}u^*(x)\phi(x) = \int_{\mathcal{O}^+} |u(x)|^{p-2}u(x)\psi(x)\mathrm{d}x. \tag{4.3}$$

Clearly $\psi \in W_0^{1,p}(\mathcal{O}^+)$ and hence

$$\int_{\mathcal{O}} |\nabla u^*(x)|^{p-2}\nabla u^*(x)\cdot\nabla\phi(x)\mathrm{d}x = \int_{\mathcal{O}^+} |u(x)|^{p-2}u(x)\psi(x)\mathrm{d}x, \tag{4.4}$$

since u solves (2.1) on $\Omega = \mathcal{O}^+$. Now (4.1) follows from (4.2), (4.3) and (4.4). $\square$

Proof of Theorem 1.2. For $n \in \mathbb{N}$, we consider the sectors S_k given by $S_k = \{x \in B_1 : \frac{(k-1)\pi}{n} < \theta_{N-1} < \frac{k\pi}{n}\}, k = 1,\dots,n$. Let H_k be the hyperplane given by $H_k = \{x \in \mathbb{R}^N : \theta_{N-1} = \frac{\pi k}{n}\}$, for $k = 1,\dots,n$. Let τ_n be the first eigenvalue for the p-Laplacian on S_1 and $u_1(x)$ be a corresponding eigenfunction. For $i = 2,\dots,n$, we define u_i recursively by $u_i = -u_{i-1}(\sigma_{H_{i-1}}(x))$, the odd reflection of u_i about H_{i-1}. Let D^+ be the sector given by $\{x \in B_1 : 0 < \theta_{N-1} < \pi\}$. Now we define u^* on $\overline{D^+}$ by

$$u^*(x) = u_i(x), \quad x \in \overline{S_i}, \quad i = 1,\dots,n.$$

From Lemma 4.1, it is clear that u^* solves (2.1) on the union of two adjacent sectors with $\lambda = \tau_n$. Let $U_i = \{x \in B_1 : \frac{(i-1)\pi}{n} < \theta_{N-1} < \frac{(i+1)\pi}{n}\}$, for $i = 1,\dots,n-1$. Then $\{U_i\}_{i=1}^{n-1}$ is an open covering of D^+. Let $\{\phi_i\}_{i=1}^{n-1}$ be a $\mathcal{C}^\infty$-partition of the unity corresponding to this open covering. Note that for each i, ϕ_i intersects at most S_i and S_{i+1}. Since $\sum_{i=1}^{n-1}\phi_i = 1$, we have

$$\begin{aligned}\int_{D^+} |\nabla u^*(x)|^{p-2}\nabla u^*(x)\cdot\nabla\phi(x)\mathrm{d}x &= \int_{D^+} |\nabla u^*(x)|^{p-2}\nabla u^*(x)\cdot\nabla\Big(\phi(x)\sum_{i=1}^{n-1}\phi_i(x)\Big)\mathrm{d}x\\ &= \sum_{i=1}^{n-1}\int_{D^+} |\nabla u^*(x)|^{p-2}\nabla u^*(x)\cdot\nabla(\phi(x)\phi_i(x))\mathrm{d}x.\end{aligned}$$

For a fixed i, the product $\phi\phi_i \in W_0^{1,p}(U_i)$. Hence by the definition of u^* and Lemma 4.1, we get

$$\begin{aligned}&\int_{D^+} |\nabla u^*(x)|^{p-2}\nabla u^*(x)\cdot\nabla(\phi(x)\phi_i(x))\mathrm{d}x\\ &\quad= \int_{U_i} |\nabla u^*(x)|^{p-2}\nabla u^*(x)\cdot\nabla(\phi(x)\phi_i(x))\mathrm{d}x\\ &\quad= \tau_n\int_{U_i} |u^*(x)|^{p-2}u^*(x)(\phi(x)\phi_i(x))\mathrm{d}x\\ &\quad= \tau_n\int_{D^+} |u^*(x)|^{p-2}u^*(x)(\phi(x)\phi_i(x))\mathrm{d}x.\end{aligned}$$

Thus we get

$$\begin{aligned}\int_{D^+} |\nabla u^*(x)|^{p-2}\nabla u^*(x)\cdot\nabla\phi(x)\mathrm{d}x &= \sum_{i=1}^{n-1}\tau_n\int_{D^+}|u^*(x)|^{p-2}u^*(x)(\phi(x)\phi_i(x))\mathrm{d}x\\ &= \tau_n\int_{D^+}|u^*(x)|^{p-2}u^*(x)\Big(\sum_{i=1}^{n-1}\phi(x)\phi_i(x)\Big)\mathrm{d}x\\ &= \tau_n\int_{D^+}|u^*(x)|^{p-2}u^*(x)\phi(x)\mathrm{d}x.\end{aligned}$$

Now define Ψ_n on B_1 by

$$\Psi_n(x) = \begin{cases} u^*(x), \quad x\in D^+,\\ 0, \quad x\in\partial(D^+)\cup\partial(D^-),\\ -u^*(\sigma_{H_0}(x)), \quad x\in D^-,\end{cases}$$

where $D^- = \{x\in B_1 : \pi < \theta_{N-1} < 2\pi\}$ is the "lower" half-ball and H_0 is the hyperplane corresponding to $\theta_{N-1}=0$. Applying Lemma 4.1 once again, we get that Ψ_n is a weak solution of (1.1). Thus we have constructed an eigenpair $\{\tau_n,\Psi_n\}$ of (1.1) such that Ψ_n has $2n$ nodal domains and each nodal domain is a sector with measure $\frac{|B_1|}{2n}$. □

In the next remark we list some of the interesting open problems related to the results of this paper:

Remark 4.2 (Open problems associated with (1.1)).

(1) Payne conjectured (Conjecture 5 of [14]) that the nodal line of a second eigenfunction of Laplacian on a bounded domain $\Omega\subset\mathbb{R}^2$ cannot be a closed curve. In [15], he proved his conjecture for the special case when Ω is convex in x and symmetric about the y axis. For a ball, his result was easily obtained by applying the Fourier method to the Laplacian in polar coordinates. We conjecture that the nodal surface of a second eigenfunction of p-Laplacian on B_1 cannot be a closed surface in B_1 for $1<p<\infty$ and for every $N\geq 2$.

(2) For $p=2$, it is easy to see that $\lambda_2=\tau_1$. We anticipate the same result for $p\neq 2$ as well. More precisely, the nodal surface of any second eigenfunction is given by the intersection of a hyperspace with B_1 and the nodal domains are the half-balls symmetric to this hyperspace.

(3) We have just shown that all the eigenfunctions corresponding to λ_2 are nonradial. Is it true that all the eigenfunctions corresponding to the second radial eigenvalue γ_2 are radial?

(4) Note that λ_2 is the least eigenvalue having an eigenfunction with two nodal domains. For $p=2$, it can also be seen that γ_2 is the maximal eigenvalue having an eigenfunction with two nodal domains. In other words, the eigenfunctions corresponding to $\lambda>\gamma_2$ must have at least three nodal domains. Is this true for $p\neq 2$?

References

[1] Jiří Benedikt, Pavel Drábek, and Petr Girg, *The second eigenfunction of the p-Laplacian on the disk is not radial*, Nonlinear Anal. **75** (2012), no. 12, 4422–4435, DOI 10.1016/j.na.2011.06.012. MR2927111

[2] A. Chorwadwala and R. Mahadevan, *A shape optimization problem for the p-laplacian*, ArXiv e-prints (accepted in Proceedings of Royal Society of Edinburgh).

[3] Manuel A. del Pino and Raúl F. Manásevich, *Global bifurcation from the eigenvalues of the p-Laplacian*, J. Differential Equations **92** (1991), no. 2, 226–251, DOI 10.1016/0022-0396(91)90048-E. MR1120904 (92g:35028)

[4] Pavel Drábek and Jaroslav Milota, *Methods of nonlinear analysis.* Applications to differential equations, 2nd ed., Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser/Springer Basel AG, Basel, 2013. MR3025694

[5] Pavel Drábek and Stephen B. Robinson, *Resonance problems for the p-Laplacian*, J. Funct. Anal. **169** (1999), no. 1, 189–200, DOI 10.1006/jfan.1999.3501. MR1726752 (2000j:35096)

[6] Pavel Drábek and Stephen B. Robinson, *On the generalization of the Courant nodal domain theorem*, J. Differential Equations **181** (2002), no. 1, 58–71, DOI 10.1006/jdeq.2001.4070. MR1900460 (2003g:35170)

[7] B. Emamizadeh and M. Zivari-Rezapour, *Monotonicity of the principal eigenvalue of the p-Laplacian in an annulus*, Proc. Amer. Math. Soc. **136** (2008), no. 5, 1725–1731, DOI 10.1090/S0002-9939-07-09153-8. MR2373602 (2009b:35105)

[8] J. P. García Azorero and I. Peral Alonso, *Existence and nonuniqueness for the p-Laplacian: nonlinear eigenvalues*, Comm. Partial Differential Equations **12** (1987), no. 12, 1389–1430, DOI 10.1080/03605308708820534. MR912211 (89e:35058)

[9] Nassif Ghoussoub, *Duality and perturbation methods in critical point theory.* With appendices by David Robinson, Cambridge Tracts in Mathematics, vol. 107, Cambridge University Press, Cambridge, 1993. MR1251958 (95a:58021)

[10] Bernd Kawohl and Peter Lindqvist, *Positive eigenfunctions for the p-Laplace operator revisited*, Analysis (Munich) **26** (2006), no. 4, 545–550, DOI 10.1524/anly.2006.26.4.545. MR2329592 (2008g:35151)

[11] Gary M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), no. 11, 1203–1219, DOI 10.1016/0362-546X(88)90053-3. MR969499 (90a:35098)

[12] Peter Lindqvist, *On the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, Proc. Amer. Math. Soc. **109** (1990), no. 1, 157–164, DOI 10.2307/2048375. MR1007505 (90h:35088)

[13] Enea Parini, *The second eigenvalue of the p-Laplacian as p goes to* 1, Int. J. Differ. Equ. (2010), Art. ID 984671, 23. MR2575290 (2010m:35357)

[14] L. E. Payne, *Isoperimetric inequalities and their applications*, SIAM Rev. **9** (1967), 453–488. MR0218975 (36 #2058)

[15] Lawrence E. Payne, *On two conjectures in the fixed membrane eigenvalue problem* (English, with German summary), Z. Angew. Math. Phys. **24** (1973), 721–729. MR0333487 (48 #11812)

[16] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), no. 3, 191–202, DOI 10.1007/BF01449041. MR768629 (86m:35018)

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