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ON THE DIRICHLET PROBLEM FOR p-HARMONIC MAPS II: TARGETS WITH SPECIAL STRUCTURE

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ABSTRACT. In this paper we develop new geometric techniques to deal with the Dirichlet problem for a p-harmonic map from a compact manifold with boundary to a Cartan-Hadamard target manifold which is either 2-dimensional or rotationally symmetric.

INTRODUCTION

Let (M, g) and (N, h) be Riemannian manifolds of dimensions m and n respectively and suppose that M is compact with smooth non-empty boundary. A C^1 map $u : \text{int}M \to N$ is said to be p-harmonic if it satisfies the p-Laplace equation

$$\Delta_p u = \text{div}(|du|^{p-2} du) = 0. \tag{0.1}$$

Here $du \in \Gamma(T^*M \otimes u^{-1}TN)$ is a vector-valued differential 1-form and $T^*M \otimes u^{-1}TN$ is endowed with its Hilbert-Schmidt scalar product. Moreover $-\text{div} = \delta$ is the formal adjoint of the exterior differential d, with respect to the standard L^2 inner product on vector-valued differential 1-forms on M. Equation (0.1) is the Euler-Lagrange equation of the p-energy functional

$$E_p(u) = \frac{1}{p} \int_\Omega |du|^p_{HS}(x) dV_M.$$

The topic of this paper is the Dirichlet problem for p-harmonic maps into non-positively curved target N. Namely, given a sufficiently regular boundary datum $f : \partial M \to N$ the corresponding Dirichlet problem consists in finding a map $u : M \to N$ which extends f to a p-harmonic map on $\text{int}M$.

In case the target manifold N is closed (i.e., compact without boundary), in [PV1] we gave a complete solution to the homotopic p-Dirichlet, i.e., the solution is found in a prescribed homotopy class. The proof therein is purely variational. Exploiting powerful techniques due to B. White [Wh], one can define the weak relative d-homotopy type of $W^{1,p}$ maps, hence minimize the p-energy in the d-homotopy class of the initial datum, and finally show how to apply R. Hardt and F.-H. Lin's regularity theory to the minimizer [HL].

In this paper we focus our attention on a non-compact, but topologically trivial, target manifold N of non-positive curvature. Such a manifold N is usually said to be Cartan-Hadamard. Under these assumptions, a solution to the Dirichlet

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problem has been given by Fuchs [Fu, Theorem 5.1]. See also the more recent [FR]. In order to win the lack of compactness of the target, the proof given in [Fu] needed to deeply exploit Fuchs' regularity theory for constrained p-minimizers. Our main purpose, here, is to show that, even if N is non-compact, one can prove a posteriori a uniform bound for the solution. In particular, thanks to a gluing and compactification argument, the problem can be reduced to the closed one, at least in case the target is either a surface or rotationally symmetric.

Actually, we feel that the geometric construction introduced in this paper will be useful in more general settings where the analytic problem is related to different functionals. Indeed, as it will be clear from the proof, the relevant properties of the p-energy required by the method we propose are: (a) the solvability of the problem when the target is compact and (b) a maximum principle for regular enough solutions.

It's worthwhile to remark that one crucial point in the previous works [Fu, FR] is a quite implicit use of a tight relation between two different notions of bounded Sobolev maps: a first one, that we could call *intrinsic*, is defined in a global coordinate chart of the target space. A second one, somewhat more standard and called *extrinsic*, uses a proper isometric embedding of the target into a Euclidean space of sufficiently large dimension. In a future paper [PV2], we shall investigate carefully the relations between these two notions and we will point out some interesting consequences.

The starting point of the present investigation is that the only interesting case involves target manifolds without compact quotients for, otherwise, the non-compact problem can be reduced to the compact one where the machinery alluded to above can be applied without changes.

Proposition A. *Let (M,g) be a compact, m-dimensional Riemannian manifold with smooth boundary $\partial M \neq \emptyset$ and let (N,h) be a complete, Riemannian manifold of dimension n such that its universal cover supports a strictly convex exhaustion function. Assume that there exists a subgroup Γ of isometries of N acting freely, properly and co-compactly on N. Then, for any $p \geq 2$ and for every $f \in C^0(M,N) \cap Lip(\partial M, N)$, the homotopy p-Dirichlet problem has a solution $u \in C^{1,\alpha}(\mathrm{int}(M),N) \cap C^0(M,N)$. Moreover, the solution is unique provided N has non-positive sectional curvature.*

We aim at facing the general situation where either we have no information on the structure of the isometry group of N or it is known that N has no compact quotients.

Theorem B. *Let (M,g) be a compact, m-dimensional Riemannian manifold with smooth boundary $\partial M \neq \emptyset$ and let N be an n-dimensional simply connected manifold of non-positive curvature N^n_σ which is either rotationally symmetric or 2-dimensional. Then, for any $p \geq 2$ and any given $f \in C^0(M,N) \cap Lip(\partial M, N)$, the p-Dirichlet problem*

$$\begin{cases} \Delta_p u = 0 & \text{on } M, \\ u = f & \text{on } \partial M, \end{cases} \tag{0.2}$$

has a unique solution $u \in C^{1,\alpha}(\mathrm{int}(M),N) \cap C^0(M)$.

1. Scheme of the proofs

We start this section giving the simple proof of Proposition A.

Proof (of Proposition A*).* By assumption, $N' = N/\Gamma$ is a compact, aspherical Riemannian manifold covered by N via the quotient projection $P : N \to N'$. The original datum f projects to a new function $P(f) : M \to N'$ which, in turn, can be used to state the corresponding p-Dirichlet problem

$$\begin{cases} \Delta_p u' = 0 & \text{on } M, \\ u' = P(f) & \text{on } \partial M. \end{cases}$$

Thanks to the analysis of the compact target case provided in [PV1], this problem admits a solution $u' \in C^{1,\alpha}(\mathrm{int}(M), N') \cap C^0(M, N')$ in the homotopy class of $P(f)$ relative to ∂M. Let $H' : [0,1] \times M \to N'$ be such a homotopy. The classical theory of fibrations (see e.g. [Ha]) then tells us that H' lifts to a homotopy $H : [0,1] \times M \to N$ satisfying $H(1,x) = f(x)$. The homotopy H is relative to ∂M because, for every $y \in \partial M$, $H([0,1] \times \{y\})$ is contained in the (discrete) fibre over $P(f)(y)$. Let $u(x) = H(0,x)$. Since P is a local isometry and $P(u) = u'$, then u is p-harmonic in M of class $C^{1,\alpha}(\mathrm{int}(M), N) \cap C^0(M, N)$. On the other hand, using the fact that H is relative to ∂M we deduce that $u = f$ on ∂M. This proves that the original homotopy p-Dirichlet problem has a solution. In case ${}^N\mathrm{Sect} \leq 0$, uniqueness follows easily from the following few facts: (a) solutions of the homotopy p-Dirichlet problem with target N project to solutions of the corresponding problem with target N'; (b) in the case of compact targets, the solution is unique; (c) liftings are uniquely determined by their values at a single point. □

The approach we propose to prove Theorem B relates to the reduction procedure used to obtain Proposition A. This latter implies that the Dirichlet problem is easily solved when N has a compact quotient, but this is not the case for a general Cartan-Hadamard model manifold N^n_σ. The possible lack of discrete, co-compact isometry subgroups is overcome by using a combination of cut and paste and periodization arguments. Namely, we will show that it is possible to perturb the metric of N^n_σ in the exterior of a fixed geodesic ball in N^n_σ such that the complete manifold thus obtained is again Cartan-Hadamard and has compact quotients. A new maximum principle for the composition of the p-harmonic map and the convex distance function of N^n_σ then gives that this perturbation does not affect the solution to the original problem. The uniqueness part of the theorem can be clearly considered as a bypass product of the reduction to the compact case. We recall also that a comprehensive uniqueness result for general complete targets with non-positive curvature was obtained in [PV1].

To perform the cut and paste procedure we need a local explicit control on the sectional curvatures of N. To this purpose, we first focus our attention on rotationally symmetric targets. That is, having fixed a smooth function $\sigma : [0, +\infty) \to [0, +\infty)$ satisfying

$$\sigma^{(2k)}(0) = 0, \forall k \in \mathbb{N}, \quad \sigma'(0) = 1, \quad \sigma(r) > 0, \forall r > 0, \tag{1.1}$$

we shall denote by N^n_σ the smooth n-dimensional Riemannian manifold given by

$$\left([0, +\infty) \times \mathbb{S}^{n-1}, dr^2 + \sigma^2(r)\, d\theta^2\right), \tag{1.2}$$

where $d\theta^2$ denotes the standard metric on $\mathbb{S}^{n-1}$. Clearly, N^n_σ is diffeomorphic to $\mathbb{R}^n$ and geodesically complete for any choice of σ. Usually, N^n_σ is called a model manifold with warping function σ and pole 0. The r-coordinate in the expression (1.2) of the metric represents the distance from the pole. Standard formulas for warped product metrics reveal that

$$\mathrm{Sect}_{rad} = -\frac{\sigma''}{\sigma}, \quad \mathrm{Sect}_{tg} = \frac{1-(\sigma')^2}{\sigma^2}. \tag{1.3}$$

Thus, in particular, the model manifold N^n_σ is Cartan-Hadamard if and only if

$$\sigma'' \geq 0.$$

We point out that, when the Cartan-Hadamard target is 2-dimensional, the first equation in (1.3) defines its Gaussian curvature in polar coordinates regardless of any rotational symmetry condition. Namely, given a 2-dimensional Cartan-Hadamard manifold (N, h_N), in the global geodesic chart (r,θ) around some fixed pole $o \in N$, the metric h_N can be expressed as

$$h_N|_{(r,\theta)} = dr^2 + \nu^2(r,\theta)d\theta^2.$$

Direct computations show that the only (radial) sectional curvature of N^2_ν satisfies at any point (r,θ) the formula

$$\mathrm{Sect}(r,\theta) = \mathrm{Sect}_{rad}(r,\theta) = -\nu^{-1}(r,\theta)\frac{\partial^2\nu(r,\theta)}{\partial r^2}. \tag{1.4}$$

2. Gluing model manifolds keeping Sect ≤ 0

In this section we show that, in some sense, it is possible to prescribe a hyperbolic infinity to a Cartan-Hadamard model, as well as to a generic Cartan-Hadamard 2-manifold, without violating the non-positive curvature condition.

Theorem 2.1. *Let N be a rotationally symmetric (resp. 2-dimensional) Cartan-Hadamard manifold. Fix $\bar{R} > 0$. Then, for every $R > \bar{R}$ there exist a $k = k(R) >> 1$ and a Cartan-Hadamard M^n_τ such that:*

(i) $B^N_{\bar{R}}(0) \subset M^n_\tau$.

(ii) $M^n_\tau \setminus B^M_R(0) = \mathbb{H}^n_k \setminus B^{\mathbb{H}^n_k}_R(0)$.

Proof. We prove the theorem in case $N = N^n_\rho$ is rotationally symmetric. Replacing (1.3) with (1.4), the 2-dimensional case can be handled in a completely analogous way.

Thanks to (1.3), it is enough to produce a warping function $\tau : [0,+\infty) \to [0,+\infty)$ satisfying the following requirements:

(a) $\tau(r) = \rho(r)$ on $[0,\bar{R})$.

(b) $\tau(r) = \sigma_k := k^{-1/2}\sinh\left(k^{1/2}r\right)$ on $(R,+\infty)$.

(c) $\tau' \geq 1$ and $\tau'' \geq 0$ on $[0,+\infty)$.

To this end, let $\bar{R} < R_1 < R_2 < R$. By the assumptions on σ, we can choose $k = k(R_1,R_2) > 0$ large enough so that

$$\rho'(R_1) \leq \frac{\sigma_k(R_2) - \rho(R_1)}{R_2 - R_1} \leq \sigma'_k(R_2). \tag{2.1}$$

Define

$$\tau_1(r) = \begin{cases} \rho(r) & \text{on } [0,R_1), \\ \rho(R_1) + \frac{\sigma_k(R_2)-\rho(R_1)}{R_2-R_1} r & \text{on } [R_1,R_2], \\ \sigma_k(r) & \text{on } (R_2,+\infty). \end{cases}$$

Then, τ_1 is a piecewise smooth, convex function with $\tau_1' \geq 1$. To complete the construction of τ, it remains to smooth out the angles with a convex function. This can be done using the approximation procedure described by M. Ghomi in [Gh]. □

Remark 2.2. As is clear from the proof, Theorem 2.1 holds for a class of "external" manifolds wider than hyperbolic spaces. In fact, all we need is relation (2.1) to hold.

3. Compact hyperbolic manifolds with large injectivity radii

It is intuitively clear that actions of small discrete groups on a complete Riemannian manifold give rise to large fundamental domains. The intuition is confirmed in the next simple result.

Lemma 3.1. *Let (N, h) be a complete Riemannian manifold. Suppose that there exists a filtration*

$$\Gamma_0 \rhd \Gamma_1 \rhd \Gamma_2 \rhd \cdots \rhd \Gamma_k \rhd \cdots \rhd \{1\}$$

of discrete groups $\Gamma_k \subset \mathrm{Iso}\,(N)$ acting freely and properly on N. Then, for every arbitrarily large ball $B_R\,(p)$, there exists $K > 0$ such that the following holds: for every $k > K$ we find a fundamental domain Ω_k of Γ_k containing p and satisfying

$$B_R^N\,(p) \subset\subset \Omega_k. \tag{3.1}$$

Proof. Let $D_k\,(p)$ be the Dirichlet domain of Γ_k centered at p. Recall that $D_k\,(p) = \bigcap_{\gamma\in\Gamma_k} H_\gamma\,(p)$ where

$$H_\gamma\,(p) = \{x \in N : d_N\,(x,p) < d_N\,(x, \gamma\cdot p)\}\,.$$

One can easily verify that if $B_R^N\,(p) \cap (N\backslash D_k\,(p)) \neq \emptyset$, then

$$B_R^N\,(p) \cap \gamma \cdot B_R^N\,(p) \neq \emptyset, \tag{3.2}$$

for some $\gamma \in \Gamma_k \subset \Gamma_0$. Since Γ_0 acts properly on N, it follows that (3.2) can be satisfied for at most a finite number of $\gamma_1, ..., \gamma_N \in \Gamma_0$. To conclude the validity of (3.1), we now use that $\bigcap \Gamma_k = \{1\}$ and, therefore, $\gamma_1, ..., \gamma_N \notin \Gamma_k$, for every large enough k. □

A case of special interest is obtained by taking $N = \mathbb{H}^n_{-k^2}$, the standard hyperbolic spaceform of constant curvature $-k^2 < 0$. If Γ is a co-compact discrete group of isometries acting freely and properly on $\mathbb{H}^n_{-k^2}$, the corresponding Riemannian orbit space $\mathbb{H}^n_{-k^2}/\Gamma$ is named a compact hyperbolic manifold (of constant curvature $-k^2$). The existence of a co-compact discrete group of isometries of $\mathbb{H}^n_{-k^2}$ with large fundamental domain is equivalent to the existence of a compact quotient manifold with large injectivity radius. The following result was first observed in [Fa], see p. 74.

Proposition 3.2. *Let $n \geq 0$, $R > 0$ and $p \in \mathbb{H}^n_{-k^2}$. Then, there exists a co-compact, discrete group Γ of isometries of $\mathbb{H}^n_{-k^2}$ acting freely and properly on $\mathbb{H}^n_{-k^2}$ and whose fundamental domain Ω containing p satisfies*

$$\mathbb{B}_R\,(p) \subset\subset \Omega.$$

Equivalently,

$$\mathrm{inj}\left(\mathbb{H}^n_{-k^2}/\Gamma\right) \geq R.$$

Proof. By a result of A. Borel [Bo], $\mathbb{H}^n_{-k^2}$ has a co-compact, discrete group of isometries Γ_0 acting freely and properly. According to a result by A. Malcev, Γ_0 is residually finite, i.e., there exists a filtration

$$\Gamma_0 \rhd \Gamma_1 \rhd \Gamma_2 \rhd \cdots \rhd \Gamma_k \rhd \cdots \rhd \{1\},$$

satisfying $[\Gamma_k : \Gamma_{k-1}] = |\Gamma_k / \Gamma_{k-1}| < +\infty$. To conclude, we now apply Lemma 3.1. □

4. A maximum principle for p-harmonic maps

It is well known, and an easy consequence of the composition law of the Hessians, that composing a harmonic map $u : M \to N$ with a convex function $h : N \to \mathbb{R}$ gives a subharmonic function $v = h \circ u : M \to \mathbb{R}$, i.e., $\Delta v \geq 0$. In particular, if M is compact with smooth boundary $\partial M \neq 0$ and N is Cartan-Hadamard, we can choose $h(x) = d^2_N(x, o)$ and apply the usual maximum principle to conclude that the image $u(M) \subset N$ is confined in a ball $B^N_R(o)$ of radius $R > 0$ depending only on the values of u on $\partial\Omega$, namely, $R = \max_{\partial\Omega} d_N(u, o)$. It was proved in [Ve] that, in general, the nice composition property of harmonic maps does not extend to p-harmonic maps, $p > 2$. Nevertheless, we are able to recover the above conclusion, thus establishing a new maximum principle for the composition of a p-harmonic map and a convex function.

Theorem 4.1. *Let M be a compact Riemannian manifold with boundary $\partial M \neq \emptyset$, and let $u \in C^1(M, N)$ be a $p(>1)$-harmonic map. Assume that N supports a smooth convex function $f : N \to \mathbb{R}$. Set $w = f \circ u : M \to \mathbb{R}$. Then*

$$\sup_M w = \sup_{\partial M} w.$$

Proof. We give the proof for $p \geq 2$ without taking care of the regularity issues. Similar distributional computations permit us to deal with the general case.

Let $w^* = \sup_{\partial M} w$ and, by contradiction, suppose that $w(x_0) > w^*$ for some $x_0 \in \mathrm{int}(M)$. Fix $0 < \varepsilon << 1$ so that $w(x_0) - w^* > 2\varepsilon$. Let $\lambda : \mathbb{R} \to [0, 1]$ satisfy $\lambda' \geq 0$, $\lambda' > 0$ on $(\varepsilon, +\infty)$, $\lambda = 0$ on $(-\infty, \varepsilon]$. Define the vector field

$$Z = |du|^{p-2} \lambda(w - w^*) \nabla w,$$

and note that $\mathrm{supp} Z \subset \mathrm{int}(M)$. Direct computations show that

$$\begin{aligned} \operatorname{div} Z &= \lambda' \circ (w - w^*) |du|^{p-2} |\nabla w|^2 \\ &\quad + \lambda \circ (w - w^*) \operatorname{tr} \operatorname{Hess}(f) \left(|du|^{p-2} du, du \right) \\ &\quad + \lambda \circ (w - w^*) df (\Delta_p u) \\ &\geq |\nabla w|^2 |du|^{p-2} \lambda' \circ (w - w^*), \end{aligned}$$

and applying the divergence theorem, we get

$$0 \leq \int_M |\nabla w|^2 |du|^{p-2} \lambda' \circ (w - w^*) \leq \int_M \operatorname{div} Z = 0.$$

This proves that

$$|\nabla w|^2 |du|^{p-2} = 0 \text{ on } M_\varepsilon, \tag{4.1}$$

where we have denoted with M_ϵ the connected component containing x_0 of the open set

$$\{x \in M : w - w^* - \varepsilon > 0\}.$$

Since, by (4.1), $dw = df(du) = 0$ where $du \neq 0$ and $dw = df(du) = 0$ where $du = 0$, it follows that w is constant on M_ε, and this easily gives the desired contradiction. □

5. Proof of the main results

In this last section we put all the previous ingredients together to get a proof of Theorem B.

The boundary datum f has image confined in a ball $B^N_{R_0}(0)$ of N^n. Using Theorem 2.1, we glue $B^N_{R_0}(0)$ to the exterior of a large ball in the hyperbolic spaceform $\mathbb{H}^n_{-k^2}$ of sufficiently negative curvature $-k^2 << -1$, say $\mathbb{H}^n_{-k^2} \backslash \mathbb{B}_{R_1}(0)$, $R_1 >> R_0$, thus obtaining a new Cartan-Hadamard rotationally symmetric (resp. 2-dimensional) manifold (N', h'). On the other hand, by Proposition 3.2, $\mathbb{H}^n_{-k^2}$ has compact quotients with arbitrarily large injectivity radii. Accordingly, we can choose a discrete subgroup Γ of isometries acting freely and co-compactly on $\mathbb{H}^n_{-k^2}$ in such a way that $\mathbb{B}_{R_1}(0)$ is contained in a relatively compact, fundamental domain of the action, say $\mathbb{B}_{R_1}(0) \subset\subset \Omega$. Making use of Γ we extend the deformed metric of $\bar{\Omega}$ periodically thus obtaining a new Riemannian manifold N'' diffeomorphic to $\mathbb{H}^m_{-k^2}$. More precisely, the metric h'' of N'' is defined by setting

$$h''_{\gamma \cdot p} = \left(\gamma^{-1}\right)^*_{\gamma \cdot p} h'_p.$$

Since h' is hyperbolic in a neighborhood of $\partial\Omega$, the definition of h'' is well posed. Moreover, (N'', h'') has non-positive curvature, hence it is Cartan-Hadamard, and, by construction, Γ acts freely and co-compactly by isometries on N''. In particular, each copy of Ω contains an isometric image of $B^N_{R_0}(0)$. Now, we take the quotient manifold N''/Γ which is compact and covered by N'' via the quotient projection $P : N'' \to N''/\Gamma$. By construction, the original datum f well defines $f'' = f : M \to N''$. Applying Proposition A we get a unique solution $u'' \in C^0(M, N'') \cap C^{1,\alpha}(\mathrm{int}(M), N'')$ to the Dirichlet problem

$$\begin{cases} \Delta_p u'' = 0 & \text{on } M, \\ u'' = f'' & \text{on } \partial M. \end{cases}$$

To complete the argument, it remains to show that, actually, u'' gives rise to a solution of the original problem. This clearly follows if we are able to show that its image is confined in $B^N_{R_0}(0) \subset N''$. To prove that this is the case, we recall that N'' is Cartan-Hadamard and, therefore, the function $d^2_{N''}(y, 0)$ is smooth and strictly convex. By means of Theorem 4.1, we deduce that $d^2_{N''}(u'', 0)$ achieves its maximum on ∂M. To conclude, it suffices to recall that $f(M) \subset B^N_{R_0}(0)$ and to use the equality $u'' = f$ on ∂M.

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