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ON WEIGHTED L^2 ESTIMATES FOR SOLUTIONS OF THE WAVE EQUATION

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Abstract. In this paper we consider weighted L^2 integrability for solutions of the wave equation. For this, we obtain some weighed L^2 estimates for the solutions with weights in Morrey-Campanato classes. Our method is based on a combination of bilinear interpolation and a localization argument which makes use of the Littlewood-Paley theorem and a property of Hardy-Littlewood maximal functions. We also apply the estimates to the problem of well-posedness for wave equations with potentials.

1. Introduction

Let us first consider the following Cauchy problem associated with the wave equation in $\mathbb{R}^{n+1}$, $n \geq 2$:

$$\begin{cases} -\partial_t^2 u + \Delta u = F(x,t), \\ u(x,0) = f(x), \\ \partial_t u(x,0) = g(x). \end{cases} \tag{1.1}$$

Then it is well known that the solution is given by

$$u(x,t) = \cos(t\sqrt{-\Delta})f + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g - \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(s)ds \tag{1.2}$$

in view of the Fourier transform.

The aim of this paper is to find a suitable relation between the Cauchy data f, g, the forcing term F and the weight $w(x,t)$ which guarantee that the solution lies in weighted L^2 spaces, $L^2_{x,t}(w)$.

A natural way of approaching this problem may be to control weighted L^2 integrability of the solution in terms of regularity of f, g, F. In fact our basic strategy is to obtain the following type of estimates:

$$\|u\|_{L^2_{x,t}(w)} \leq C_{s,\widetilde{s},q,r}(w)\big(\|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}} + \|F\|_{L^q_t \dot{B}^{\widetilde{s}}_{r,2}}\big), \tag{1.3}$$

where $\dot{B}^{\widetilde{s}}_{r,2}$ is the usual homogeneous Besov space (cf. [3]) and $C_{s,\widetilde{s},q,r}(w)$ is a suitable constant depending on the regularity indexes $s, \widetilde{s}, q, r$ and the weight w. The point here is that $C_{s,\widetilde{s},q,r}(w)$ reflects some information about the relation between

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the weight and the regularity of f, g, F. By considering the operator $e^{it\sqrt{-\Delta}}$, the estimate (1.3) will consist of the homogeneous estimate

$$\left\|e^{it\sqrt{-\Delta}}f\right\|_{L^2_{x,t}(w)} \le C_s(w)\|f\|_{\dot{H}^s}$$

and the inhomogeneous estimate

$$\left\|\int_0^t e^{i(t-s)\sqrt{-\Delta}}F(\cdot,s)ds\right\|_{L^2_{x,t}(w)} \le C_{\widetilde{s},q,r}(w)\|F\|_{L^q_t\dot{B}^{\widetilde{s}+1}_{r,2}}.$$

Before stating our results, we need to introduce a function class. For $\alpha > 0$ and $1 \le p \le n/\alpha$, a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be in the Morrey-Campanato class $\mathfrak{L}^{\alpha,p}$ if

$$\|f\|_{\mathfrak{L}^{\alpha,p}} := \sup_{Q \text{ cubes in } \mathbb{R}^n} |Q|^{\alpha/n}\Big(\frac{1}{|Q|}\int_Q |f(y)|^p dy\Big)^{\frac{1}{p}} < \infty.$$

In particular, $\mathfrak{L}^{\alpha,p} = L^p$ when $p = n/\alpha$, and even $L^{n/\alpha,\infty} \subset \mathfrak{L}^{\alpha,p}$ for $p < n/\alpha$.

Then we have the following result which can be regarded as the weighted L^2 Strichartz estimates for the wave equation. Strichartz estimates on weighted L^2 spaces have been studied for the Schrödinger equation ([1, 2, 15, 17]).

Theorem 1.1. *Let $n \ge 2$. Then we have for $(n+1)/4 \le s < n/2$ and $1 < p \le (n+1)/(2s+1)$*

$$\left\|e^{it\sqrt{-\Delta}}f\right\|_{L^2(w(x,t))} \le C\|w\|^{1/2}_{\mathfrak{L}^{2s+1,p}}\|f\|_{\dot{H}^s}, \tag{1.4}$$

and for $\widetilde{s} > (n-1)/2$ and $1 \le q, r \le 2$

$$\left\|\int_0^t e^{i(t-s)\sqrt{-\Delta}}F(s)ds\right\|_{L^2(w(x,t))} \le C\|w\|^{1/2}_{\mathfrak{L}^{\alpha,p}}\|F\|_{L^q_t\dot{B}^{\widetilde{s}+1}_{r,2}} \tag{1.5}$$

with $1 < p \le (n+1)/\alpha$ and

$$\alpha = 2(\widetilde{s} - \frac{1}{q} - \frac{n}{r} + \frac{n+4}{2}) + 1. \tag{1.6}$$

Remarks. (i) From the definition it is clear that $\|f(\lambda\cdot)\|_{\mathfrak{L}^{\alpha,p}} = \lambda^{-\alpha}\|f\|_{\mathfrak{L}^{\alpha,p}}$. Using this one can see that $\alpha = 2s+1$ is the only possible index which allows (1.4) to be invariant under the scaling $(x,t) \to (\lambda x, \lambda t)$, $\lambda > 0$.

(ii) In the special case where $\widetilde{s}+1 = 0$, one can see that $L^r \subset \dot{B}^{\widetilde{s}+1}_{r,2}$ for $1 < r \le 2$, and (1.6) is just the scaling condition for (1.5) with L^r instead of $\dot{B}^{\widetilde{s}+1}_{r,2}$.

(iii) From the classical Strichartz's estimate ([19]), one can see that

$$\left\|e^{it\sqrt{-\Delta}}f\right\|_{L^r_{x,t}} \le C\|f\|_{\dot{H}^s}$$

for $2(n+1)/(n-1) \le r < \infty$ and $s = n/2 - (n+1)/r$. Then, by this and Hölder's inequality, it follows that for $1/2 \le s < n/2$,

$$\left\|e^{it\sqrt{-\Delta}}f\right\|^2_{L^2(w(x,t))} \le \|w\|_{L^{\frac{r}{r-2}}}\|e^{it\sqrt{-\Delta}}f\|^2_{L^r} \le C\|w\|_{L^{\frac{n+1}{2s+1}}}\|f\|^2_{\dot{H}^s}. \tag{1.7}$$

Hence our estimate (1.4) can be seen as natural extensions to the Morrey-Campanato classes of (1.7).

Some weighted $L^2_{x,t}$ estimates are known for the wave equation. In particular, (1.4) can be found in [15] for $w(x,t)$ satisfying $\sup_t w(x,t) \in \mathfrak{L}^{2,p}(\mathbb{R}^n)$ with $p > (n-1)/2$, $n \ge 3$. In fact, it is easy to see that this condition implies $w \in \mathfrak{L}^{2,p}(\mathbb{R}^{n+1})$ for the same p. For a specific weight $w(x,t) = |x|^{-(2s+1)}$, $0 < s < (n-1)/2$,

(1.4) is proved in [8] (see (3.6) there). Note that $|x|^{-(2s+1)} \in \mathfrak{L}^{2s+1,p}(\mathbb{R}^{n+1})$ for $p < n/(2s+1)$. But, our theorem gives estimates for more general time-dependent weights $w(x,t)$. Also, the smoothing estimates (known as Morawetz estimates)

$$\big\||x|^{-b/2}e^{itD^a}f\big\|_{L^2_{t,x}} \le C\|D^{(b-a)/2}f\|_2 \tag{1.8}$$

have been proved by many authors for the wave equation ($a = 1$) [14] and for the Schrödinger equation ($a = 2$) [10,17,22]. For more general Morawetz estimates with angular smoothing, see [6,9,17]. These estimates can be compared with (1.4) for the wave equation ($a = 1$). Indeed, since $w(x)$ is in $\mathfrak{L}^{\alpha,p}(\mathbb{R}^{n+1})$ if $w(x) \in \mathfrak{L}^{\alpha,p}(\mathbb{R}^n)$ for $1 \le p \le n/\alpha$, the following corollary is directly deduced from (1.4) with $s = (b-1)/2$. Then, (1.8) with $a = 1$ and $(n+3)/2 \le b < n$ is just a particular case of (1.9) because $|x|^{-b} \in \mathfrak{L}^{b,p}(\mathbb{R}^n)$.

Corollary 1.2. *Let $n > 3$. Then we have*

$$\big\||w(x)|^{1/2}e^{it\sqrt{-\Delta}}f\big\|_{L^2_{t,x}} \le C\|D^{(b-1)/2}f\|_2 \tag{1.9}$$

if $w \in \mathfrak{L}^{b,p}(\mathbb{R}^n)$ for $(n+3)/2 \le b < n$ and $1 < p \le n/b$.

Compared with the index $\alpha = 2s+1$ in (1.4), we can have the same index of α in (1.5) if

$$\frac{n+1}{4} \le \widetilde{s} - \frac{1}{q} - \frac{n}{r} + \frac{n+4}{2} < \frac{n}{2}$$

(see (1.6)). Hence, setting $s = \widetilde{s} - 1/q - n/r + (n+4)/2$ and using the fact that

$$\|(\sqrt{-\Delta})^{-1}f\|_{\dot{B}^{\widetilde{s}+1}_{r,2}} \le C\|f\|_{\dot{B}^{\widetilde{s}}_{r,2}},$$

the following corollary is deduced from a simple combination of (1.2), (1.4) and (1.5).

Corollary 1.3. *Let $n \ge 2$. If u is a solution of the Cauchy problem* (1.1)*, then*

$$\|u\|_{L^2(w(x,t))} \le C\|w\|^{1/2}_{\mathfrak{L}^{2s+1,p}}\Big(\|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}} + \|F\|_{L^q_t\dot{B}^{\widetilde{s}}_{r,2}}\Big)$$

if $\frac{n+1}{4} \le s < \frac{n}{2}$, $1 < p \le \frac{n+1}{2s+1}$, $\widetilde{s} > \frac{n-1}{2}$, and $1 \le q, r \le 2$, with $s = \widetilde{s} - \frac{1}{q} - \frac{n}{r} + \frac{n+4}{2}$.

Let us sketch the organization of the paper. In Section 2, we obtain a property of the Morrey-Campanato class regarding the Hardy-Littlewood maximal function, which is to be used for the proof of the weighted L^2 estimates in Theorem 1.1. Then, using bilinear interpolation and a localization argument based on the Littlewood-Paley theorem in weighted L^2 spaces, we prove Theorem 1.1 in Section 3. Finally in Section 4, we apply the estimates to the well-posedness theory for wave equations with potentials.

Throughout this paper, we will use the letter C to denote a constant which may be different at each occurrence. We also denote by $\widehat{f}$ the Fourier transform of f and by $\langle f, g\rangle$ the usual inner product of f, g on L^2. Given two complex Banach spaces A_0 and A_1, we denote by $(A_0, A_1)_{\theta,q}$ the real interpolation spaces for $0 < \theta < 1$ and $1 \le q \le \infty$. In particular, $(A_0, A_1)_{\theta,q} = A_0 = A_1$ if $A_0 = A_1$. See [3,21] for details.

2. Preliminary lemmas

In this section we present some preliminary lemmas which will be used for the proof of Theorem 1.1.

A weight $w : \mathbb{R}^n \to [0, \infty]$ is a locally integrable function that is allowed to be zero or infinite only on a set of Lebesgue measure zero, and we denote by $w \in A_2(\mathbb{R}^n)$ that w is in the Muckenhoupt $A_2(\mathbb{R}^n)$ class which is defined by

$$\sup_{Q \text{ cubes in } \mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q w(x)dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1} dx \right) < C_{A_2}.$$

Also, w is said to be in the class A_1 if there is a constant C_{A_1} such that for almost every x

$$M(w)(x) \le C_{A_1} w(x),$$

where $M(w)$ is the Hardy-Littlewood maximal function of w given by

$$M(w)(x) = \sup_Q \frac{1}{|Q|} \int_Q w(y)dy,$$

where the sup is taken over all cubes Q in $\mathbb{R}^n$ with center x. Then,

$$A_1 \subset A_2 \quad \text{with} \quad C_{A_2} \le C_{A_1}. \tag{2.1}$$

See, for example, [7] for more details. Also, the following lemma can be found in Chapter 5 of [18]. (See also Proposition 2 in [5].)

Lemma 2.1. *If $M(w)(x) < \infty$ for almost every $x \in \mathbb{R}^n$, then for $0 < \delta < 1$*

$$(M(w))^\delta \in A_1,$$

with C_{A_1} independent of w.

Next we obtain the following property of the Morrey-Campanato class regarding the Hardy-Littlewood maximal function. A similar property when $\alpha = 2$ can also be found in [4].

Lemma 2.2. *Let $w \in \mathfrak{L}^{\alpha,p}$ be a weight on $\mathbb{R}^{n+1}$, and let $w_*(x,t)$ be the n-dimensional maximal function defined by*

$$w_*(x,t) = \sup_{Q'} \left(\frac{1}{|Q'|} \int_{Q'} w(y,t)^\rho dy \right)^{\frac{1}{\rho}}, \quad \rho > 1,$$

where Q' denotes a cube in $\mathbb{R}^n$ with center x. Then, if $p > \rho$ and $p > 1/\alpha$, we have

$$\sup_Q |Q|^{\frac{\alpha}{n+1}} \left(\frac{1}{|Q|} \int_Q w_*(x,t)^p dxdt \right)^{\frac{1}{p}} \le C \sup_Q |Q|^{\frac{\alpha}{n+1}} \left(\frac{1}{|Q|} \int_Q w(y,t)^p dydt \right)^{\frac{1}{p}}.$$

Namely, if $p > \rho$ and $p > 1/\alpha$, $\|w_*\|_{\mathfrak{L}^{\alpha,p}} \le C\|w\|_{\mathfrak{L}^{\alpha,p}}$. Furthermore, $w_*(\cdot,t) \in A_2(\mathbb{R}^n)$ in the x variable with a constant C_{A_2} uniform in almost every $t \in \mathbb{R}$.

Proof. Fix a cube Q in $\mathbb{R}^{n+1}$ with center at (z,τ) and side length δ. Let us define the rectangles R_k, $k \ge 1$, such that $(y,t) \in R_k$ if $|t-\tau| < 2\delta$ and $y \in \widetilde{Q}(z, 2^{k+1}\delta) \setminus \widetilde{Q}(z, 2^k\delta)$. Here, $\widetilde{Q}(z,r)$ denote a cube in $\mathbb{R}^n$ with center z and side length r. Also, let $R_0 = 4Q$.

Now, setting $w^{(k)} = w\chi_{R_k}$ with the characteristic function χ_{R_k} of the set R_k, we may write

$$w(y,t) = \sum_{k \ge 0} w^{(k)}(y,t) + \phi(y,t),$$

where $\phi(y,t)$ is supported on $\mathbb{R}^{n+1} \setminus \bigcup_{k\geq 0} R_k$. Then it is easy to see that

$$w_*(x,t) \leq \sum_{k\geq 0} w_*^{(k)}(x,t) + \phi_*(x,t)$$

and

$$\Big(\frac{1}{|Q|}\int_Q w_*(x,t)^p dxdt\Big)^{\frac{1}{p}} \leq \sum_{k\geq 0}\Big(\frac{1}{|Q|}\int_Q w_*^{(k)}(x,t)^p dxdt\Big)^{\frac{1}{p}} + \Big(\frac{1}{|Q|}\int_Q \phi_*(x,t)^p dxdt\Big)^{\frac{1}{p}}.$$

Since $(x,t)\in Q$, it is clear that $\phi_*(x,t)=0$. Also, applying the well-known maximal theorem, $\|M(f)\|_q \leq C\|f\|_q$, $q>1$, with $q=p/\rho$, we see that if $p>\rho$,

$$\begin{aligned} |Q|^{\frac{\alpha}{n+1}}\Big(\frac{1}{|Q|}\int_Q w_*^{(0)}(x,t)^p dxdt\Big)^{\frac{1}{p}} &\leq C|Q|^{\frac{\alpha}{n+1}}\Big(\frac{1}{|Q|}\int_{4Q} w(y,t)^p dydt\Big)^{\frac{1}{p}} \\ &\leq C\|w\|_{\mathfrak{L}^{\alpha,p}}. \end{aligned} \tag{2.2}$$

Consequently, we only need to consider the case where $k\geq 1$.

Let $k\geq 1$. Since $(x,t)\in Q$, it follows that

$$\begin{aligned} w_*^{(k)}(x,t) &= \sup_{Q'}\Big(\frac{1}{|Q'|}\int_{Q'} w(y,t)^\rho \chi_{R_k}(y,t)dy\Big)^{\frac{1}{\rho}} \\ &\leq C\Big(\frac{1}{(2^k\delta)^n}\int_{\widetilde{Q}(z,2^{k+1}\delta)\setminus\widetilde{Q}(z,2^k\delta)} w(y,t)^\rho dy\Big)^{\frac{1}{\rho}} \\ &\leq C\Big(\frac{1}{(2^k\delta)^n}\int_{\widetilde{Q}(z,2^{k+1}\delta)\setminus\widetilde{Q}(z,2^k\delta)} w(y,t)^p dy\Big)^{\frac{1}{p}}, \end{aligned}$$

where we used Hölder's inequality for the last inequality since $p\geq\rho$. Thus,

$$\begin{aligned} \int_Q w_*^{(k)}(x,t)^p dxdt &\leq \frac{C}{(2^k\delta)^n}\int_{|\tau-t|<\delta}\int_{\widetilde{Q}(z,2^{k+1}\delta)\setminus\widetilde{Q}(z,2^k\delta)} w(y,t)^p \int_{|z-x|<\delta} 1\, dxdydt \\ &\leq \frac{C}{2^{kn}}\int_{R_k} w(y,t)^p dydt. \end{aligned}$$

Since we can clearly choose a cube Q_k in $\mathbb{R}^{n+1}$ such that $R_k\subset Q_k$ and $|Q_k|\sim (2^k\delta)^{n+1}$, we now get

$$\begin{aligned} |Q|^{\frac{\alpha}{n+1}}\Big(\frac{1}{|Q|}\int_Q w_*^{(k)}(x,t)^p dxdt\Big)^{\frac{1}{p}} &\leq C|Q|^{\frac{\alpha}{n+1}}\Big(\frac{1}{2^{kn}|Q|}\int_{R_k} w(y,t)^p dydt\Big)^{\frac{1}{p}} \\ &\leq C|Q|^{\frac{\alpha}{n+1}}\Big(\frac{2^k}{|Q_k|}\int_{Q_k} w(y,t)^p dydt\Big)^{\frac{1}{p}} \\ &\leq C2^{-\alpha k+\frac{k}{p}}|Q_k|^{\frac{\alpha}{n+1}}\Big(\frac{1}{|Q_k|}\int_{Q_k} w(y,t)^p dydt\Big)^{\frac{1}{p}}. \end{aligned}$$

Hence, if $p>1/\alpha$ and $p\geq\rho$, we conclude that

$$\sum_{k\geq 1}|Q|^{\frac{\alpha}{n+1}}\Big(\frac{1}{|Q|}\int_Q w_*^{(k)}(x,t)^p dxdt\Big)^{\frac{1}{p}} \leq C\|w\|_{\mathfrak{L}^{\alpha,p}}.$$

By combining this and (2.2), if $p>\rho$ and $p>1/\alpha$, we get $\|w_*\|_{\mathfrak{L}^{\alpha,p}}\leq C\|w\|_{\mathfrak{L}^{\alpha,p}}$ as desired.

Finally, to show the last assertion in the lemma, note first that

$$w_*(x,t) = (M(w(\cdot,t)^\rho))^{1/\rho}.$$

Since $w \in \mathfrak{L}^{\alpha,p}$ and $p \geq \rho$, it is not difficult to see that $M(w(\cdot,t)^\rho) < \infty$ for almost every $x \in \mathbb{R}^n$. Now, applying Lemma 2.1 with $\delta = 1/\rho$, we conclude that $w_*(\cdot,t) \in A_1$ with C_{A_1} uniform in $t \in \mathbb{R}$, which in turn implies that $w_*(\cdot,t) \in A_2$ with C_{A_2} uniform in $t \in \mathbb{R}$ (see (2.1)). □

3. Proof of Theorem 1.1

Since $w \leq w_*$ and $\|w_*\|_{\mathfrak{L}^{\alpha,p}} \leq C\|w\|_{\mathfrak{L}^{\alpha,p}}$ for $p > \rho > 1$ and $p > 1/\alpha$ (see Lemma 2.2), it is enough to prove Theorem 1.1 by replacing w with w_*. The motivation behind this replacement is that $w_*(\cdot,t) \in A_2(\mathbb{R}^n)$ in the x variable with a constant C_{A_2} uniform in almost every $t \in \mathbb{R}$ (see Lemma 2.2). This A_2 condition will enable us to use a localization argument in weighted L^2 spaces. Consequently, we may assume that w satisfies the same A_2 condition in proving the theorem.

3.1. **Homogeneous estimates.** First we prove the homogeneous estimate (1.4). Let ϕ be a smooth function supported in $(1/2, 2)$ such that

$$\sum_{k=-\infty}^{\infty} \phi(2^k t) = 1, \quad t > 0.$$

For $k \in \mathbb{Z}$, we define the multiplier operators $P_k f$ by

$$\widehat{P_k f}(\xi) = \phi(2^{-k}|\xi|)\widehat{f}(\xi).$$

First we claim that if $(n+1)/4 \leq s < n/2$ and $1 < p \leq (n+1)/(2s+1)$,

$$\left\|e^{it\sqrt{-\Delta}} P_0 f\right\|_{L^2(w(x,t))} \leq C\|w\|^{1/2}_{\mathfrak{L}^{2s+1,p}} \|f\|_2. \tag{3.1}$$

Then it follows from scaling that

$$\begin{aligned}
\left\|e^{it\sqrt{-\Delta}} P_k f\right\|^2_{L^2(w(x,t))} &\leq C 2^{-kn} 2^{-k} \left\|e^{it\sqrt{-\Delta}} P_0(f(2^{-k}\cdot))\right\|^2_{L^2(w(2^{-k}x, 2^{-k}t))} \\
&\leq C 2^{-kn} 2^{-k} \|w(2^{-k}x, 2^{-k}t)\|_{\mathfrak{L}^{2s+1,p}} \|f(2^{-k}\cdot)\|_2^2 \\
&\leq C 2^{2ks} \|w\|_{\mathfrak{L}^{2s+1,p}} \|f\|_2^2.
\end{aligned}$$

Since $w(\cdot,t) \in A_2(\mathbb{R}^n)$ uniformly for almost every $t \in \mathbb{R}$, by the Littlewood-Paley theorem on weighted L^2 spaces (see Theorem 1 in [12]), we see that

$$\begin{aligned}
\left\|e^{it\sqrt{-\Delta}} f\right\|^2_{L^2(w(x,t))} &= \int \left\|e^{it\sqrt{-\Delta}} f\right\|^2_{L^2(w(\cdot,t))} dt \\
&\leq C \int \left\| \left(\sum_k \left|P_k e^{it\sqrt{-\Delta}} f\right|^2 \right)^{1/2} \right\|^2_{L^2(w(\cdot,t))} dt \\
&= C \sum_k \left\|e^{it\sqrt{-\Delta}} P_k f\right\|^2_{L^2(w(x,t))}.
\end{aligned}$$

On the other hand, since $P_kP_jf=0$ if $|j-k|\geq 2$, it follows that

$$\begin{aligned}\sum_k \big\|e^{it\sqrt{-\Delta}}P_kf\big\|^2_{L^2(w(x,t))} &= \sum_k \big\|e^{it\sqrt{-\Delta}}P_k\big(\sum_{|j-k|\leq 1}P_jf\big)\big\|^2_{L^2(w(x,t))}\\ &\leq C\|w\|_{\mathfrak{L}^{2s+1,p}}\sum_k 2^{2ks}\big\|\sum_{|j-k|\leq 1}P_jf\big\|_2^2\\ &\leq C\|w\|_{\mathfrak{L}^{2s+1,p}}\|f\|^2_{\dot H^s}.\end{aligned}$$

Consequently, we get the desired estimate

$$\big\|e^{it\sqrt{-\Delta}}f\big\|_{L^2(w(x,t))}\leq C\|w\|^{1/2}_{\mathfrak{L}^{2s+1,p}}\|f\|_{\dot H^s}$$

for $(n+1)/4\leq s<n/2$ and $1<p\leq (n+1)/(2s+1)$.

Now it remains to show (3.1). By duality (3.1) is equivalent to

$$\Big\|\int e^{-is\sqrt{-\Delta}}P_0F(\cdot,s)ds\Big\|_{L^2_x}\leq C\|w\|^{1/2}_{\mathfrak{L}^{2s+1,p}}\|F\|_{L^2(w^{-1})}.$$

Hence it suffices to show the following bilinear form estimate:

$$\Big|\Big\langle\int_{\mathbb{R}}e^{i(t-s)\sqrt{-\Delta}}P_0^2F(\cdot,s)ds,G(x,t)\Big\rangle\Big|\leq C\|w\|_{\mathfrak{L}^{2s+1,p}}\|F\|_{L^2(w^{-1})}\|G\|_{L^2(w^{-1})}.$$

To show this, we first write

$$\int_{\mathbb{R}}e^{i(t-s)\sqrt{-\Delta}}P_0^2F(\cdot,s)ds=K*F,$$

where

$$K(x,t)=\int_{\mathbb{R}^n}e^{i(x\cdot\xi+t|\xi|)}\phi(|\xi|)^2d\xi.$$

Next, we decompose the kernel K in the following way:

$$\big|\big\langle K*F,G\big\rangle\big|\leq\sum_{j\geq 0}\big|\big\langle(\psi_jK)*F,G\big\rangle\big|,$$

where $\psi_j:\mathbb{R}^{n+1}\to[0,1]$ is a smooth function which is supported in $B(0,2^j)\setminus B(0,2^{j-2})$ for $j\geq 1$ and in $B(0,1)$ for $j=0$, such that $\sum_{j\geq 0}\psi_j=1$. Then it suffices to show that

$$\sum_{j\geq 0}\big|\big\langle(\psi_jK)*F,G\big\rangle\big|\leq C\|w\|_{\mathfrak{L}^{2s+1,p}}\|F\|_{L^2(w^{-1})}\|G\|_{L^2(w^{-1})}. \tag{3.2}$$

For this, we assume for the moment that

$$\big|\big\langle(\psi_jK)*F,G\big\rangle\big|\leq C2^{j(\frac{n+3}{2}-(2s+1)p)}\|w\|^p_{\mathfrak{L}^{2s+1,p}}\|F\|_{L^2(w^{-p})}\|G\|_{L^2(w^{-p})}, \tag{3.3}$$

$$\big|\big\langle(\psi_jK)*F,G\big\rangle\big|\leq C2^{j(\frac{n+3}{2}-\frac{2s+1}{2}p)}\|w\|^{p/2}_{\mathfrak{L}^{2s+1,p}}\|F\|_{L^2(w^{-p})}\|G\|_2, \tag{3.4}$$

$$\big|\big\langle(\psi_jK)*F,G\big\rangle\big|\leq C2^{j(\frac{n+3}{2}-\frac{2s+1}{2}p)}\|w\|^{p/2}_{\mathfrak{L}^{2s+1,p}}\|F\|_2\|G\|_{L^2(w^{-p})}, \tag{3.5}$$

and use the following bilinear interpolation lemma (see [3, Section 3.13, Exercise 5(b)]) as in [1, 11].

Lemma 3.1. *For $i=0,1$, let A_i,B_i,C_i be Banach spaces and let T be a bilinear operator such that $T:A_0\times B_0\to C_0$, $T:A_0\times B_1\to C_1$, and $T:A_1\times B_0\to C_1$. Then one has for $\theta=\theta_0+\theta_1$ and $1/q+1/r\geq 1$,*

$$T:(A_0,A_1)_{\theta_0,q}\times(B_0,B_1)_{\theta_1,r}\to(C_0,C_1)_{\theta,1}.$$

Here $0 < \theta_i < \theta < 1$ *and* $1 \le q, r \le \infty$.

Indeed, let us first define the bilinear vector-valued operator T by

$$T(F,G) = \big\{\big\langle (\psi_j K) * F, G\big\rangle\big\}_{j\ge 0}. \tag{3.6}$$

Then (3.2) is equivalent to

$$T : L^2(w^{-1}) \times L^2(w^{-1}) \to \ell_1^0(\mathbb{C}) \tag{3.7}$$

with the operator norm $C\|w\|_{\mathfrak{L}^{2s+1,p}}$. Here, for $a \in \mathbb{R}$ and $1 \le p \le \infty$, $\ell_p^a(\mathbb{C})$ denotes the weighted sequence space with the norm

$$\|\{x_j\}_{j\ge 0}\|_{\ell_p^a} = \begin{cases} \big(\sum_{j\ge 0} 2^{jap}|x_j|^p\big)^{\frac{1}{p}}, & \text{if} \quad p \ne \infty, \\ \sup_{j\ge 0} 2^{ja}|x_j|, & \text{if} \quad p = \infty. \end{cases}$$

Note that the above three estimates (3.3), (3.4) and (3.5) become

$$\|T(F,G)\|_{\ell_\infty^{\beta_0}(\mathbb{C})} \le C\|w\|^p_{\mathfrak{L}^{2s+1,p}}\|F\|_{L^2(w^{-p})}\|G\|_{L^2(w^{-p})},$$

$$\|T(F,G)\|_{\ell_\infty^{\beta_1}(\mathbb{C})} \le C\|w\|^{p/2}_{\mathfrak{L}^{2s+1,p}}\|F\|_{L^2(w^{-p})}\|G\|_{2}, \tag{3.8}$$

$$\|T(F,G)\|_{\ell_\infty^{\beta_1}(\mathbb{C})} \le C\|w\|^{p/2}_{\mathfrak{L}^{2s+1,p}}\|F\|_{2}\|G\|_{L^2(w^{-p})}, \tag{3.9}$$

respectively, with $\beta_0 = -(\frac{n+3}{2} - (2s+1)p)$ and $\beta_1 = -(\frac{n+3}{2} - \frac{2s+1}{2}p)$. Then, applying Lemma 3.1 with $\theta_0 = \theta_1 = 1/p'$ and $q = r = 2$, we get for $1 < p < 2$,

$$T : (L^2(w^{-p}), L^2)_{1/p',2} \times (L^2(w^{-p}), L^2)_{1/p',2} \to (\ell_\infty^{\beta_0}(\mathbb{C}), \ell_\infty^{\beta_1}(\mathbb{C}))_{2/p',1},$$

with the operator norm $C\|w\|_{\mathfrak{L}^{2s+1,p}}$. Now, we use the following real interpolation space identities (see Theorems 5.4.1 and 5.6.1 in [3]):

Lemma 3.2. *Let* $0 < \theta < 1$. *Then one has*

$$(L^2(w_0), L^2(w_1))_{\theta,2} = L^2(w), \quad w = w_0^{1-\theta} w_1^{\theta},$$

and for $1 \le q_0, q_1, q \le \infty$ *and* $s_0 \ne s_1$,

$$(\ell_{q_0}^{s_0}, \ell_{q_1}^{s_1})_{\theta,q} = \ell_q^s, \quad s = (1-\theta)s_0 + \theta s_1.$$

Then, for $1 < p < 2$, we have

$$(L^2(w^{-p}), L^2)_{1/p',2} = L^2(w^{-1}),$$

and

$$(\ell_\infty^{\beta_0}(\mathbb{C}), \ell_\infty^{\beta_1}(\mathbb{C}))_{2/p',1} = \ell_1^0(\mathbb{C})$$

if $(1-\frac{2}{p'})\beta_0 + \frac{2}{p'}\beta_1 = 0$ (i.e., $s = (n+1)/4$). Hence, we get (3.7) if $s = (n+1)/4$ and $1 < p \le \frac{n+1}{2s+1}$ (< 2). When $s > (n+1)/4$, note that $\gamma := \frac{p'}{2}(2s+1-\frac{n+3}{2}) > 0$. Since $j \ge 0$ and $\beta_1 < 0$, the estimates (3.8) and (3.9) are trivially satisfied for β_1 replaced by $\beta_1 - \gamma$. Hence, by the same argument we only need to check that $(1-\frac{2}{p'})\beta_0 + \frac{2}{p'}(\beta_1 - \gamma) = 0$. But this is an easy computation. Consequently we get (3.7) if $(n+1)/4 \le s < n/2$ and $1 < p \le \frac{n+1}{2s+1}$, and so the proof is completed.

Finally, we have to show the estimates (3.3), (3.4) and (3.5). For $j \geq 0$, let $\{Q_\lambda\}_{\lambda \in 2^j\mathbb{Z}^{n+1}}$ be a collection of cubes $Q_\lambda \subset \mathbb{R}^{n+1}$ centered at λ with side length 2^j. Then by disjointness of cubes, we see that

$$\begin{aligned}\big|\big\langle (\psi_j K) * F, G\big\rangle\big| &\leq \sum_{\lambda,\mu \in 2^j\mathbb{Z}^{n+1}} \big|\big\langle (\psi_j K) * (F\chi_{Q_\lambda}), G\chi_{Q_\mu}\big\rangle\big| \\ &\leq \sum_{\lambda \in 2^j\mathbb{Z}^{n+1}} \big|\big\langle (\psi_j K) * (F\chi_{Q_\lambda}), G\chi_{\widetilde{Q}_\lambda}\big\rangle\big|,\end{aligned}$$

where $\widetilde{Q}_\lambda$ denotes the cube with side length 2^{j+2} and the same center as Q_λ. By the Young and Cauchy-Schwartz inequalities, it follows that

$$\begin{aligned}\big|\big\langle (\psi_j K) * F, G\big\rangle\big| &\leq \sum_{\lambda \in 2^j\mathbb{Z}^{n+1}} \|(\psi_j K) * (F\chi_{Q_\lambda})\|_\infty \|G\chi_{\widetilde{Q}_\lambda}\|_1 \\ &\leq \sum_{\lambda \in 2^j\mathbb{Z}^{n+1}} \|\psi_j K\|_\infty \|F\chi_{Q_\lambda}\|_1 \|G\chi_{\widetilde{Q}_\lambda}\|_1 \\ &\leq \|\psi_j K\|_\infty \Big(\sum_{\lambda \in 2^j\mathbb{Z}^{n+1}} \|F\chi_{Q_\lambda}\|_1^2\Big)^{\frac{1}{2}} \Big(\sum_{\lambda \in 2^j\mathbb{Z}^{n+1}} \|G\chi_{\widetilde{Q}_\lambda}\|_1^2\Big)^{\frac{1}{2}}. \qquad (3.10)\end{aligned}$$

Now we need to bound the terms

$$\|\psi_j K\|_\infty, \qquad \sum_{\lambda \in 2^j\mathbb{Z}^{n+1}} \|F\chi_{Q_\lambda}\|_1^2, \qquad \sum_{\lambda \in 2^j\mathbb{Z}^{n+1}} \|G\chi_{\widetilde{Q}_\lambda}\|_1^2.$$

For the first term we use the following well-known lemma, Lemma 3.3, which is essentially due to Littman [13]. (See also [18, VIII, Section 5, B].) Indeed, by applying the lemma with $\psi(\xi) = |\xi|$, it follows that

$$|K(x,t)| = \left| \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|)} \phi(|\xi|)^2 d\xi \right| \leq C(1 + |(x,t)|)^{-\frac{n-1}{2}},$$

since the Hessian matrix $H\psi$ has $n-1$ non-zero eigenvalues for each $\xi \in \{\xi \in \mathbb{R}^n : |\xi| \sim 1\}$. Thus we get

$$\|\psi_j K\|_\infty \leq C 2^{-j\frac{n-1}{2}}. \qquad (3.11)$$

Lemma 3.3. *Let $H\psi$ be the Hessian matrix given by $(\frac{\partial^2 \psi}{\partial \xi_i \partial \xi_j})$. Suppose that ϕ is a compactly supported smooth function on $\mathbb{R}^n$ and ψ is a smooth function which satisfies rank $H\psi \geq k$ on the support of ϕ. Then, for $(x,t) \in \mathbb{R}^{n+1}$,*

$$\left| \int e^{i(x\cdot\xi + t\psi(\xi))} \phi(\xi) d\xi \right| \leq C(1 + |(x,t)|)^{-\frac{k}{2}}.$$

Next, we have the bound

$$\begin{aligned}\sum_{\lambda \in 2^j\mathbb{Z}^{n+1}} \|F\chi_{Q_\lambda}\|_1^2 &= \sum_{\lambda \in 2^j\mathbb{Z}^{n+1}} \Big(\int_{Q_\lambda} |F\chi_{Q_\lambda}| w^{-\frac{p}{2}} w^{\frac{p}{2}} dxdt\Big)^2 \\ &\leq \sum_{\lambda \in 2^j\mathbb{Z}^{n+1}} \Big(\int_{Q_\lambda} |F\chi_{Q_\lambda}|^2 w^{-p} dxdt\Big)\Big(\int_{Q_\lambda} w^p dxdt\Big) \\ &\leq \sup_{\lambda \in 2^j\mathbb{Z}^{n+1}} \Big(\int_{Q_\lambda} w^p dxdt\Big) \sum_{\lambda \in 2^j\mathbb{Z}^{n+1}} \Big(\int_{Q_\lambda} |F\chi_{Q_\lambda}|^2 w^{-p} dxdt\Big) \\ &\leq C 2^{j(n+1-(2s+1)p)} \|w\|_{\mathfrak{L}^{2s+1,p}}^p \|F\|_{L^2(w^{-p})}^2, \qquad (3.12)\end{aligned}$$

while

$$\sum_{\lambda\in 2^j\mathbb{Z}^{n+1}} \|F\chi_{Q_\lambda}\|_1^2 \le \sum_{\lambda\in 2^j\mathbb{Z}^{n+1}} \|F\chi_{Q_\lambda}\|_2^2 \|\chi_{Q_\lambda}\|_2^2$$
$$\le C2^{j(n+1)}\|F\|_2^2. \tag{3.13}$$

Similarly for $\sum_{\lambda\in 2^j\mathbb{Z}^{n+1}} \|G\chi_{\widetilde{Q}_\lambda}\|_1^2$. Now, combining (3.10), (3.11), (3.12) and (3.13), we get the desired estimates (3.3), (3.4) and (3.5).

3.2. **Inhomogeneous estimates.** Now we prove the inhomogeneous estimate (1.5). First we claim that for $\alpha > 2n+4-2n/r-2/q$ and $1<p\le (n+1)/\alpha$,

$$\Big\| \int_0^t e^{i(t-s)\sqrt{-\Delta}} P_0 F(\cdot,s)ds \Big\|_{L^2(w(x,t))} \le C\|w\|_{\mathfrak{L}^{\alpha,p}}^{1/2} \|F\|_{L_t^q L_x^r} \tag{3.14}$$

if $1\le r,q\le 2$. Then, by the Littlewood-Paley theorem on weighted L^2 spaces as before, it follows that

$$\Big\| \int_0^t e^{i(t-s)\sqrt{-\Delta}} F(\cdot,s)ds \Big\|_{L^2(w(x,t))}^2$$
$$\le C\sum_k \Big\| \int_0^t e^{i(t-s)\sqrt{-\Delta}} P_k\big(\sum_{|j-k|\le 1} P_j F(\cdot,s)\big)ds \Big\|_{L^2(w(x,t))}^2 .$$

By using (3.14) and scaling, the right-hand side in the above is bounded by

$$C\|w\|_{\mathfrak{L}^{\alpha,p}} \sum_k 2^{k(\alpha-n-3+\frac{2n}{r}+\frac{2}{q})} \big\| \sum_{|j-k|\le 1} P_j F(\cdot,s)\big\|_{L_t^q L_x^r}^2,$$

which is in turn bounded by

$$C\|w\|_{\mathfrak{L}^{\alpha,p}} \sum_k 2^{k(\alpha-n-3+\frac{2n}{r}+\frac{2}{q})} \|P_k F(\cdot,s)\|_{L_t^q L_x^r}^2 .$$

Since $q\le 2$, by Minkowski's integral inequality we see that

$$\sum_k 2^{k(\alpha-n-3+\frac{2n}{r}+\frac{2}{q})} \|P_k F(\cdot,t)\|_{L_t^q L_x^r}^2 \le \Big\| \Big(\sum_k 2^{k(\alpha-n-3+\frac{2n}{r}+\frac{2}{q})} \|P_k F(\cdot,t)\|_{L_x^r}^2 \Big)^{\frac12} \Big\|_{L_t^q}^2$$
$$= \|F\|_{L_t^q \dot{B}_{r,2}^{\widetilde{s}+1}}^2$$

with $\widetilde{s}+1 = \frac12(\alpha-n-3+\frac{2n}{r}+\frac{2}{q})$. Thus, we get the desired estimate

$$\Big\| \int_0^t e^{i(t-s)\sqrt{-\Delta}} F(\cdot,s)ds \Big\|_{L^2(w(x,t))} \le C\|w\|_{\mathfrak{L}^{\alpha,p}}^{1/2} \|F\|_{L_t^q \dot{B}_{r,2}^{\widetilde{s}+1}}$$

for $\alpha > 2n+4-2n/r-2/q$ and $1<p\le (n+1)/\alpha$ if $1\le q,r\le 2$ and

$$\widetilde{s}+1 = \frac12(\alpha-n-3+\frac{2n}{r}+\frac{2}{q}). \tag{3.15}$$

Note that from (3.15) the condition $\alpha > 2n+4-2n/r-2/q$ is equivalent to the condition $\widetilde{s} > (n-1)/2$ in Theorem 1.1, and so the proof is completed.

Now it remains to show (3.14). We will show the estimate

$$\Big\| \int_{-\infty}^t e^{i(t-s)\sqrt{-\Delta}} P_0 F(\cdot,s)ds \Big\|_{L^2(w(x,t))} \le C\|w\|_{\mathfrak{L}^{\alpha,p}}^{1/2} \|F\|_{L_t^q L_x^r}, \tag{3.16}$$

which implies (3.14). Indeed, to obtain (3.14) from (3.16), we first decompose the L^2_t norm in the left-hand side of (3.14) into two parts, $t \geq 0$ and $t < 0$. Then the second part can be reduced to the first one by changing the variable $t \mapsto -t$, and so we only need to consider the first part. But, since $[0,t) = (-\infty,t) \cap [0,\infty)$, applying (3.16) with F replaced by $\chi_{[0,\infty)}(s)F$, we can bound the first part as desired. To show (3.16), by duality it suffices to show that

$$\left|\left\langle \int_{-\infty}^{t} e^{i(t-s)\sqrt{-\Delta}} P_0 F(\cdot,s)ds, G \right\rangle\right| \leq C\|w\|^{1/2}_{\mathfrak{L}^{\alpha,p}} \|F\|_{L^q_t L^r_x} \|G\|_{L^2(w^{-1})}.$$

Let us first write

$$\begin{aligned}\int_{-\infty}^{t} e^{i(t-s)\sqrt{-\Delta}} P_0 F(\cdot,s)ds &= \int_{\mathbb{R}} \chi_{(0,\infty)}(t-s) e^{i(t-s)\sqrt{-\Delta}} P_0 F(\cdot,s)ds \\ &= K * F,\end{aligned}$$

where

$$K(x,t) = \int_{\mathbb{R}^n} \chi_{(0,\infty)}(t) e^{i(x\cdot\xi + t|\xi|)} \phi(|\xi|)d\xi.$$

Then, with the same notation as in the previous section, it is enough to show that

$$\sum_{j\geq 0} \left|\left\langle (\psi_j K) * F, G\right\rangle\right| \leq C\|w\|^{1/2}_{\mathfrak{L}^{\alpha,p}} \|F\|_{L^q_t L^r_x} \|G\|_{L^2(w^{-1})}. \tag{3.17}$$

For this, we assume for the moment that

$$\left|\left\langle (\psi_j K) * F, G\right\rangle\right| \leq C 2^{j(\frac{n}{r'} + \frac{1}{q'} + 1)} \|F\|_{L^q_t L^r_x} \|G\|_{L^2} \tag{3.18}$$

and

$$\left|\left\langle (\psi_j K) * F, G\right\rangle\right| \leq C 2^{j(\frac{n}{r'} + \frac{1}{q'} + 1 - \frac{\alpha}{2}p)} \|w\|^{p/2}_{\mathfrak{L}^{\alpha,p}} \|F\|_{L^q_t L^r_x} \|G\|_{L^2(w^{-p})}. \tag{3.19}$$

Then these estimates say that for $\beta_0 = -(\frac{n}{r'} + \frac{1}{q'} + 1)$ and $\beta_1 = -(\frac{n}{r'} + \frac{1}{q'} + 1 - \frac{\alpha}{2}p)$,

$$\|T(F,G)\|_{\ell^{\beta_0}_\infty(\mathbb{C})} \leq C\|F\|_{L^q_t L^r_x} \|G\|_{L^2}$$

and

$$\|T(F,G)\|_{\ell^{\beta_1}_\infty(\mathbb{C})} \leq C\|w\|^{p/2}_{\mathfrak{L}^{\alpha,p}} \|F\|_{L^q_t L^r_x} \|G\|_{L^2(w^{-p})},$$

where T is given as in (3.6). Now, by the bilinear interpolation (see [3, Section 3.13, Exercise 5(a)]) between these two estimates, it follows that for $1 < p < \infty$,

$$T : (L^q_t L^r_x, L^q_t L^r_x)_{1/p,2} \times (L^2, L^2(w^{-p}))_{1/p,2} \to (\ell^{\beta_0}_\infty(\mathbb{C}), \ell^{\beta_1}_\infty(\mathbb{C}))_{1/p,\infty}$$

with the operator norm $C\|w\|^{1/2}_{\mathfrak{L}^{\alpha,p}}$. Using Lemma 3.2, we get

$$\|T(F,G)\|_{\ell^{\beta}_\infty(\mathbb{C})} \leq C\|w\|^{1/2}_{\mathfrak{L}^{\alpha,p}} \|F\|_{L^q_t L^r_x} \|G\|_{L^2(w^{-1})}$$

with $\beta = (1-\frac{1}{p})\beta_0 + \frac{1}{p}\beta_1 = -(\frac{n}{r'} + \frac{1}{q'} + 1 - \frac{\alpha}{2})$. That is to say,

$$\left|\left\langle (\psi_j K) * F, G\right\rangle\right| \leq C 2^{j(\frac{n}{r'} + \frac{1}{q'} + 1 - \frac{\alpha}{2})} \|w\|^{1/2}_{\mathfrak{L}^{\alpha,p}} \|F\|_{L^q_t L^r_x} \|G\|_{L^2(w^{-1})}.$$

Thus, when $\alpha > 2n + 4 - 2n/r - 2/q$, we get the desired estimate (3.17).

Finally, we have to show (3.18) and (3.19). Recall from (3.10) that

$$\left|\left\langle (\psi_j K) * F, G\right\rangle\right| \leq \|\psi_j K\|_\infty \Big(\sum_{\lambda \in 2^j\mathbb{Z}^{n+1}} \|F\chi_{Q_\lambda}\|_1^2\Big)^{\frac{1}{2}} \Big(\sum_{\lambda \in 2^j\mathbb{Z}^{n+1}} \|G\chi_{\widetilde{Q}_\lambda}\|_1^2\Big)^{\frac{1}{2}}.$$

First, using Lemma 3.3 as before, we have

$$\|\psi_j K\|_\infty \le C2^{-j\frac{n-1}{2}}. \tag{3.20}$$

Next, we may write $Q_\lambda = Q_{\lambda(x)} \times Q_{\lambda(t)}$, where $Q_{\lambda(x)}$ is a cube in $\mathbb{R}^n$ with respect to the x variable, and $Q_{\lambda(t)}$ is an interval in $\mathbb{R}$ with respect to the t variable. Then, since $q, r \le 2$, by Minkowski's integral inequality it follows that

$$\begin{aligned}\Big(\sum_{\lambda\in 2^j\mathbb{Z}^{n+1}} \|F\chi_{Q_\lambda}\|_1^2\Big)^{\frac12} &\le \Big(\sum_{\lambda\in 2^j\mathbb{Z}^{n+1}} \Big(\int_{Q_{\lambda(t)}} \|F\chi_{Q_\lambda}\|_{L^r_x}|Q_{\lambda(x)}|^{\frac{1}{r'}}dt\Big)^2\Big)^{\frac12}\\ &\le \Big(\sum_{\lambda\in 2^j\mathbb{Z}^{n+1}} \|F\chi_{Q_\lambda}\|^2_{L^q_tL^r_x}|Q_{\lambda(x)}|^{\frac{2}{r'}}|Q_{\lambda(t)}|^{\frac{2}{q'}}\Big)^{\frac12}\\ &\le C2^{j(\frac{n}{r'}+\frac{1}{q'})}\|F\|_{L^q_tL^r_x}.\end{aligned}$$

On the other hand, we have the following bound:

$$\begin{aligned}\sum_{\lambda\in 2^j\mathbb{Z}^{n+1}} \|F\chi_{Q_\lambda}\|_1^2 &= \sum_{\lambda\in 2^j\mathbb{Z}^{n+1}} \Big(\int_{Q_\lambda} |F\chi_{Q_\lambda}|w^{-\frac{p}{2}}w^{\frac{p}{2}}dxdt\Big)^2\\ &\le \sum_{\lambda\in 2^j\mathbb{Z}^{n+1}} \Big(\int_{Q_\lambda} |F\chi_{Q_\lambda}|^2w^{-p}dxdt\Big)\Big(\int_{Q_\lambda} w^p dxdt\Big)\\ &\le \sup_{\lambda\in 2^j\mathbb{Z}^{n+1}} \Big(\int_{Q_\lambda} w^p dxdt\Big)\sum_{\lambda\in 2^j\mathbb{Z}^{n+1}} \Big(\int_{Q_\lambda} |F\chi_{Q_\lambda}|^2w^{-p}dxdt\Big)\\ &\le C2^{j(n+1-\alpha p)}\|w\|^p_{\mathfrak{L}^{\alpha,p}}\|F\|^2_{L^2(w^{-p})}.\end{aligned} \tag{3.21}$$

Combining (3.13), (3.20) and (3.21), we get the desired estimates (3.18) and (3.19).

4. Further applications

In this final section we consider the following Cauchy problem for wave equations with potentials:

$$\begin{cases}\partial_t^2 u - \Delta u + V(x,t)u = 0,\\ u(x,0) = f(x),\\ \partial_t u(x,0) = g(x),\end{cases} \tag{4.1}$$

where $(x,t) \in \mathbb{R}^{n+1}$, $n = 2, 3$. The well-posedness for this problem in the space $L^2_{x,t}(|V|)$ was studied for $n \ge 3$ in [15], when $V = V_1 + V_2$ with $V_1 \in L^\infty_t\mathfrak{L}^{2,p}_x$, $(n-1)/2 < p \le n/2$, $V_2 \in L^r_tL^\infty_x$, $r > 1$, and $\|V_1\|_{L^\infty_t\mathfrak{L}^{2,p}_x}$ small enough. Here our main aim is to deal with the two-dimensional case $n = 2$. Our result is the following theorem.

Theorem 4.1. *Let $n = 2, 3$. If $n = 2$ we assume that $V \in L^1_t\mathfrak{L}^{1-s,r}_x \cap \mathfrak{L}^{2s+1,p}_{x,t}$ for $3/4 \le s < 1$, $1 < r \le 2/(1-s)$ and $1 < p \le (n+1)/(2s+1)$, with $\|V\|_{L^1_t\mathfrak{L}^{1-s,r}_x}$ and $\|V\|_{\mathfrak{L}^{2s+1,p}_{x,t}}$ small enough. Similarly, we assume for $n = 3$ the same conditions with $s = 1$ and $\mathfrak{L}^{1-s,r}_x$ replaced by L^∞_x. Then, if $f \in \dot H^s$ and $g \in \dot H^{s-1}$, there exists a unique solution of (4.1) in the space $L^2_{x,t}(|V|)$. Furthermore,*

$$\|u\|_{L^2_{x,t}(|V|)} \le C(\|f\|_{\dot H^s} + \|g\|_{\dot H^{s-1}}). \tag{4.2}$$

Proof. As a preliminary step, we recall from [16] that the fractional integral I_α of convolution with $|x|^{-n+\alpha}$, $0<\alpha<n$, satisfies the inequality

$$\|I_\alpha f\|_{L^2(w(x))} \le C\|w\|^{1/2}_{\mathfrak{L}^{\alpha,r}}\|f\|_{L^2}, \tag{4.3}$$

where $\alpha>0$ and $1<r\le n/\alpha$. Indeed, this inequality follows directly from [16, (1.10)] with $w=v^{-1}$ and $p=2$. Then, using this inequality, we see that

$$\begin{aligned}\left\|\int_0^t e^{i(t-s)\sqrt{-\Delta}}|\nabla|^{-\alpha}F(\cdot,s)ds\right\|_{L^2_x(w(x,t))} &\le C\|w\|^{1/2}_{\mathfrak{L}^{\alpha,r}_x}\left\|\int_0^t e^{i(t-s)\sqrt{-\Delta}}F(\cdot,s)ds\right\|_{L^2_x}\\ &\le C\|w\|^{1/2}_{\mathfrak{L}^{\alpha,r}_x}\left\|\int e^{-is\sqrt{-\Delta}}\chi_{(0,t)}(s)F(\cdot,s)ds\right\|_{L^2_x}\end{aligned}$$

because the integral kernel of the multiplier operator $|\nabla|^{-\alpha}$ is given by $|x|^{-n+\alpha}$, $0<\alpha<n$. Combining this estimate and the following dual estimate of (1.4),

$$\left\|\int e^{-is\sqrt{-\Delta}}F(\cdot,s)ds\right\|_{L^2_x} \le C\|w\|^{1/2}_{\mathfrak{L}^{2s+1,p}}\||\nabla|^sF\|_{L^2_{x,t}(w^{-1})},$$

we get for $\alpha>0$, $1<r\le n/\alpha$, $(n+1)/4\le s<n/2$, and $1<p\le(n+1)/(2s+1)$,

$$\left\|\int_0^t e^{i(t-s)\sqrt{-\Delta}}F(\cdot,s)ds\right\|_{L^2_{x,t}(w(x,t))} \le C\|w\|^{1/2}_{L^1_t\mathfrak{L}^{\alpha,r}_x}\|w\|^{1/2}_{\mathfrak{L}^{2s+1,p}}\||\nabla|^{\alpha+s}F\|_{L^2_{x,t}(w^{-1})}. \tag{4.4}$$

Now we consider the solution of (4.1) which is given by

$$u(x,t)=\cos(t\sqrt{-\Delta})f+\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g-\int_0^t\frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}(Vu)(s)ds, \tag{4.5}$$

where $f\in\dot H^s$ and $g\in\dot H^{s-1}$. Then by the homogeneous estimate (1.4), we only need to show that

$$\left\|\int_0^t\frac{e^{i(t-s)\sqrt{-\Delta}}}{\sqrt{-\Delta}}(Vu)(\cdot,s)ds\right\|_{L^2_{x,t}(|V|)} \le \frac12\|u\|_{L^2_{x,t}(|V|)} \tag{4.6}$$

in order to obtain the well-posedness using the standard fixed-point argument. But, when $n=2$, one can use (4.4), with $\alpha=1-s$ and $w=|V|$, to conclude that

$$\begin{aligned}\left\|\int_0^t\frac{e^{i(t-s)\sqrt{-\Delta}}}{\sqrt{-\Delta}}(Vu)(\cdot,s)ds\right\|_{L^2_{x,t}(V)} &\le C\|V\|^{1/2}_{L^1_t\mathfrak{L}^{1-s,r}_x}\|V\|^{1/2}_{\mathfrak{L}^{2s+1,p}}\|Vu\|_{L^2_{x,t}(|V|^{-1})}\\ &\le \frac12\|u\|_{L^2_{x,t}(|V|^{-1})}\end{aligned}$$

if $\|V\|_{L^1_t\mathfrak{L}^{1-s,r}_x}$ and $\|V\|_{\mathfrak{L}^{2s+1,p}}$ are sufficiently small. Also, (4.2) follows easily from combining (4.5), (1.4) and (4.6). Now the proof is complete for $n=2$.

When $n=3$, by repeating the above argument with $\alpha=0$ using (4.3) replaced by the trivial inequality $\|f\|_{L^2(w(x))}\le C\|w\|^{1/2}_{L^\infty}\|f\|_{L^2}$, one can similarly obtain (4.6) under the conditions given in the theorem. We omit the details. □

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