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MOCK MODULAR FORMS AND QUANTUM MODULAR FORMS

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ABSTRACT. In his last letter to Hardy, Ramanujan introduced mock theta functions. For each of his examples $f(q)$, Ramanujan claimed that there is a collection $\{G_j\}$ of modular forms such that for each root of unity ζ, there is a j such that

$$\lim_{q\to\zeta}(f(q)-G_j(q)) = O(1).$$

Moreover, Ramanujan claimed that this collection must have size larger than 1. In his 2001 PhD thesis, Zwegers showed that the mock theta functions are the holomorphic parts of harmonic weak Maass forms. In this paper, we prove that there must exist such a collection by establishing a more general result for all holomorphic parts of harmonic Maass forms. This complements the result of Griffin, Ono, and Rolen that shows such a collection cannot have size 1. These results arise within the context of Zagier's theory of quantum modular forms. A linear injective map is given from the space of mock modular forms to quantum modular forms. Additionally, we provide expressions for "Ramanujan's radial limits" as L-values.

1. Introduction

In his 1920 letter to Hardy [1], Ramanujan constructed seventeen examples which he called mock theta functions. Ramanujan claimed that these functions satisfy the following:

(1) They have infinitely many singularities at rational numbers.
(2) It is impossible to construct a single modular form to cut out all of the singularities.
(3) It is possible to find a collection of modular forms to cut out the singularities.

Watson [17], in 1936, was the first to prove that there is a collection of modular forms cutting out the singularities of Ramanujan's examples. Very recently, in 2013, it was proved by Griffin, Ono, and Rolen [11], that there is no single modular form which cuts out all of the singularities simultaneously.

Zwegers [21,22] showed that each of Ramanujan's examples, denoted $f(z)$, is, up to a power of $e^{2\pi iz}$ with $z\in\mathbb{H}=\{x+iy : x,y\in\mathbb{R} \text{ and } y>0\}$, the holomorphic part of some harmonic weak Maass forms of weight $\frac{1}{2}$ for some modular group $\Gamma_0(N)$,

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denoted $M_f(z)$ (defined by Bruinier and Funke [5], see Section 2). Additionally, $\xi_{1/2}(M_f)(z)$ is a cusp form of weight $\frac{3}{2}$, where $\xi_k := 2iy^k\overline{\frac{\partial}{\partial \bar{z}}}$. This led Zagier to define a *mock modular form*, called $f(z)$, as the holomorphic part of any harmonic weak Maass form $M(z)$ of weight k such that $\xi_k(M)(z)$, called the *shadow of* $f(z)$, is a cusp form. Thus, the mock theta functions of Ramanujan are examples of weight $\frac{1}{2}$ of Zagier's mock modular forms. For the applications of mock modular forms, we recommend Ono's survey paper [15]. We denote by $\mathbb{M}_{k,\chi}(\Gamma_0(N))$ the space of mock modular forms of weight k and multiplier system χ on $\Gamma_0(N)$.

Our first theorem proves that any mock modular form has the three properties described by Ramanujan in his last letter.

Theorem 1.1. *Assume $f(z)$ is a mock modular form so that $M_f(z) = f(z)+f^-(z)$ is a harmonic Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$ for $\Gamma_0(N)$ such that $\Gamma_0(N)$ has $t > 1$ inequivalent cusps $\{q_1, \dots, q_t\} \subset \mathbb{Q} \cup \{\infty\}$ and $f^-(z)$ is non-zero. Then:*

1. *$f(z)$ has exponential singularities at infinitely many rational numbers,*
2. *for every weakly holomorphic modular form $g(z)$ of weight k, $f(z) - g(z)$ has exponential singularities at infinitely many rational numbers,*
3. *there is a collection $\{G_j\}_{j=1}^t$ of weakly holomorphic modular forms such that $f(z) - G_j(z)$ is bounded toward all rational numbers equivalent to the cusp q_j.*

Remark 1.2. Parts (1) and (2) follow from the work of Griffin, Ono, and Rolen [11] and are based on the results of Bruinier and Funke [5]. Part (3) is established using a generalization of a result of Borcherds [2].

Remark 1.3. The only N for which $\Gamma_0(N)$ has one cusp is $N = 1$. However, if f is a mock modular form for $\Gamma_0(1)$, then f is also a mock modular form for $\Gamma_0(N)$ for any $N > 1$. Thus, in fact, Theorem 1.1 implies that Ramanujan's claim is also true for mock modular forms on $\Gamma_0(1)$.

Remark 1.4. Ramanujan referred to "theta functions" rather than modular forms in his letter. However, we understand today that the theta functions Ramanujan referred to may be taken to be weakly holomorphic modular forms. A weakly holomorphic modular form is a meromorphic modular form whose poles (if any) are supported at the cusps of $\Gamma_0(N)$.

Using Theorem 1.1, for a mock modular form $f(z)$, associated to a harmonic weak Maass form on $\Gamma_0(N)$ with $N > 1$, define a map $Q_f : \mathbb{Q} \to C$ the following way: if a rational number x is equivalent to q_j under the action of $\Gamma_0(N)$, then

$$Q_f(x) := \lim_{t \to 0^+} (f - G_j)(x + it). \tag{1.1}$$

This map Q_f is well defined by Theorem 1.1(3) because for each rational number x there is a unique q_i equivalent to x under the action of $\Gamma_0(N)$ (see Section 3 for details).

Motivated by a number of examples from quantum invariants of 3-manifolds, Vassiliev invariants of knots, and period functions of Maass wave forms, Zagier [20] introduced a new modular object defined on $\mathbb{Q}$. A quantum modular form of weight k and multiplier system χ on $\Gamma_0(N)$ is a complex-valued function $f(x)$ on $\mathbb{Q}$ such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ the function

$$h_{\gamma,\chi}(x) := f(x) - (f \mid_{k,\chi} \gamma)(x), \quad \text{where } (f \mid_{k,\chi} \gamma)(x) := \overline{\chi(\gamma)}(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right),$$

can be extended smoothly to $\mathbb{R}$ except for finitely many points. Let $Q_{k,\chi}(\Gamma_0(N))$ be the space of such quantum modular forms. For $x \in \mathbb{Q}$, the twisted L-function $L(g, x; s)$ associated to a cusp form $g(z) = \sum_{m+\kappa>0} a(m)e^{2\pi i(m+\kappa)z}$ is given by the analytic continuation of

$$\sum_{m+\kappa>0} \frac{a(m)e^{2\pi i(m+\kappa)x}}{(m+\kappa)^s}$$

(see Lemma 1 in [12]).

The following theorem shows that we have a linear map from the space of mock modular forms on $\Gamma_0(N)$ to the space of quantum modular forms whose kernel is the space of weakly holomorphic modular forms, denoted $M^!_{k,\chi}(\Gamma_0(N))$.

Theorem 1.5. *Let $k \in \frac{1}{2}\mathbb{Z}$. Let $\Gamma = \Gamma_0(N)$ with $N > 1$ and χ be a multiplier system of weight k on Γ. Then we have a linear injective map*

$$\mu : \mathbb{M}_{k,\chi}(\Gamma)/M^!_{k,\chi}(\Gamma) \to Q_{k,\chi}(\Gamma),\ f \mapsto Q_f.$$

Moreover, if $k < 1$, then $\mu(f)(x) = (4\pi)^{k-1}\Gamma(1-k)\overline{L(g, x; 1-k)}$, where $g = \xi_k(f)$.

Remark 1.6. Hints of the quantum modular properties of the mock theta functions appeared in Ramanujan's letter. The quantum modular properties of the mock theta functions in the context of Ramanujan's examples were discussed by Lawrence and Zagier [14] based on observations of Zwegers (see also [20]) and were described by Folsom, Ono, and the third author [9].

One of the first examples of a quantum modular form is the strange function of Kontsevich (see [18]) defined by

$$F(q) = 1 + (1-q) + (1-q)(1-q^2) + (1-q)(1-q^2)(1-q^3) + \cdots$$

which is defined only at roots of unity. Here, q denotes $e^{2\pi iz}$. Zagier computed and proved

$$\begin{aligned} e^{-u/24}F(e^{-u}) =& 1 + \frac{23}{1!}\left(\frac{u}{24}\right) + \frac{1681}{2!}\left(\frac{u}{24}\right)^2 + \frac{257543}{3!}\left(\frac{u}{24}\right)^3 + \cdots \\ =& \sum_{n=0}^{\infty} \frac{T_n}{n!}\left(\frac{u}{24}\right)^n, \end{aligned}$$

where T_n, defined by Glaisher in 1898, are the "algebraic" parts of certain L-values

$$T_n = \frac{(2n+1)!}{2\sqrt{3}}\left(\frac{6}{\pi}\right)^{2n+2} \cdot L\left(\left(\frac{12}{\cdot}\right), 2n+2\right),$$

where $\left(\frac{12}{n}\right)$ is 1 if $n \equiv \pm 1 \pmod{12}$, -1 if $n \equiv \pm 5 \pmod{12}$, and 0 otherwise. The following result shows that the behavior of a period function of the non-holomorphic Eichler integral associated with a shadow $g(z)$ of a mock modular form $f(z)$ has similar behavior. We let

$$\tilde{g}(z) := (-2i)^{k-1}\overline{\int_z^{i\infty} g(\tau)(\tau - \bar{z})^{-k} d\tau}$$

for $z \in \mathbb{H}$. Then $Q_f(x)$ is the same as the limit point $\tilde{g}(z)$ at $z = x$ for $x \in \mathbb{Q}$.

Theorem 1.7. *Let $k \in \frac{1}{2}\mathbb{Z}, \Gamma = \Gamma_0(N)$ with $N > 1$ and χ be a multiplier system of weight k on Γ. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $c > 0$ and a rational number $x_0 > -\frac{d}{c}$*

$$\tilde{g}(x_0 + it) - (\tilde{g}|_{k,\chi}\gamma)(x_0 + it) = (-2i)^{k-1} \sum_{l=0}^{\infty} \frac{(k)_l}{l!} \overline{\Phi\left(g, -\frac{d}{c}, x_0; -k-l\right)} (it)^l,$$

where $\Phi(g, a, b; s) = \int_a^{\infty} g(\tau)(\tau - b)^s d\tau$ and $(a)_n$ is the Pochhammer symbol defined by $(a)_n = a(a+1)\dots(a+n-1)$ for $n \geq 1$ and $(a)_0 = 1$.

Remark 1.8. The above theorem gives the same formulas for mock modular forms since

$$f(z) - (f|_{k,\chi}\gamma)(z) = \tilde{g}(z) - (\tilde{g}|_{k,\chi}\gamma)(z).$$

Zagier's computations were motivated by computations of Kontsevich [18]

$$F(e^{2\pi i/k}) \sim \zeta(k) k^{\frac{3}{2}} F(e^{-2\pi i k}) + 1 + \left(-\frac{2\pi i}{k}\right) + \frac{3}{2!}\left(-\frac{2\pi i}{k}\right)^2 + \frac{19}{3!}\left(-\frac{2\pi i}{k}\right)^3 + \frac{207}{4!}\left(-\frac{2\pi i}{k}\right)^4 + \cdots,$$

where $\zeta(k)$ is a root of unity that depends on k.

Remark 1.9. The coefficients of this expansion agree with those computed by Zagier for $F(e^{-t})$.

The following results generalize this for quantum modular forms associated to mock modular forms.

Theorem 1.10. *Let $k \in \frac{1}{2}\mathbb{Z}$, $\Gamma = \Gamma_0(N)$ with $N > 1$ and χ be a multiplier system of weight k on Γ. Suppose that $f(z)$ is a mock modular form of weight k and multiplier system χ on Γ whose shadow is $g(z) \in S_{2-k,\bar{\chi}}(\Gamma)$. Let $x_0 = \frac{-d}{c} \in \mathbb{Q}$ such that $(c, d) = 1, N|c$ and $c > 0$. Assume that $\chi(T)^m = 1$ for a positive integer m. Then*

$$\left(\frac{1}{mnc}\right)^k Q_f\left(x_0 + \frac{1}{mnc^2}\right) \sim \bar{\chi}(\gamma) Q_f\left(\frac{a}{c}\right) + \left(\frac{-1}{4\pi}\right)^{1-k} \left(\frac{1}{mnc}\right)^k \times \sum_{l=0}^{\infty} \frac{(k)_l \Gamma(1-k-l)}{l!} \overline{L(g, x_0; 1-k-l)} \left(\frac{-2\pi i}{mnc^2}\right)^l$$

as $n \to \infty$, where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

1.1. **An example.** In his letter, Ramanujan defined

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots.$$

It is clear that $f(q)$ has singularities exactly when q is an even order root of unity. Ramanujan compared $f(q)$ to $b(q) = (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-\cdots)$. The function $q^{-1/24}b(q)$ is a weakly holomorphic modular form. Moreover, Ramanujan claimed, and Watson [17] proved, that $f(q) - b(q) = O(1)$ as q heads radially toward a fourth root of unity and $f(q) - (-b(q)) = O(1)$ as q heads radially toward a root of unity that is 2 modulo 4. Thus, the pair $b(q), -b(q)$ cuts out the singularities of $f(q)$. Zwegers [21] showed that $q^{-1/24}f(q)$ is the holomorphic part of a harmonic weak Maass form of weight $\frac{1}{2}$. More precisely, Bringmann and Ono [3] established that $q^{-1}f(q^{24})$ is the holomorphic part of a harmonic weak Maass form of weight $\frac{1}{2}$ on $\Gamma_0(144)$ with Nebentypus character $\left(\frac{12}{\cdot}\right)$.

We define

$$Q_f(x) = \begin{cases} q^{-1/24}f(q) & q \text{ is an odd order root of unity,} \\ q^{-1/24}(f(q)-b(q)) & q \text{ is a } 4k\text{th root of unity,} \\ q^{-1/24}(f(q)+b(q)) & q \text{ is a } (4k{+}2)\text{th root of unity,} \end{cases}$$

where the right hand side is interpreted as the radial limit as q approaches the root of unity. The following gives a small table of values for $Q_f(\zeta_n)$, where $\zeta_n := e^{2\pi i/n}$.

n	1	2	3	4	5	6
$\zeta_{24n}Q_f(\zeta_n)$	$\frac{4}{3}$	4	$\frac{4}{3}(1-\zeta_3^2)$	$4i$	$\frac{4}{3}(\zeta_5^3-\zeta_5^4)$	$-4\zeta_6$

Moreover, we have the asymptotic expansions

$$\begin{aligned} Q_f\left(-\frac{t}{2\pi i}\right) =& 4\left(\frac{1}{3}+\frac{5}{3^2}\left(-\frac{t}{24}\right)+\frac{153}{3^3\cdot 2!}\left(-\frac{t}{24}\right)^2+\frac{12285}{3^4\cdot 3!}\left(-\frac{t}{24}\right)^3+\cdots\right) \\ =& \frac{4}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{\alpha_n}{n!}\left(-\frac{t}{6}\right)^n \end{aligned}$$

and

$$\begin{aligned} Q_f\left(\frac{1}{2}-\frac{t}{2\pi i}\right) =& 4\left(1+23\left(-\frac{t}{24}\right)+\frac{3985}{2!}\left(-\frac{t}{24}\right)^2+\frac{1743623}{3!}\left(-\frac{t}{24}\right)^3+\cdots\right) \\ =& 4\sum_{n=0}^{\infty}\frac{\beta_n}{n!}\left(-\frac{t}{24}\right)^n, \end{aligned}$$

where $\alpha_n = \int_0^\infty x^{2n}\frac{\cosh\left(\frac{\pi x}{3}\right)}{\cosh(\pi x)}dx$ and β_n are given in Theorem 4.1 of [10] in terms of the Euler numbers. Zwegers [22] showed that the integrals appearing in the definition of α_n and β_n may be written in terms of integrals of unary theta functions of weight $\frac{3}{2}$.

The remaining parts of the paper are organized as follows. Section 2 contains background on modular forms and harmonic weak Maass forms. Section 3 contains the proofs of the main theorems.

2. Modular objects

In this section we recall definitions and basic facts about modular forms and harmonic weak Maass forms. Let $k \in \frac{1}{2}\mathbb{Z}$ and let $\Gamma = \Gamma_0(N)$ with $N > 1$. Let χ be a multiplier system of weight k on Γ, i.e., $\chi : \Gamma \to \mathbb{C}$ satisfies the following conditions:

(1) $|\chi(\gamma)| = 1$ for all $\gamma \in \Gamma$,
(2) for $z \in \mathbb{H}$, χ satisfies the consistency condition

$$\chi(\gamma_3)(c_3z+d_3)^k = \chi(\gamma_1)\chi(\gamma_2)(c_1\gamma_2z+d_1)^k(c_2z+d_2)^k,$$

where $\gamma_3 = \gamma_1\gamma_2$ and $\gamma_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$ for $i = 1,2,3$.

We recall the slash operator

$$(f|_{k,\chi}\gamma)(z) := \bar{\chi}(\gamma)(cz+d)^{-k}f(\gamma z)$$

for any function $f(z)$ on $\mathbb{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Note that $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ generates the subgroup Γ_∞ of translations in Γ. If a holomorphic function $f(z)$ satisfies $(f|_{k,\chi}T)(z) = f(z)$, then $f(z)$ has the Fourier expansion at $i\infty$

$$f(z) = \sum_{n=-\infty}^{\infty} a(n) e^{2\pi i(n+\kappa)z}, \tag{2.1}$$

where κ is a constant in $[0,1)$ such that $\chi(T) = e^{2\pi i\kappa}$. Suppose that Γ has $t \geq 1$ inequivalent cusps. Let $q_1 = i\infty$ and $q_2, \ldots, q_t$ be the inequivalent cusps of Γ. Suppose also that $Q_j = \left(\begin{smallmatrix} * & * \\ c_j & d_j \end{smallmatrix}\right)$ is a generator of Γ_j, which is the cyclic subgroup of Γ fixing q_j for $2 \leq j \leq t$. For $2 \leq j \leq t$, put $\chi(Q_j) = e^{2\pi i\kappa_j}$, $0 \leq \kappa_j < 1$. If a holomorphic function $f(z)$ satisfies $(f|_{k,\chi}Q_j)(z) = f(z)$, then $f(z)$ has the Fourier expansion at q_j

$$(-z)^{-k} f(A_j^{-1}z) = \sum_{n=-\infty}^{\infty} a_j(n) e^{2\pi i(n+\kappa_j)z/\lambda_j}. \tag{2.2}$$

Here, $A_j = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -q_j \end{smallmatrix}\right)$ and λ_j is a positive real number called the width of the cusp q_j, which is chosen so that $A_j^{-1} \left(\begin{smallmatrix} 1 & \lambda_j \\ 0 & 1 \end{smallmatrix}\right) A_j$ generates Γ_j. For convenience, let $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Definition 2.1. Suppose $f(z)$ is holomorphic in $\mathbb{H}$ and satisfies the functional equation $(f|_{k,\chi}\gamma)(z) = f(z)$ for all $\gamma \in \Gamma$.

(1) If $f(z)$ has only finitely many terms with $n + \kappa < 0$ in (2.1) and with $n + \kappa_j < 0$, $2 \leq j \leq t$, in (2.2), then $f(z)$ is called a weakly holomorphic modular form of weight k and multiplier system χ on Γ. The set of all such weakly holomorphic modular forms is denoted by $M^!_{k,\chi}(\Gamma)$.
(2) Let $f(z) \in M^!_{k,\chi}(\Gamma)$. Suppose in addition $f(z)$ has only terms with $n+\kappa \geq 0$ in (2.1) and $n + \kappa_j \geq 0$, $2 \leq j \leq t$, in (2.2). Then $f(z)$ is called a holomorphic modular form. The set of holomorphic modular forms in $M^!_{k,\chi}(\Gamma)$ is denoted by $M_{k,\chi}(\Gamma)$.
(3) If $f(z) \in M_{k,\chi}(\Gamma)$ and has only terms with $n + \kappa > 0$, $n + \kappa_j > 0$ in the expansions (2.1), (2.2), respectively, then $f(z)$ is called a cusp form. The collection of cusp forms in $M_{k,\chi}(\Gamma)$ is denoted by $S_{k,\chi}(\Gamma)$.

A harmonic weak Maass form of weight k and multiplier system χ on Γ is a smooth function on $\mathbb{H}$ with possible singularities at cusps that transforms like a modular form and is annihilated by the weight k hyperbolic Laplacian

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Now we give the precise definition of a harmonic weak Maass form.

Definition 2.2. A harmonic weak Maass form of weight k and multiplier system χ on Γ is any smooth function on $\mathbb{H}$ satisfying:

(1) $(f|_{k,\chi}\gamma)(z) = f(z)$ for all $\gamma \in \Gamma$,
(2) $\Delta_k f = 0$,
(3) a linear exponential growth condition in terms of y at every cusp.

We write $H_{k,\chi}(\Gamma)$ for the space of harmonic weak Maass forms of weight k and multiplier system χ on Γ.

Recall that any harmonic weak Maass form $f(z)$ of weight k has the unique decomposition $f(z) = f^+(z) + f^-(z)$, where

$$f^+(z) = \sum_{n \gg -\infty} a^+(n) e^{2\pi i(n+\kappa)z/\lambda}, \tag{2.3}$$

$$f^-(z) = \delta_{\kappa,0} a^-(0) y^{1-k} + \sum_{\substack{n \ll \infty \\ n+\kappa \neq 0}} a^-(n) \Gamma(4\pi(n+\kappa)y/\lambda, -k+1) e^{2\pi i(n+\kappa)z/\lambda}.$$

Here, $\delta_{\kappa,0} = 1$ if $\kappa = 0$, and $\delta_{\kappa,0} = 0$ otherwise (for this decomposition see [5, Section 3]). The function $f^+(z)$ (resp. $f^-(z)$) is called the holomorphic (resp. non-holomorphic) part of $f(z)$. By Zagier's definition in [19], $f^+(z)$ is called a mock modular form and $f(z)$ is its completion.

There is a differential operator $\xi_k := 2iy^k \overline{\frac{\partial}{\partial \bar{z}}}$ which plays an important role in the theory of harmonic weak Maass forms. The assignment $f(z) \mapsto \xi_k(f)(z)$ defines an anti-linear mapping

$$\xi_k : H_{k,\chi}(\Gamma) \to M^!_{2-k,\bar{\chi}}(\Gamma).$$

Moreover, the kernel of ξ_k is $M^!_{k,\chi}(\Gamma)$ (see [5, Proposition 3.2]). We let $H^*_{k,\chi}(\Gamma)$ denote the inverse image of the space of cusp forms $S_{2-k,\bar{\chi}}(\Gamma)$ under the mapping ξ_k. Hence, if $f(z) \in H^*_{k,\chi}(\Gamma)$, then the Fourier coefficients $a^-(n)$ vanish if $n+\kappa$ is non-negative. It is known that for a harmonic weak Maass form $f(z) \in H^*_{k,\chi}(\Gamma)$ the non-holomorphic part $f^-(z)$ is given by the non-holomorphic Eichler integral associated with the shadow $g(z) := \xi_k(f)$ (see [4] or [7])

$$f^-(z) = -(-2i)^{k-1} \overline{\int_z^{i\infty} g(\tau)(\tau - \bar{z})^{-k} d\tau}. \tag{2.4}$$

Here, we determine the branch of $(\tau - \bar{z})^{-k}$ by means of the convention $z^{-k} = |z|^{-k} e^{-ik \arg z}$, where $-\pi \leq \arg z < \pi$.

3. Proof of the main theorems

In this section we prove the main results. To prove Theorem 1.1 we need the following lemma.

Lemma 3.1. *Let $k \in \frac{1}{2}\mathbb{Z}$ and $\Gamma = \Gamma_0(N)$. We take a cusp q_j in $\{q_1, \ldots, q_t\}$. Suppose that $f(z)$ is a harmonic Maass form in $H^*_{k,\chi}(\Gamma)$ which has a Fourier expansion at the cusp q_j of the form*

$$\sum_{n+\kappa_j > 0} a_q^+(n) e^{2\pi i(n+\kappa_j)z/\lambda_j} + \sum_{n+\kappa_j < 0} a_q^-(n) \Gamma(-4\pi(n+\kappa_j)y/\lambda_j, -k+1) e^{2\pi i(n+\kappa_j)z/\lambda_j}.$$

Let C_j be a set of rational numbers which are equivalent to the cusp q_j under the action of Γ. Then for any $x \in C_j$

$$\lim_{t \to 0^+} f^+(x+it) = -\lim_{t \to 0^+} f^-(x+it) = (-2i)^{k-1} \overline{\int_x^{i\infty} g(\tau)(\tau - x)^{-k} d\tau}, \tag{3.1}$$

where $g(z) := \xi_k(f)(z)$.

Proof. Let $x = \gamma q_j$ for some $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ. Since $g(z)$ is a cusp form, we have by (2.4)

$$-\lim_{t \to 0^+} f^-(x+it) = \lim_{t \to 0^+} \tilde{g}(x+it) = (-2i)^{k-1} \overline{\int_x^{i\infty} g(\tau)(\tau - x)^{-k} d\tau}.$$

Thus, the proof can be completed by showing $\lim_{t\to 0^+} f(x+it) = 0$. To do so, we consider the action of $A_j\gamma^{-1}$ for a special geodesic

$$(3.2)\qquad \begin{array}{ll} A_j\gamma^{-1}(x+it) = \frac{1}{(cq_j+d)^2}(c(cq_j+d)+i\frac{1}{t}) & \text{if } j\neq 1,\\ A_j\gamma^{-1}(x+it) = \frac{1}{c^2}(-cd+i\frac{1}{t}) & \text{if } j=1. \end{array}$$

Recall that $A_j = \left(\begin{smallmatrix} 0 & -1\\ 1 & -q_j\end{smallmatrix}\right)$ if $j\neq 1$ and $A_j = \left(\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right)$ if $j=1$. In the first equation, cq_j+d is not zero since q_j is not equivalent to $i\infty$ under the action of Γ, and in the second equation c is also not zero since $x\in\mathbb{Q}$ and $x=\gamma i\infty$. Moreover, a modular property of $f(z)$ implies

$$f(x+it) = f|_{k,\chi}\gamma^{-1}(x+it) = \bar{\chi}(\gamma^{-1})(-c(x+it)+d)^{-k} f(A_j^{-1}A_j\gamma^{-1}(x+it)).$$

Combining this with (3.2), we obtain $\lim_{t\to 0^+} f(x+it) = 0$ since by the assumption $f(A_j^{-1}z)$ decays exponentially when z approaches $i\infty$. Since we have

$$\lim_{t\to 0^+} f^+(x+it) = \lim_{t\to 0^+} f(x+it) - \lim_{t\to 0^+} f^-(x+it),$$

we complete the proof. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Parts (1) and (2) follow from the work of Griffin, Ono, and Rolen [11] and are based on the results of Bruinier and Funke [5]. Now we prove part (3) using Lemma 3.1. Suppose that $f(z)$ is a mock modular form in $\mathbb{M}_{k,\chi}(\Gamma_0(N))$. Then, we can find a weakly holomorphic modular form $G_j(z)\in M^!_{k,\chi}(\Gamma_0(N))$ such that $(f-G_j)(z)$ has neither a principal part nor a constant term in the Fourier expansion at the cusp q_j by a variant of Theorem 3.1 in [2]. By Lemma 3.1, the function $(f-G_j)(z)$ is bounded toward all rational numbers equivalent to the cusp q_j. □

Suppose that $f(z)$ is a mock modular form whose shadow is $g(z)\in S_{2-k,\bar{\chi}}(\Gamma_0(N))$ and that x is a rational. Then, x is equivalent to a cusp q_j for some $1\leq j\leq t$ and by (3) of Theorem 1.1 there is a weakly holomorphic modular form $G_j(z)$ in $M^!_{k,\chi}(\Gamma_0(N))$ which has the same singularity and constant term with a mock modular form $f(z)$ at the cusp q_j. Although $G_j(z)$ is not unique, Lemma 3.1 shows that the value $\lim_{x\to 0^+}(f-G_j)(x+it)$ for $x\in C_j$ is the same as $\tilde{g}(x)$ and hence it is independent of the choice of $G_j(z)$. Therefore, we have a well-defined function Q_f on $\mathbb{Q}$ which is given by $Q_f(x) = \lim_{x\to 0^+}(f-G_j)(x+it)$ for $x\in\mathbb{Q}$.

The next theorem shows that $Q_f(x)$ is a quantum modular form for any mock modular forms $f(z)$.

Theorem 3.2. *Let $k\in\frac{1}{2}\mathbb{Z}$. Let $\Gamma=\Gamma_0(N)$ with $N>1$ and χ be a multiplier system of weight k on Γ. If $f(z)$ is a mock modular form of weight k and multiplier system χ on Γ, then for $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma$ the function $Q_f(x)$ satisfies*

$$Q_f(x) - (Q_f|_{k,\chi}\gamma)(x) = (-2i)^{k-1}\overline{\int_{\gamma^{-1}(i\infty)}^{i\infty} g(\tau)(\tau-x)^{-k}d\tau},$$

where $g(z)$ is the shadow of $f(z)$. Hence, $Q_f(x)$ is a quantum modular form of weight k and multiplier system χ on Γ.

To prove the theorem, we need the following lemma on the analytic property for period functions of $Q_f(x)$.

Lemma 3.3. *Under the same notation as in Theorem* 3.2, *let m denote an integer or a half integer, and let $p_{\gamma,m}(x)$ be a $\mathbb{C}$-valued function on $\mathbb{R}$ defined by*

$$p_{\gamma,m}(x) := \int_{\gamma^{-1}(i\infty)}^{i\infty} g(\tau)(\tau - x)^m d\tau.$$

Then, $p_{\gamma,m}(x)$ is a smooth function on $\mathbb{R}$ except $x = \gamma^{-1}(i\infty)$ and has an analytic extension to $\{u + iv \in \mathbb{C} \mid u > 0$ or $v > 0\}$.

The proof of this lemma can be easily obtained by standard methods and direct computations. Thus we give just a sketch of the proof.

Proof of Lemma 3.3. To prove smoothness of $p_{\gamma,m}(x)$ on $\mathbb{R}\backslash\{\gamma^{-1}(i\infty)\}$, it is enough to show that the derivative of $p_{\gamma,m}(x)$ is $mp_{\gamma,m-1}(x)$ if $m \neq 0$ and $x \neq \gamma^{-1}(i\infty)$. Let $x_0 := \gamma^{-1}(i\infty)$. Taking a special contour, we have

$$\begin{aligned}\lim_{t\to 0} &\frac{(p_{\gamma,m}(x+t) - p_{\gamma,m}(x))}{t} \\ &= \lim_{t\to 0} \int_0^{\infty} g(x_0 + ir) \frac{(x_0 + ir - x - t)^m - (x_0 + ir - x)^m}{t} idr.\end{aligned}$$

Assume that t is sufficiently small. By direct computation, we have

$$((\gamma^{-1}(i\infty) + ir - x - t)^m - (\gamma^{-1}(i\infty) + ir - x)^m)t^{-1}$$

has polynomial growth independent of t when r approaches ∞ or 0. Since $g(z)$ is a cusp form,

$$g(\gamma^{-1}(i\infty) + ir)((\gamma^{-1}(i\infty) + ir - x - t)^m - (\gamma^{-1}(i\infty) + ir - x)^m)t^{-1}$$

decays exponentially independent of t as r goes to 0 or ∞. Thus, by Lebesgue's dominated convergence theorem, we can exchange the order of limit and integration. This implies the smoothness of $p_{\gamma,m}(x)$ on $\mathbb{R} \setminus \{\gamma^{-1}(i\infty)\}$. Considering the branch of $(\tau - \bar{z})^m$, one can prove by a similar argument to the above that $p_{\gamma,m}(x)$ has an analytic extension to $\{u + iv \in \mathbb{C} \mid u > 0$ or $v > 0\}$. Thus we omit it. ☐

Now we prove Theorem 3.2.

Proof of Theorem 3.2. From Lemma 3.1, we know that

$$Q_f(x) = (-2i)^{k-1} \overline{\int_x^{i\infty} g(\tau)(\tau - x)^{-k} d\tau}. \tag{3.3}$$

Thus, by the definition of the slash operator, we have

$$(Q_f|_{k,\chi}\gamma)(x) = -(-2i)^{k-1}\overline{\chi}(\gamma)(cx + d)^{-k} \overline{\int_{\gamma x}^{i\infty} g(\tau)(\tau - \gamma x)^{-k} d\tau}.$$

So we can compute the function $(Q_f|_{k,\chi}\gamma)(x)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ using the argument in Lemma 2.2 of [13]. By simple computation, we have $\frac{\tau - x}{(c\tau+d)(cx+d)} = \gamma\tau - \gamma x$. Since $\tau - x$ and $\gamma\tau - \gamma x$ are in $\mathbb{H}$, we see that

$$\left(\frac{\tau - x}{(c\tau + d)(cx + d)}\right)^{-k} = \frac{(\tau - x)^{-k}}{\{(c\tau + d)(cx + d)\}^{-k}}.$$

Furthermore, by comparing arguments we have

$$\{(c\tau + d)(cx + d)\}^{-k} = \delta_{k,\gamma,x}^{-1}(c\tau + d)^{-k}(cx + d)^{-k},$$

where

$$\delta_{k,\gamma,x} := \begin{cases} e^{2\pi ik} & \text{if } c \leq 0 \text{ and } cx+d<0, \\ 1 & \text{otherwise.} \end{cases}$$

Using the substitution $\tau \to \gamma\tau$ in the above integral and the fact that $g(z)$ is a cusp form in $S_{2-k,\bar{\chi}}(\Gamma)$, we see that

$$(Q_f|_{k,\chi}\gamma)(x) = (-2i)^{k-1}\overline{\int_{\gamma^{-1}(i\infty)}^{x} g(\tau)(\tau - x)^{-k}d\tau}.$$

Thus, we have

$$Q_f(x) - (Q_f|_{k,\chi}\gamma)(x) = -(-2i)^{k-1}\overline{\int_{\gamma^{-1}(i\infty)}^{i\infty} g(\tau)(\tau - x)^{-k}d\tau}.$$

Finally, to complete the proof, it must be shown that period functions of $Q_f(x)$ are smooth on $\mathbb{R}$ except for finitely many points. This follows from Lemma 3.3. □

Thanks to the Eichler-Shimura cohomology, we can prove by using Theorem 3.2 that the map $\mu : \mathbb{M}_{k,\chi}(\Gamma)/M^!_{k,\chi}(\Gamma) \to Q_{k,\chi}(\Gamma)$ is injective.

Proof of Theorem 1.5. It is shown that Q_f is well defined for mock modular forms. Moreover, from (3.3), it is clear that Q_f is a zero function if $f \in M^!_{k,\chi}(\Gamma)$. Thus, one can check that the map $\mu : \mathbb{M}_{k,\chi}(\Gamma)/M^!_{k,\chi}(\Gamma) \to Q_{k,\chi}(\Gamma)$ is a well-defined linear map.

Next, we will prove the injectivity of the map μ. Let $g(z)$ be the shadow of $f(z)$. The non-holomorphic part of $M_f(z)$ is given by a non-holomorphic Eichler integral associated with $g(z)$ as in (2.4). Thus, for the injectivity of μ, it is sufficient to show that if $Q_f(x)$ is a zero function, then $g(z)$ is also a zero function.

Suppose that $Q_f(x)$ is a zero function. Let $\gamma_0 := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element in Γ. Then, by Theorem 3.2, we have

$$Q_f(x) - (Q_f|_{k,\chi}\gamma_0)(x) = -(-2i)^{k-1}\overline{\int_{\gamma_0^{-1}(i\infty)}^{i\infty} g(\tau)(\tau - x)^{-k}d\tau}.$$

Since $Q_f(x) = 0$ on $\mathbb{Q}$ by the assumption, a function

$$g_{\gamma_0}(z) := \overline{\int_{\gamma_0^{-1}(i\infty)}^{i\infty} g(\tau)(\tau - \bar{z})^{-k}d\tau} \tag{3.4}$$

is zero on the real line. On the other hand, Lemma 3.3 implies that this function is holomorphic on $E := \{u + iv \in \mathbb{C} \mid u > 0 \text{ or } v > 0\}$. Thus, the function $g_{\gamma_0}(z)$ should be a zero function on $\mathbb{H}$, $\mathbb{H} \subset E$, by the identity theorem for holomorphic functions. From the Eichler-Shimura cohomology theory, there is an isomorphism η between $S_{2-k,\bar{\chi}}(\Gamma)$ and the Eichler-Shimura cohomology group $\tilde{H}^1_{-k,\chi}(\Gamma)$. The image of $g(z)$ under the map η is a 1-cocycle element defined by $\langle g_\gamma(z) | \ \gamma \in \Gamma\rangle$ (for more details, see [13, Section 2]). We already checked that $g_\gamma(z)$ is a zero function for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Since η is an isomorphism, this implies that $g(z)$ is a zero function.

Suppose that $k < 1$. If $g(z)$ has a Fourier expansion of the form

$$g(z) = \sum_{m+\kappa>0} a(m)e^{2\pi i(m+\kappa)z},$$

then by (3.3)

$$\begin{aligned} Q_f(x) &= (-2i)^{k-1} \sum_{m+\kappa>0} \overline{a(m)} e^{2\pi i(m+\kappa)(-x)} \overline{\int_0^{i\infty} e^{2\pi i(m+\kappa)\tau} \tau^{-k} d\tau} \\ &= (-2i)^{k-1} \sum_{m+\kappa>0} \overline{a(m)} e^{2\pi i(m+\kappa)(-x)} (-i)^{1-k} \frac{1}{(2\pi(m+\kappa))^{1-k}} \Gamma(1-k). \end{aligned}$$

If we use the definition of the twisted L-function $L(g, x; s)$, then one can check that

$$Q_f(x) = (4\pi)^{k-1} \Gamma(1-k) \overline{L(g, x; 1-k)}.$$

This completes the proof. □

Using the analytic property for period functions of Q_f, we prove Theorems 1.7 and 1.10.

Proof of Theorem 1.7. By Theorem 3.2, we have

$$\tilde{g}(z) - (\tilde{g}|_{k,\chi}\gamma)(z) = (-2i)^{k-1} \overline{\int_{\gamma^{-1}(i\infty)}^{i\infty} g(\tau)(\tau - \bar{z})^{-k} d\tau}$$

for $z \in \mathbb{H}$. Thus, since $x_0 \neq \gamma^{-1} i\infty$, it is analytic at $z = x_0$. Therefore, if we let $p_\gamma(z) = \tilde{g}(z) - (\tilde{g}|_{k,\chi}\gamma)(z)$, then

$$p_\gamma(z) = \sum_{l=0}^{\infty} \frac{p_\gamma^{(l)}(x_0)}{l!} (z - x_0)$$

around x_0. Here, $p_\gamma^{(l)}(x_0)$ denotes the lth derivative of $p_\gamma(x)$ at $x = x_0$. Since

$$p_\gamma^{(l)}(x_0) = (-2i)^{k-1} (-k)_l \overline{\int_{-\frac{d}{c}}^{i\infty} g(\tau)(\tau - x_0)^{-k-l} d\tau},$$

we have

$$\tilde{g}(x_0 + it) - (\tilde{g}|_{k,\chi}\gamma)(x_0 + it) = (-2i)^{k-1} \sum_{l=0}^{\infty} \frac{(-k)_l}{l!} \overline{\int_{-\frac{d}{c}}^{i\infty} g(\tau)(\tau - x_0)^{-k-l} d\tau} (it)^l.$$

This completes the proof. □

Proof of Theorem 1.10. Let

$$p_\gamma(x) := Q_f(x) - (Q_f|_{k,\chi}\gamma)(x)$$

and

$$z_n := x_0 + \frac{1}{mnc^2}$$

for integers n. Note that $z_n \to x_0$ as $n \to \infty$ and $z_n = (\gamma^{-1} T^{-mn} \gamma)(i\infty)$. One can check that $cz_n + d = \frac{1}{mnc}$ and

$$Q_f(\gamma z_n) = Q_f((T^{-mn}\gamma)(i\infty)) = Q_f(\gamma(i\infty)) = Q_f\left(\frac{a}{c}\right)$$

since $Q_f(T^m x) = Q_f(x)$ for any $x \in \mathbb{Q}$ and $m \in \mathbb{Z}$. Therefore, we obtain

$$p_\gamma(z_n) = Q_f\left(x_0 + \frac{1}{mnc^2}\right) - \bar{\chi}(\gamma)\left(\frac{1}{mnc}\right)^{-k} Q_f\left(\frac{a}{c}\right). \tag{3.5}$$

On the other hand, by Taylor's Theorem, we have an asymptotic expansion at $x = x_0$

$$p_\gamma(x) \sim \sum_{l=0}^{\infty} \frac{p_\gamma^{(l)}(x_0)}{l!}(x-x_0)^l$$

as $x \to x_0$. Here, $p_\gamma^{(l)}(x_0)$ denotes the lth derivative of $p_\gamma(x)$ at $x = x_0$.

If s is a complex number whose real part is sufficiently large, then

$$\begin{aligned}\int_{x_0}^{i\infty} g(\tau)(\tau - x_0)^s d\tau &= \int_0^{i\infty} g(\tau + x_0)\tau^s d\tau = i^{1+s}\int_0^{\infty} g(iy + x_0)y^s dy \\ &= i^{1+s} L\left(g, x_0; 1+s\right)\Gamma(1+s)\left(\frac{1}{2\pi}\right)^{1+s}.\end{aligned}$$

A function $L\left(g, x_0; 1+s\right)\Gamma(1+s)$ has an analytic continuation to $\mathbb{C}$ (see Lemma in [12]). Using the integral expression of $p_\gamma(x)$ in Theorem 3.2, we have

$$\begin{aligned}p_\gamma^{(l)}(x_0) &= (-2i)^{k-1}(-k)_l \overline{\int_{x_0}^{i\infty} g(\tau)(\tau - x_0)^{-k-l} d\tau} \\ &= i^{1-k-l} L\left(g, x_0; 1-k-l\right)\Gamma(1-k-l)\left(\frac{1}{2\pi}\right)^{1-k-l}.\end{aligned}$$

Therefore,

$$p_\gamma(z_n) \sim \left(\frac{-1}{4\pi}\right)^{1-k} \sum_{l=0}^{\infty} \frac{(-k)_l \Gamma(1-k-l)}{l!} \overline{L(g, x_0; 1-k-l)} \left(\frac{-2\pi i}{mnc^2}\right)^l. \tag{3.6}$$

Combining (3.5) and (3.6), we get the desired result. □

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