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# ENTIRE $s$-HARMONIC FUNCTIONS ARE AFFINE

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ABSTRACT. In this paper, we prove that solutions to the equation $(-\Delta)^s u = 0$ in $\mathbb{R}^N$, for $s \in (0,1)$, are affine. This allows us to prove the uniqueness of the Riesz potential $|x|^{2s-N}$ in Lebesgue spaces.

## 1. INTRODUCTION

The classical Liouville theorem for harmonic functions states that *a bounded harmonic function in $\mathbb{R}^N$ is constant*; see for instance the particularly short proof by E. Nelson in [13]. The stronger version of it states that a nonnegative harmonic function on $\mathbb{R}^N$ is constant. In the case of the fractional Laplacian $-(-\Delta)^s$, for $s \in (0,1)$, (see Section 2) the strong form of the Liouville theorem holds as well and was proved by K. Bogdan et al. in [5]. Applications of Liouville theorems for nonlocal operators in the study of nonlocal elliptic systems of equations can be found in [2, 9, 14, 15].

The aim of this paper is to classify all $s$-harmonic functions in $\mathbb{R}^N$, thereby obtaining the Liouville theorem for the fractional Laplacian as a particular case.

**Theorem 1.1.** *Every $s$-harmonic function in $\mathbb{R}^N$ is affine, and constant if $s \in (0, 1/2]$.*

The proof of this theorem is mainly based on a Cauchy-type estimate for the derivatives of an $s$-harmonic function. More precisely, given $s \in (0,1)$, $\gamma \in \mathbb{N}^N$ and a function $u$ which is $s$-harmonic in the ball $B(0,R)$, we have the estimate

$$|D^\gamma u(0)| \le C R^{2s-|\gamma|} \int_{|y| \ge R/4} |u(y)||y|^{-N-2s}\, dy, \tag{1.1}$$

for some positive constant $C$ depending only on $N, \gamma$ and $s$; see Section 3. This estimate is obtained from the Poisson kernel representation formula for $s$-harmonic functions. We refer to Section 2 for more details.

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In the following, we denote by $\mathcal{D}'(\mathbb{R}^N)$ the dual of $C_c^\infty(\mathbb{R}^N)$ endowed with the usual topology. An iteration argument based on Theorem 1.1 allows us to state the following result.

**Theorem 1.2.** *Assume that $s \in (0,1)$ and let $u$ be a solution to the equation*

$$(-\Delta)^s u = P \qquad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

*where $P$ is a polynomial. Then $u$ is affine and $P = 0$.*

Another consequence of the main theorem which is of independent interest is the following result.

**Corollary 1.3.** *Let $p \in [1,\infty)$ and $u \in L^p(\mathbb{R}^N)$ be such that*

$$(-\Delta)^s u = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

*Then $u \equiv 0$.*

Combining Corollary 1.3 and the Hardy-Littlewood-Sobolev inequality, we have a uniqueness result.

**Corollary 1.4** (Uniqueness of Riesz potential)**.** *Let $s \in (0,1)$, $1 < p < \frac{N}{2s}$ and $f \in L^p(\mathbb{R}^N)$. Then there exists a unique $u \in L^{\frac{Np}{N-2sp}}(\mathbb{R}^N)$ such that*

$$(-\Delta)^s u = f \qquad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

*and $u$ is given by*

$$u(x) = \alpha_{N,s} \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2s}}\, dy,$$

*where*

$$\alpha_{N,s} = \pi^{N/2} 2^{2s} \frac{\Gamma(s)}{\Gamma((N-2s)/2)}.$$

The paper is organized as follows. In Section 2 we collect some basic facts concerning the fractional Laplacian $-(-\Delta)^s$ and $s$-harmonic functions. Finally, in Section 3 we prove the Cauchy-type estimate (1.1), the main result and its corollaries.

*Note added in proof.* We mention that after this paper was submitted, Liouville-type results for a class of nonlocal operators were proved in [8] and [10] using Fourier transform. □

## 2. Preliminaries

This section is devoted to recalling some basic notions about $s$-harmonic functions. We refer the reader to [7, Section 3]. Let $\mathcal{L}_s^1$ denote the space of all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} \frac{|u(x)|}{1+|x|^{N+2s}} dx < \infty.$$

For functions $\varphi \in C^2(\mathbb{R}^N) \cap \mathcal{L}_s^1$, the fractional Laplacian $-(-\Delta)^s$ is defined by

$$-(-\Delta)^s \varphi(x) = C_{N,s} \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{\varphi(y)-\varphi(x)}{|y-x|^{N+2s}}\, dy \qquad \text{for all } x \in \mathbb{R}^N, \tag{2.1}$$

where $C_{N,s} = s(1-s)\pi^{-N/2} 4^s \frac{\Gamma(\frac{N}{2}+s)}{\Gamma(2-s)}$.

For $u \in \mathcal{L}^1_s$, the expression $(-\Delta)^s u$ defines a distribution on every open set $\Omega \subset \mathbb{R}^N$ by

$$\langle (-\Delta)^s u, \varphi \rangle = \int_{\mathbb{R}^N} u(x)(-\Delta)^s \varphi(x)\, dx \qquad \text{for every } \varphi \in C^\infty_c(\Omega).$$

In the case where $(-\Delta)^s u = 0$ in $\mathcal{D}'(\Omega)$, we will say that $u$ *is $s$-harmonic* in $\Omega$. We note that affine functions $u$ belong to $\mathcal{L}^1_s$ if $s > 1/2$ and constant functions $u$ belong to $\mathcal{L}^1_s$ if $s \in (0, 1/2]$. Moreover, by using (2.1), in both cases, we can see that $(-\Delta)^s u(x) = 0$ for every $x \in \mathbb{R}^N$. Furthermore, thanks to [7, Lemma 3.3], we have $(-\Delta)^s u = 0$ in $\mathcal{D}'(\mathbb{R}^N)$.

The fractional Laplacian has an explicit Poisson kernel with respect to the ball $B(x, r)$ (see [4]). It is given by

$$P_r(x, y) = \begin{cases} \beta_{N,s} \dfrac{(r^2 - |x|^2)^s}{(|y|^2 - r^2)^s} |y - x|^{-N} & \text{for } |x| < r,\ |y| > r, \\ 0 & \text{otherwise}, \end{cases} \tag{2.2}$$

where $\beta_{N,s} = \Gamma(N/2)\pi^{-N/2-1}\sin(s\pi)$. Therefore (see also [5]), if $u$ is $s$-harmonic in $\Omega$, then for every ball $B(a, r) \Subset \Omega$ we have

$$u(x) = \int_{\mathbb{R}^N} P_r(x - a, y - a) u(y)\, dy \qquad \text{for all } x \in B(a, r).$$

We now consider the regularization of $P_r$ as in [7]. To this end, we pick a function $\phi \in C^\infty_c(1, 4)$ such that $\int_{\mathbb{R}} \phi(r)\, dr = 1$ and define $\Psi : \mathbb{R}^N \to \mathbb{R}$ by

$$\Psi(y) = \int_1^4 P_r(0, y)\phi(r)\, dr = \beta_{N,s}|y|^{-N} \int_{\min(1,|y|)}^{\min(4,|y|)} r^{2s}(|y|^2 - r^2)^{-s}\phi(r)\, dr.$$

Observe that, if $|y| \le 1$, then $(0, |y|) \cap (1, 4) = \emptyset$ and thus

$$\Psi(y) = 0 \qquad \text{for every } y \in B(0, 1). \tag{2.3}$$

Furthermore, as shown e.g. in [7, Lemma 3.11], we have $\Psi \in C^\infty(\mathbb{R}^N)$. Moreover, for every $\gamma \in \mathbb{N}^N$ there holds

$$|D^\gamma \Psi(y)| \le C\, |y|^{-N-2s-|\gamma|} \qquad \text{for every } y \in \mathbb{R}^N \setminus \{0\}, \tag{2.4}$$

where $C = C(N, \gamma, s)$ denotes, here and in the following, a positive constant depending only on $N$, $\gamma$ and $s$.

We define $\Psi_{r_0}(y) = r_0^{-N}\Psi(y/r_0)$, for $y \in \mathbb{R}^N$ and $r_0 > 0$. Then, for any $s$-harmonic function $u$ in an open set $\Omega \subset \mathbb{R}^N$, we have

$$u(x) = u \star \Psi_{r_0}(x) \qquad \text{for all almost every } x \in \Omega_{4r_0}, \tag{2.5}$$

where $\Omega_{4r_0} = \{x \in \Omega : \operatorname{dist}(x, \mathbb{R}^N \setminus \Omega) > 4r_0\}$; see [11, Lemma 2.6] or [7, Page 65]. We will therefore assume, in the sequel, that $s$-harmonic functions in some open set are smooth in that set.

## 3. Proof of the main result and its consequences

The following result (from which we will derive our main result) can be seen as a nonlocal version of the Cauchy estimate for bounded harmonic functions; see e.g. [3, Chapter 2].

**Lemma 3.1.** *For every $\gamma \in \mathbb{N}^N$, there exists a constant $C > 0$ only depending on $N$, $\gamma$ and $s$ such that for every function $u$ which is $s$-harmonic in $B(0, R)$,*

$$|D^\gamma u(0)| \leq C\, R^{2s-|\gamma|} \int_{|y|\geq R/4} |u(y)||y|^{-N-2s}\, dy.$$

*Proof.* Let $u$ be an $s$-harmonic function in $B(0, R)$ and $r_0 \in (0, R/4)$. Then by (2.5) we have

$$u(x) = u \star \Psi_{r_0}(x) \qquad \text{for all } x \in B(0, R - 4r_0).$$

By (2.3), (2.4) and the dominated convergence theorem, we deduce that

$$D^\gamma u(0) = u \star D^\gamma \Psi_{r_0}(0) \qquad \text{for all } \gamma \in \mathbb{N}^N.$$

Using once more (2.3) and (2.4), we get

$$|D^\gamma u(0)| = \left| \int_{|y|\geq r_0} u(y) D^\gamma \Psi_{r_0}(-y)\, dy \right| \leq C\, r_0^{2s} \int_{|y|\geq r_0} |u(y)|\, |y|^{-N-2s-|\gamma|}\, dy.$$

It follows that

$$|D^\gamma u(0)| \leq C\, r_0^{2s-|\gamma|} \int_{|y|\geq r_0} |u(y)|\, |y|^{-N-2s}\, dy.$$

Letting $r_0 \to R/4$, we get the desired estimate. □

As a consequence of Lemma 3.1, we have the following result.

**Corollary 3.2.** *Let $\Omega$ be a nonempty open set of $\mathbb{R}^N$ such that $\Omega \neq \mathbb{R}^N$. Then for every $\gamma \in \mathbb{N}^N$, there exists a constant $C > 0$ only depending on $N$, $\gamma$ and $s$ such that for every function $u$ which is $s$-harmonic in $\Omega$,*

$$|D^\gamma u(x)| \leq C\, \delta_\Omega^{2s-|\gamma|}(x) \int_{|y|\geq \frac{\delta_\Omega(x)}{4}} |u(y)||y|^{-N-2s}\, dy \qquad \textit{for all } x \in \Omega,$$

*where $\delta_\Omega(x) = \mathrm{dist}(x, \mathbb{R}^N \setminus \Omega)$.*

*Proof.* By assumption, $\delta_\Omega(x) < \infty$ for every $x \in \Omega$. Since $B(x, \delta_\Omega(x)) \subset \Omega$, applying Lemma 3.1 to the function $y \mapsto u(y + x)$ and $R = \delta_\Omega(x)$, we get the desired result. □

*Proof of Theorem* 1.1. Let $x \in \mathbb{R}^N$ and $r > 0$. We apply Corollary 3.2 with $\Omega = B(x, r)$ and $|\gamma| \geq 2s$. Then, letting $r \to \infty$, we get $|D^\gamma u(x)| = 0$ for every $|\gamma| \geq 2s$ and $x \in \mathbb{R}^N$. The proof of the theorem is thus completed. □

*Remark* 3.3. It is well known that there are smooth functions $u$ — hence in $L^1_{loc}(\mathbb{R}^N)$ — satisfying $\Delta u = 0$ in $\mathcal{D}'(\mathbb{R}^N)$ for $N \geq 2$ which are not polynomials. Therefore a natural question arises: does there exist a larger space of distributions, strictly containing $\mathcal{L}^1_s$, where the fractional Laplacian is appropriately defined and where there are nontrivial entire $s$-harmonic functions which are not affine?

*Proof of Theorem* 1.2. The proof will be done by induction. Suppose $\ell$ is the degree of $P$. Assume that $\ell = 0$ so that $P$ is a constant. Let $h \in \mathbb{R}^N$ and $u_h(x) = u(x + h) - u(x)$. It is clear that $u_h \in \mathcal{L}^1_s$. In addition $(-\Delta)^s u_h = 0$. It follows from Theorem 1.1 that $\partial_{i,j} u_h(0) = 0$ and therefore $\partial_{i,j} u(h) = \partial_{i,j} u(0)$ for every $h \in \mathbb{R}^N$. This implies that $u$ is a second order polynomial and since it belongs to $\mathcal{L}^1_s$, it is affine.

Now assume that the result holds true for a polynomial of degree up to $\ell \geq 0$ and suppose that $(-\Delta)^s u = P_{\ell+1}$, a polynomial of degree $\ell+1$. Then, for $h \in \mathbb{R}^N$, using the binomial formula we can see that $(-\Delta)^s u_h = P_{\ell,h}$, where $P_{\ell,h}$ is a polynomial of degree $\ell$. It follows from our assumption that $u_h$ is affine for any $h \in \mathbb{R}^N$. This again implies that $u$ is a second order polynomial and thus an affine function, since it belongs to $\mathcal{L}^1_s$. □

*Proof of Corollary* 1.3. We just note that $L^q(\mathbb{R}^N) \subset \mathcal{L}^1_s$ for every $q \in [1,\infty]$ by Hölder's inequality. □

*Proof of Corollary* 1.4. We define the function $\widetilde{u}(x) = \alpha_{N,s} \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2s}}\, dy$. Let $f_n \in C^\infty_c(\mathbb{R}^N)$ be such that $f_n \to f$ in $L^p(\mathbb{R}^N)$. Define $u_n(x) = \alpha_{N,s} \int_{\mathbb{R}^N} \frac{f_n(y)}{|x-y|^{N-2s}}\, dy$. By the Hardy-Littlewood-Sobolev inequality (see [12, Theorem 4.3]), we have $u_n \to \widetilde{u}$ in $L^{\frac{Np}{N-2sp}}(\mathbb{R}^N)$. In particular, $u_n \to \widetilde{u}$ in $\mathcal{L}^1_s$ by Hölder's inequality. Thanks to [6, Lemma 5.3], we have $(-\Delta)^s u_n = f_n$ in $\mathcal{D}'(\mathbb{R}^N)$. Passing to the limit as $n \to \infty$, we deduce that $(-\Delta)^s \widetilde{u} = f$ in $\mathcal{D}'(\mathbb{R}^N)$. Finally, if $u \in L^{\frac{Np}{N-2sp}}(\mathbb{R}^N)$ is an arbitrary solution to $(-\Delta)^s u = f$ in $\mathcal{D}'(\mathbb{R}^N)$, then $(-\Delta)^s(u - \widetilde{u}) = 0$ in $\mathcal{D}'(\mathbb{R}^N)$. We thus conclude, from Corollary 1.3, that $u = \widetilde{u}$. □

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