LINEAR CONGRUENCES WITH RATIOS

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Abstract. We use new bounds of double exponential sums with ratios of integers from prescribed intervals to get an asymptotic formula for the number of solutions to congruences

$$\sum_{j=1}^{n} a_j \frac{x_j}{y_j} \equiv a_0 \pmod{p},$$

with variables from rather general sets.

1. Introduction

1.1. Motivation. We count the number of solutions to a linear congruence with rational variables with restricted numerators and denominators. This includes solutions with rationals of a bounded height or more generally with numerators and denominators from a certain large class of sets with a regular boundary. For example, this class of sets includes all convex sets. In some special cases, the corresponding equation over $\mathbb{Q}$ has recently been considered by Blomer and Brüdern [2] and also by Blomer, Brüdern and Salberger [3]. However, in positive characteristic this natural question has never been studied before.

More precisely, for a prime $p$ we consider the equation

$$(1) \sum_{j=1}^{n} a_j \frac{x_j}{y_j} = a_0,$$

with coefficients $\mathbf{a} = (a_0, a_1, \ldots, a_n) \in \mathbb{F}_p^{n+1}$ and variables $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{F}_p^n$,

where $\mathbb{F}_p$ denotes the finite fields of $p$ elements.

Given a set $S \subseteq [0, p-1]^{2n}$, we use $N(\mathbf{a}; S)$ to denote the number of solutions to the equation (1) with variables $(x_1, y_1, \ldots, x_n, y_n) \in S$.

The equation (1) can be considered over the integers. In particular, recently Blomer, Brüdern and Salberger [3] have studied it for $n = 3$, $a_0 = 0$ and $a_1 = a_2 = a_3 = 1$. In particular, by [3, Theorem 1], the number of integer solutions with $(\mathbf{x}, \mathbf{y}) \in [-H,H]^6$ to the analogue of (1) with variables over $\mathbb{Z}$ is given by

$H^3 Q(H) + O(H^{3-\delta}),$ where $Q \in \mathbb{Q}[X]$ is a polynomial of degree 4 and $\delta > 0$ is some absolute constant. Blomer and Brüdern [2] have also suggested an alternative approach which yields a tight upper bound for the same equation but for a slightly
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different way of ordering and counting solutions. The methods of [2,3] can probably
be extended to arbitrary $n$ (see, for example, the comment in [3, Section 1.3]).

In [15], a different approach has been suggested, which is based on some argu-
ments from [13] and leads to bounds that are weaker by a logarithmic factor than
those expected to be produced by the methods of [2,3]; however it seems to be
more robust and is able to work in more general situations.

Here we combine some ideas from [13] with several other arguments and apply
them to the case of the equation (1) over a finite field.

Throughout the paper, any implied constants in the symbols $O$, $\ll$ and $\gg$
may depend on the integer parameter $n \geq 1$. We recall that the notation
$U = O(V)$, $U \ll V$ and $V \gg U$ are all equivalent to the statement that the inequality $|U| \leq cV$
holds with some constant $c > 0$.

1.2. Solutions in boxes. We fix some intervals

\begin{align*}
I_j &= [A_j + 1, A_j + K_j], \quad J_j = [B_j + 1, B_j + L_j] \subseteq [0, p - 1],
\end{align*}

with integers $A_j$, $B_j$, $K_j$ and $L_j$, $j = 1, \ldots, n$, and obtain the following asymptotic formula.

**Theorem 1.** For $n \geq 3$ and arbitrary intervals (2) for the box $B = I_1 \times J_1 \times \cdots \times I_n \times J_n$ we have

$$
\left| N(a; B) - \frac{1}{p} \prod_{j=1}^{n} (K_j L_j) \right| \leq \sqrt{K_1 L_1 K_2 L_2} \prod_{j=3}^{n} (K_j + \sqrt{pL_j}) p^{o(1)}.
$$

We now consider the case when $B$ is a cube with the side length $H$.

**Corollary 2.** For $n \geq 3$ and intervals (2) with $K_j = L_j = H$, $j = 1, \ldots, n$, for the cubic box $C = I_1 \times J_1 \times \cdots \times I_n \times J_n$ we have

$$
\left| N(a; C) - \frac{H^{2n}}{p} \right| \leq p^{n/2 - 1 + o(1)} H^{n/2 + 1}.
$$

In particular, the asymptotic formula of Corollary 2 is nontrivial starting from
the values of $H$ of order $p^{n/(3n-2)+\delta}$ for any fixed $\delta > 0$ and sufficiently large $p$.
We also record the following result which is convenient for further applications.

For a set $\Omega \subseteq [0,1]^{2n}$ we use $p\Omega$ to denote its blowup by $p$, that is,

$$
p\Omega = \{ p\omega : \omega \in \Omega \}.
$$

Rounding up and down the sides of $p\Gamma$ for a cubic box

\begin{align*}
\Gamma &= [\alpha_1, \alpha_1 + \xi] \times [\beta_1, \beta_1 + \xi] \times \cdots \times [\alpha_n, \alpha_n + \xi] \times [\beta_n, \beta_n + \xi] \in [0,1]^{2n},
\end{align*}

we derive

**Corollary 3.** For $n \geq 3$ and a cubic box (3) with $\xi > 1/p$ we have

$$
\left| N(a; p\Gamma) - \xi^{2n} p^{2n-1} \right| \leq \left( \xi^{2n-1} p^{2n-2} + \xi^{n/2+1} p^n \right) p^{o(1)}.
$$
1.3. Solutions in well-shaped sets. We combine Corollary 2 with some ideas of Schmidt [12] to get an asymptotic formula for \( N(a; \Omega) \) for a rather general class of sets, which includes all convex sets.

First we need to introduce some definitions. We define the distance between a vector \( \alpha \in [0, 1]^m \) and a set \( \Xi \subseteq [0, 1]^m \) by

\[
\text{dist}(\alpha, \Xi) = \inf_{\beta \in \Xi} \| \alpha - \beta \|,
\]

where \( \| \gamma \| \) denotes the Euclidean norm of \( \gamma \). Given \( \varepsilon > 0 \) and a set \( \Xi \subseteq [0, 1]^m \), we define the sets

\[
\Xi^+ = \{ \alpha \in [0, 1]^m \setminus \Xi : \text{dist}(\alpha, \Xi) < \varepsilon \}
\]

and

\[
\Xi^- = \{ \alpha \in \Xi : \text{dist}(\alpha, [0, 1]^m \setminus \Xi) < \varepsilon \}.
\]

We note that in the definition of \( \Xi^+ \) we discard the part of the outer \( \varepsilon \)-neighbourhood that does not belong to \([0, 1]^m \). These parts can also be included in \( \Xi^+ \) but this does not affect our argument as we work only with inner \( \varepsilon \)-neighbourhoods \( \Xi^- \) and \([0, 1]^m \setminus \Xi)^- = \Xi^+ \).

Following [15] (see also [9,10]), we say that a set \( \Xi \) is well-shaped if for every \( \varepsilon > 0 \) the Lebesgue measures \( \mu(\Xi^-) \) and \( \mu(\Xi^+) \) exist, for some constant \( C \), and satisfy

\[
\mu(\Xi^\pm) \leq C\varepsilon.
\]

As we have mentioned, all convex sets are well-shaped.

Theorem 4. For \( n \geq 3 \) and an arbitrary well-shaped set \( \Omega \subseteq [0, 1]^{2n} \) of Lebesgue measure \( \mu(\Omega) \), we have

\[
\left| N(a; p\Omega) - p^{2n-1}\mu(\Omega) \right| \leq p^{2n-(5n-4)/(3n-2)+o(1)}.
\]

2. Preliminaries

2.1. Multiplicative congruences. We recall the following special case of a result of Ayyad, Cochrane and Zheng [1, Theorem 1].

Lemma 5. Let \( I_j, J_j, j = 1, 2 \), be four intervals of the form (2). Then

\[
x_1y_2 \equiv x_2y_1 \pmod{p}, \quad x_i \in I_i, \ y_i \in J_i, \ i = 1, 2,
\]

has \( K_1K_2L_1L_2/p + O(\sqrt{K_1K_2L_1L_2p^{\alpha(1)}}) \) solutions.

We also need a version of the result of Cilleruelo and Garaev [5, Theorem 1].

Lemma 6. For any integers \( B, L \) and \( M \) with \( 0 \leq L, M < p \), the congruence

\[
(B + y)z \equiv 1 \pmod{p}, \quad 1 \leq y \leq L, \ 1 \leq z \leq M,
\]

has at most \( p^{-1/2+o(1)}L^{1/2}M + p^{\alpha(1)} \) solutions.

Proof. As in the proof of [5, Theorem 1] we note that by the Dirichlet principle, for any positive integers \( U < p \) and \( V \) with \( UV \geq p \) one can choose integers \( u \) and \( v \) with

\[
1 \leq u \leq U, \quad |v| = O(V), \quad uB \equiv v \pmod{p}
\]

(see also [6, Lemma 3.2] for a more general statement). With this choice of \( u \) and \( v \) the above congruence can be written as

\[
vz + uyz \equiv u \pmod{p}.
\]
We now take  

\[ U = \left\lceil \left(\frac{p}{L}\right)^{1/2} \right\rceil \]  

and  

\[ V = \left\lceil \left(\frac{pL}{1}^{1/2} \right) \right\rceil \]  

(thus  \( UV \geq p \)).

Since the left hand side is at most  \( O(MV + LMU) = O((pL)^{1/2}M) \), we see that for every solution \((y, z)\) we have

\[ vz + uyz = u + kp \]

with some integer  \( k = O \left( \left(\frac{pL}{1}^{1/2}M/p \right) = O \left( \left(\frac{p}{L} \right)^{1/2}M/p \right) \right) \).

We now recall the well-known bound

\[ \tau(m) \leq m^{o(1)} \]

on the number of integer positive divisors \( \tau(m) \) of an integer \( m \neq 0 \); see, for example, [7, Theorem 317]. Since by (5) we have the divisibility  \( z \mid |u + kp| \) and also  \( 0 < |u + kp| = O(p^2) \), we conclude that for each of the  \( O \left( \left(\frac{p}{L} \right)^{1/2}M/p \right) \) possible values of \( k \), there are at most  \( p^{o(1)} \) possible values for \( z \), and thus for \( y \). The result now follows.

\[ \Box \]

2.2. Exponential sums with ratios.

For a prime \( p \), we denote  \( e_p(z) = \exp(2\pi iz/p) \). Clearly for  \( p \nmid u \) the expression  \( e_p(\alpha x/u) \) is correctly defined (as  \( e_p(aw) \) for  \( w \equiv v/u \pmod{p} \)).

Let  

\[ I = [A + 1, A + K], \quad J = [B + 1, B + L] \subseteq [0, p - 1] \]

be two intervals with integers \( A, B, K \) and \( L \).

The following result is a variation of [13, Lemma 3]. We present it in the slightly more general form that we need for our applications.

**Lemma 7.** Let \( I \) and \( J \) be two intervals of the form (6) and let  \( W \subseteq I \times J \) be an arbitrary convex set. Then uniformly over the integers \( a \) with  \( \gcd(a, p) = 1 \), we have

\[ \sum_{(x,y) \in W} e_p(ax/y) \ll (K + p^{1/2}L^{1/2})p^{o(1)}, \]

where the summation is over all integral points \((x, y) \in W\).

**Proof.** Since \( W \) is convex, for each \( y \) there are integers  \( K \geq K_y > H_y \geq 1 \) such that

\[ \sum_{(x,y) \in W} e_p(ax/y) = \sum_{y \in J} \sum_{x = A + H_y}^{A + K_y} e_p(ax/y). \]

Following the proof of [13, Lemma 3], we define

\[ I = \lfloor \log(2p/K) \rfloor \quad \text{and} \quad J = \lfloor \log(2p) \rfloor . \]

Furthermore, for a rational number  \( \alpha = u/v \) with  \( \gcd(v, p) = 1 \), we denote by  \( \rho(\alpha) \) the unique integer \( w \) with  \( w \equiv u/v \pmod{p} \) and  \( |w| < p/2 \) (we can assume that  \( p \geq 3 \)). Using the bound

\[ \sum_{x = A + H_y}^{A + K_y} e_p(ax) \ll \min \left\{ K, \frac{p}{\rho(\alpha)} \right\}, \]

which holds for any rational  \( \alpha \) with the denominator that is not a multiple of  \( p \) (see [5, Bound (8.6)]), we obtain a version of [13, Equation (1)],

\[ \sum_{(x,y) \in W} e_p(ax/y) \ll KR + p \sum_{j=I+1}^{J} T_j e^{-j}, \]
where
\[ R = \# \{ y : B + 1 \leq y \leq B + L, \ |\rho(a/y)| < e^I \} , \]
\[ T_j = \# \{ y : B + 1 \leq y \leq B + L, \ e^j \leq |\rho(a/y)| < e^{j+1} \} . \]

We now see that Lemma 6 implies the bounds
\[ R \leq p^{-1/2+o(1)} L^{1/2} e^I + p^{o(1)} \leq p^{1/2+o(1)} L^{1/2} K^{-1} + p^{o(1)} \]
and
\[ T_j \leq p^{-1/2+o(1)} L^{1/2} e^j + p^{o(1)} . \]
Substituting these bounds in (7), we obtain
\[ \left| \sum_{(x,y) \in W} e_p(ax/y) \right|^2 \ll p^{1/2+o(1)} L^{1/2} + Kp^{o(1)} + p \sum_{j=I+1}^J p^{-1/2+o(1)} L^{1/2} e^j + p^{o(1)} e^{-j} \]
\[ = p^{1/2+o(1)} L^{1/2} + Kp^{o(1)} + Jp^{1/2+o(1)} L^{1/2} + p^{1+o(1)} e^{-I} \]
\[ = p^{1/2+o(1)} L^{1/2} + Kp^{o(1)} , \]
which concludes the proof.

We also need a version of Lemma 7 on average over \( a \).

**Lemma 8.** Let \( I \) and \( J \) be two intervals of the form (6). Then we have
\[ \sum_{a=1}^{p-1} \left| \sum_{x \in I} \sum_{y \in J} e_p(ax/y) \right|^2 \leq KLp^{1+o(1)} . \]

**Proof.** First we write
\[ \left( \sum_{a=1}^{p-1} \sum_{x \in I} \sum_{y \in J} e_p(ax/y) \right)^2 = \sum_{a=0}^{p-1} \left| \sum_{x \in I} \sum_{y \in J} e_p(ax/y) \right|^2 - K^2 L^2 . \]
Expanding the square of the inner sum on the right hand side of (8), changing the order of summations and using the orthogonality of characters, we obtain
\[ \sum_{a=0}^{p-1} \left| \sum_{x \in I} \sum_{y \in J} e_p(ax/y) \right|^2 = \sum_{x_1, x_2 \in I} \sum_{y_1, y_2 \in J} \sum_{a=0}^{p-1} e_p(a(x_1/y_1 - x_2/y_2)) = pT , \]
where \( T \) is the number of solutions to the congruence
\[ x_1/y_1 \equiv x_2/y_2 \pmod{p}, \quad x_1, x_2 \in I, y_1, y_2 \in J . \]
Extending the admissible region of solutions to \( I \times J \) and evoking Lemma 5 we conclude that
\[ T = \frac{K^2 L^2}{p} + O \left( KLp^{o(1)} \right) , \]
which together with (8) completes the proof.

\[ \square \]
3. Proofs of the main results

3.1. Proof of Theorem 1. Using the orthogonality of the exponential function, we write

\[ N(a; B) = \sum_{(x_1, y_1, \ldots, x_n, y_n) \in B} \frac{1}{p} \sum_{\lambda=0}^{p-1} e_p \left( \lambda \left( \sum_{j=1}^{n} a_j \frac{x_j}{y_j} - a_0 \right) \right). \]

Changing the order of summation, and recalling that \( B \) is a direct product of the intervals \( I_j \) and \( J_j \), \( j = 1, \ldots, n \), we obtain

\[ N(a; B) = \frac{1}{p} \prod_{j=1}^{n} \sum_{x_j \in I_j} \sum_{y_j \in J_j} e_p \left( \lambda a_j x_j / y_j \right). \]

Now, the contribution from \( \lambda = 0 \) gives the main term

\[ \frac{1}{p} \prod_{j=1}^{n} \sum_{x_j \in I_j} \sum_{y_j \in J_j} 1 = \frac{1}{p} \prod_{j=1}^{n} (K_j L_j). \]

To estimate the error term, we apply Lemma 7 to \( n - 2 \) sums with \( j = 3, \ldots, n \), getting

\[ (10) \quad N(a; B) - \frac{1}{p} \prod_{j=1}^{n} (K_j L_j) \leq p^{-1+o(1)} \prod_{j=3}^{n} (K_j + p^{1/2} L_j^{1/2}) W, \]

where

\[ W = \sum_{\lambda=1}^{p-1} \left| \sum_{x_1 \in I_1} \sum_{y_1 \in J_1} e_p (\lambda a_1 x_1 / y_1) \right| \left| \sum_{x_2 \in I_2} \sum_{y_2 \in J_2} e_p (\lambda a_2 x_2 / y_2) \right|. \]

Hence, by the Cauchy inequality,

\[ (11) \quad W \leq \sqrt{W_1 W_2}, \]

where, for \( \nu = 1, 2 \),

\[ W_\nu = \sum_{\lambda=1}^{p-1} \left| \sum_{x_\nu \in I_\nu} \sum_{y_\nu \in J_\nu} e_p (\lambda a_\nu x_\nu / y_\nu) \right|^2 = \sum_{a=1}^{p-1} \left| \sum_{x_\nu \in I_\nu} \sum_{y_\nu \in J_\nu} e_p (a x_\nu / y_\nu) \right|^2. \]

We now apply Lemma 8 to estimate \( W_1 \) and \( W_2 \) and see from (11) that

\[ W \leq \sqrt{K_1 L_1 K_2 L_2} p^{1+o(1)}, \]

which together with (10) concludes the proof.

3.2. Proof of Corollaries 2 and 3. For Corollary 2, we see that the first term appearing in the bound of Theorem 1 is \( H^2 \) while each term in the product becomes \( O(p^{1/2} H^{1/2}) \). The result now follows.

For Corollary 3, we approximate the set \( pI \) by two cubes with side lengths \( \lfloor \xi p \rfloor \) and \( \lceil \xi p \rceil \). Since \( \xi > 1/p \), we have \((\xi p + O(1))^{2n} = (\xi p)^{2n} + O((\xi p)^{2n-1})\). The result now follows from Corollary 2.
3.3. **Proof of Theorem 4.** The proof follows the arguments of the proofs of [10, Theorem 1] or [14, Theorem 3.1] (however the concrete details are different).

First we observe that since the complementary set \([0, 1]^{2n} \setminus \Omega\) is also well-shaped, it is enough to establish only the lower bound

\[
N(a; p\Omega) \geq \frac{N(p\Omega)}{p} + O\left(p^{2n-4/3+o(1)}\right).
\]

We now recall some constructions and arguments from the proof of [12, Theorem 2]. Pick a point \(\alpha = (\alpha_1, \ldots, \alpha_{2n}) \in [0, 1]^{2n}\) such that all its coordinates are irrational. For a positive integer \(k\), let \(C(k)\) be the set of cubes of the form

\[
\left[\frac{\alpha_1 + u_1}{k}, \frac{\alpha_1 + 1}{k}\right] \times \ldots \times \left[\frac{\alpha_{2n} + u_{2n}}{k}, \frac{\alpha_{2n} + 1}{k}\right],
\]

with \(u_1, \ldots, u_{2n} \in \mathbb{Z}\).

We consider the set of points

\[
(x_1 \frac{p}{p}, y_1 \frac{p}{p}, \ldots, x_n \frac{p}{p}, y_n \frac{p}{p}) \in [0, 1]^{2n}
\]

taken over all solutions \((x, y) \in \mathbb{F}_p^{2n}\) to the equation \((a; \Omega)\).

Note that the above irrationality condition on \(\alpha\) guarantees that the points \((x, y)\) all belong to the interior of the cubes from \(C(k)\).

Furthermore, let \(C_0(k)\) be the set of cubes from \(C(k)\) that are contained inside \(\Omega\). By [12, Equation (9)], for any well-shaped set \(\Omega \in [0, 1]^{2n}\), we have

\[
\#C_0(k) = k^{2n}\mu(\Omega) + O(k^{2n-1}).
\]

Let \(B_1 = C_0(2)\) and for \(i = 2, 3, \ldots\), let \(B_i\) be the set of cubes \(\Gamma \in C_0(2^i)\) that are not contained in any cube from \(C_0(2^{i-1})\). Clearly

\[
2^{-2n}\#B_i + 2^{-2(i-1)n}\#C_0(2^{i-1}) \leq \mu(\Omega), \quad i = 2, 3, \ldots.
\]

We now infer from (14) that

\[
\mu(\Omega) - 2^{-2(i-1)n}\#C_0(2^{i-1})
\]

\[
= \mu(\Omega) - 2^{-2(i-1)n}\left(2^{2(i-1)n}\mu(\Omega) + O(2^{2(i-1)(2n-1)})\right)
\]

\[
\ll 2^{(i-1)(2n-1)-2(i-1)n} = 2^{-i+1}.
\]

Therefore, the inequality (15) implies the bound

\[
\#B_i \ll 2^{i(2n-1)}.
\]

We also see that for any integer \(M \geq 1\),

\[
\Omega \setminus \Omega_{M-1} \subseteq \bigcup_{i=1}^{M} \bigcup_{\Gamma \subseteq \Omega_i} \Gamma \subseteq \Omega,
\]

with \(\varepsilon = (2n)^{1/2}2^{-M}\). Indeed, for any point \(\gamma \in \Omega \setminus \Omega_{M-1}\) there is a cube \(\Gamma_\gamma \in C(2^M)\) with \(\gamma \in \Gamma\) (since for any integer \(k \geq 1\), the cubes from \(C(k)\) tile the whole space \(\mathbb{F}_p^{2n}\)). Because the diameter (that is, the largest distance between the points) of \(\Gamma_\gamma\)
is \((2n)^{1/2}2^{-M}\), we see from the definition of \(\Omega^{-\varepsilon}\) that \(\Gamma \cap [0,1]^{2n} \setminus \Omega = \emptyset\). Thus \(\Gamma \subseteq \Omega\). This implies
\[
\Gamma \subseteq \bigcup_{i=1}^{2n} \bigcup_{\Gamma \in \mathcal{B}_i} \Gamma
\]
and \((17)\) follows.

Since \(\Omega\) is well-shaped, from \((4)\) we deduce that
\[
\mu \left( \bigcup_{i=1}^{2n} \bigcup_{\Gamma \in \mathcal{B}_i} \Gamma \right) = \mu(\Omega) + O(2^{-M}).
\]

We now assume that
\[
2^{M} < p,
\]
so Corollary \(3\) applies to all cubes \(\Gamma \in \mathcal{C}_0(2^i), \ i = 1, \ldots, M\). Together with \((17)\), this implies the inequality:
\[
N(a; p\Omega) \geq \sum_{i=1}^{M} \sum_{\Gamma \in \mathcal{B}_i} N(a; p\Gamma) = p^{2n-1} \sum_{i=1}^{M} \sum_{\Gamma \in \mathcal{B}_i} \mu(\Gamma) + O(R),
\]
where
\[
R = \sum_{i=1}^{M} \# \mathcal{B}_i \left( 2^{-i(2n-1)} p^{2n-2} + 2^{-i(n/2+1)} p^n \right) p^{o(1)}.
\]

We see from \((18)\) that
\[
p^{2n-1} \sum_{i=1}^{M} \sum_{\Gamma \in \mathcal{B}_i} \mu(\Gamma) = p^{2n-1} \mu \left( \bigcup_{i=1}^{M} \bigcup_{\Gamma \in \mathcal{B}_i} \Gamma \right) = p^{2n-1} \mu(\Omega) + O \left( p^{2n-1} 2^{-M} \right).
\]

Furthermore, using \((16)\), we derive
\[
R \leq \sum_{i=1}^{M} \left( p^{2n-2} + 2^{i(3n/2-2)} p^n \right) p^{o(1)} = \left( Mp^{2n-2} + 2^M (3n/2-2) p^n \right) p^{o(1)}.
\]

Substituting \((21)\) and \((22)\) in \((20)\) with the above choice of \(M\), noticing that \((19)\) implies \(M = O(\log p)\), we obtain
\[
N(a; p\Omega) \geq p^{2n-1} \mu(\Omega) - Qp^{o(1)},
\]
where
\[
Q \leq p^{2n-1} 2^{-M} + p^{2n-2} + 2^M (3n/2-2) p^n.
\]

We now choose \(M\) to satisfy
\[
2^M \leq p^{2(n-1)/(3n-2)} < 2^{M+1},
\]
which asymptotically optimises the right hand side of the bound \((24)\), verifies \((19)\) and produces the bound \(Q \ll p^{2n-5n/(3n-2)} + p^{2n-2} \ll p^{2n-5n/(3n-2)}\). We now see from \((23)\) that \((12)\) holds, which concludes the proof.
4. Comments

We note that for $B_1 = \ldots = B_n = 0$, using [13, Lemma 3] instead of Lemma [7] in this special case, one can improve Theorem [1] as follows:

\[
\left| N(a; B) - \frac{1}{p} \prod_{j=1}^{n} (K_j L_j) \right| \\
\leq \left( \frac{K_1 L_1}{p^{1/2}} + \sqrt{K_1 L_1} \right) \left( \frac{K_2 L_2}{p^{1/2}} + \sqrt{K_2 L_2} \right) \prod_{j=3}^{n} (K_j + L_j)p^{o(1)}.
\]

Furthermore, it is easy to see that one can get a version of Lemma 8 for the more general sums of Lemma 7, which now becomes

\[
\left| \sum_{a=1}^{p-1} \sum_{(x,y) \in W} e_p(ax/y) \right|^2 \leq K^2 L^2 + KLP^{1+o(1)},
\]

that is, there is no cancellation of the main term for the number of solutions to the congruence (9) anymore. Thus the same arguments lead to the following result. For $n \geq 3$ and arbitrary intervals and arbitrary convex sets $W_j \subseteq I_j \times J_j$, $j = 1, \ldots, n$, for the set $S = W_1 \times \ldots \times W_n$ we have

\[
\left| N(a; S) - N(S) \right| p \\
\leq \left( \frac{K_1 L_1}{p^{1/2}} + \sqrt{K_1 L_1} \right) \left( \frac{K_2 L_2}{p^{1/2}} + \sqrt{K_2 L_2} \right) \prod_{j=3}^{n} (K_j + \sqrt{pL_j})p^{o(1)},
\]

where $N(S) = \# (S \cap \mathbb{Z}^{2n})$. For example, this can be used for counting solutions to the equation (11) with variables in disks

\[(x_j - b_j)^2 + (y_j - c_j)^2 \leq r_j^2, \quad j = 1, \ldots, n.\]

One can also ask about solutions to (11) with the additional co-primality condition $\gcd(x_j, y_j) = 1$, $j = 1, \ldots, n$, that is, essentially in Farey fractions. Using simple inclusion-exclusion arguments, one can easily derive relevant asymptotic formulas from our results.

Finally, we remark that Lemma 4 can be viewed as a statement about cancellations among short Kloosterman sums of the form

\[\mathcal{K}(\lambda; J) = \sum_{u \in J} e_p(\lambda/u)\]

over an interval $J = [B+1, B+L]$ when $\lambda$ runs over an interval $I = [A+1, A+K]$. Say, for $K = L$ we have a nontrivial cancellation starting with $L \geq p^{1/3+\delta}$ for any fixed $\delta > 0$, which is beyond the range of modern bounds of individual short Kloosterman sums over intervals that are not at the origin; we refer to the recent work of Bourgain and Garaev [4] for an outline of the state of art and several results.

References

[1] Anwar Ayyad, Todd Cochrane, and Zhiyong Zheng, The congruence $x_1x_2 \equiv x_3x_4 \pmod{p}$, the equation $x_1x_2 = x_3x_4$, and mean values of character sums, J. Number Theory 59 (1996), no. 2, 398–413, DOI 10.1006/jnth.1996.0105. MR1402616 (97f:11091)


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