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ON FLOW EQUIVALENCE OF ONE-SIDED TOPOLOGICAL MARKOV SHIFTS

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ABSTRACT. We introduce notions of suspension and flow equivalence on one-sided topological Markov shifts, which we call one-sided suspension and one-sided flow equivalence, respectively. We prove that one-sided flow equivalence is equivalent to continuous orbit equivalence on one-sided topological Markov shifts. We also show that the zeta function of the flow on a one-sided suspension is a dynamical zeta function with some potential function and that the set of certain dynamical zeta functions is invariant under one-sided flow equivalence of topological Markov shifts.

1. INTRODUCTION

Flow equivalence relation on two-sided topological Markov shifts is one of the most important and interesting equivalence relations on symbolic dynamical systems. It has a close relationship to classifications of not only continuous time dynamical systems but also associated C^*-algebras. For an irreducible square matrix $A = [A(i,j)]_{i,j=1}^N$ with its entries in $\{0,1\}$, the two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$ is defined as a compact Hausdorff space $\bar{X}_A$ consisting of bi-infinite sequences $(\bar{x}_n)_{n\in\mathbb{Z}}$ of $\bar{x}_n \in \{1,2,\dots,N\}$ such that $A(\bar{x}_n, \bar{x}_{n+1}) = 1, n \in \mathbb{Z}$ with shift homeomorphism $\bar{\sigma}_A$ defined by $\bar{\sigma}_A((\bar{x}_n)_{n\in\mathbb{Z}}) = (\bar{x}_{n+1})_{n\in\mathbb{Z}}$. For a positive continuous function g on $\bar{X}_A$, let us denote by $\bar{S}_A^g$ the compact Hausdorff space obtained from $\{(\bar{x}, r) \in \bar{X}_A \times \mathbb{R} \mid \bar{x} \in \bar{X}_A, 0 \le r \le g(\bar{x})\}$ by identifying $(\bar{x}, g(\bar{x}))$ with $(\bar{\sigma}_A(\bar{x}), 0)$ for each $\bar{x} \in \bar{X}_A$. Let $\bar{\phi}_{A,t}, t \in \mathbb{R}$ be the flow on $\bar{S}_A^g$ defined by $\bar{\phi}_{A,t}([(\bar{x}, r)]) = [(\bar{x}, r+t)]$ for $[(\bar{x}, r)] \in \bar{S}_A^g$. The dynamical system $(\bar{S}_A^g, \bar{\phi}_A)$ is called the suspension of $(\bar{X}_A, \bar{\sigma}_A)$ by a ceiling function g. Two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are said to be flow equivalent if there exists a positive continuous function g on $\bar{X}_A$ such that $(\bar{S}_A^g, \bar{\phi}_A)$ is topologically conjugate to $(\bar{S}_B^1, \bar{\phi}_B)$. It is well known that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if $\mathbb{Z}^N/(\mathrm{id}-A)\mathbb{Z}^N$ is isomorphic to $\mathbb{Z}^M/(\mathrm{id}-B)\mathbb{Z}^M$ as abelian groups and $\det(\mathrm{id}-A) = \det(\mathrm{id}-B)$, where N, M are the sizes of the matrices A, B, respectively ([3], [5], [12]). Let us denote by $\mathcal{K}$ the C^*-algebra of compact operators on the separable infinite dimensional Hilbert space $\ell^2(\mathbb{N})$ and $\mathcal{C}$ its maximal abelian C^*-subalgebra consisting of diagonal elements on $\ell^2(\mathbb{N})$. Let us denote by $\mathcal{O}_A$ the Cuntz–Krieger algebra and by $\mathcal{D}_A$ its canonical maximal abelian C^*-subalgebra. Since the group $\mathbb{Z}^N/(\mathrm{id}-A)\mathbb{Z}^N$ is a complete invariant of the isomorphism class of the tensor product C^*-algebra $\mathcal{O}_A \otimes \mathcal{K}$ ([16]), the C^*-algebra $\mathcal{O}_A \otimes \mathcal{K}$ with $\det(\mathrm{id}-A)$ is a complete invariant for flow equivalence of the two-sided topological

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Markov shift $(\bar{X}_A, \bar{\sigma}_A)$. It has been recently shown in [9] that the isomorphism class of the pair $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})$ is a complete invariant for flow equivalence class of $(\bar{X}_A, \bar{\sigma}_A)$.

One-sided topological Markov shifts (X_A, σ_A) are also an important and interesting class of dynamical systems. The space X_A is defined as a compact Hausdorff space consisting of right infinite sequences $(x_n)_{n\in\mathbb{N}}$ of $x_n \in \{1, 2, \dots, N\}$ such that $A(x_n, x_{n+1}) = 1, n \in \mathbb{N}$ with continuous map σ_A defined by $\sigma_A((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}$. In [8], the author has introduced a notion of continuous orbit equivalence between one-sided topological Markov shifts. One-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are said to be continuously orbit equivalent if there exists a homeomorphism $h : X_A \to X_B$ and continuous functions $k_1, l_1 : X_A \to \mathbb{Z}_+$ and $k_2, l_2 : X_B \to \mathbb{Z}_+$ such that

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \text{for} \quad x \in X_A, \tag{1.1}$$

$$\sigma_A^{k_2(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l_2(y)}(h^{-1}(y)) \quad \text{for} \quad y \in X_B. \tag{1.2}$$

In [9], it has been proved that the isomorphism class of $\mathcal{O}_A$ with $\det(\mathrm{id} - A)$ is a complete invariant for continuous orbit equivalence of the one-sided topological Markov shift (X_A, σ_A). We have already known in [8] that the isomorphism class of the pair $(\mathcal{O}_A, \mathcal{D}_A)$ is a complete invariant for continuous orbit equivalence of (X_A, σ_A). Hence we may regard continuous orbit equivalence of one-sided topological Markov shifts as a one-sided counterpart of flow equivalence of two-sided topological Markov shifts.

In this paper, we will introduce a notion of flow equivalence on one-sided topological Markov shifts (X_A, σ_A). We will first introduce one-sided suspension $S_{A,b}^{l,k}$ with a flow ϕ_A associated to three real valued continuous functions $l, k, b \in C(X_A, \mathbb{R})$ on X_A for a one-sided topological Markov shift (X_A, σ_A). The space $S_{A,b}^{l,k}$ is determined by a base map $b : X_A \to \mathbb{R}$ and a ceiling function $l : X_A \to \mathbb{R}_+$ by identifying (x, r) with $(\sigma_A(x), r - (l-k))$ for $r \geq l(x)$. By using the one-sided suspension, we will define one-sided flow equivalence on one-sided topological Markov shifts in Definition 3.1. As a main result of the paper, we will prove the following theorem.

Theorem 1.1 (Theorem 3.4)**.** *Assume that matrices A and B are irreducible and not permutation matrices. One-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are one-sided flow equivalent if and only if they are continuously orbit equivalent.*

By using [10, Theorem 3.6], we see the following characterization of one-sided flow equivalence which is a corollary of the above theorem.

Corollary 1.2 (Corollary 3.5)**.** *Assume that matrices A and B are irreducible and not permutation matrices. One-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are one-sided flow equivalent if and only if there exists an isomorphism $\Phi : \mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N \to \mathbb{Z}^M/(\mathrm{id} - B)\mathbb{Z}^M$ of abelian groups such that $\Phi([u_A]) = [u_B]$ and $\det(\mathrm{id} - A) = \det(\mathrm{id} - B)$, where $[u_A]$ (resp. $[u_B]$) is the class of the vector $u_A = [1, \dots, 1]$ in $\mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N$ (resp. $u_B = [1, \dots, 1]$ in $\mathbb{Z}^M/(\mathrm{id} - B)\mathbb{Z}^M$).*

The zeta function $\zeta_\phi(s)$ of a flow $\phi_t : S \to S$ on a compact metric space S with at most countably many closed orbits is defined by

$$\zeta_\phi(s) = \prod_{\tau \in P_{orb}(S,\phi)} (1 - e^{-s\ell(\tau)})^{-1} \qquad \text{(see [11], [17], [19], etc.)} \tag{1.3}$$

where $P_{orb}(S,\phi)$ denotes the set of primitive periodic orbits of the flow $\phi_t : S \to S$ and $\ell(\tau)$ is the primitive length of the closed orbit defined by $\ell(\tau) = \min\{t \in \mathbb{R}_+ \mid \phi_t(u) = u\}$ for any point $u \in \tau$. For a one-sided suspension $S^{l,k}_{A,b}$ of (X_A, σ_A), there exists a bijective correspondence between primitive periodic orbits $\tau \in P_{orb}(S^{l,k}_{A,b}, \phi_A)$ and periodic orbits $\gamma_\tau \in P_{orb}(X_A)$ of (X_A, σ_A) such that the length $\ell(\tau)$ of the orbit τ is $\sum_{i=0}^{p-1} c(\sigma_A^i(x))$ for $c = l - k$ and $\gamma_\tau = \{x, \sigma_A(x), \dots, \sigma_A^{p-1}(x)\}$. Therefore we have

Proposition 1.3 (Proposition 4.2)**.** *Assume that a matrix A is irreducible and not any permutation matrix. The zeta function $\zeta_{\phi_A}(s)$ of the flow ϕ_A of the one-sided suspension $S^{l,k}_{A,b}$ of (X_A, σ_A) is given by the dynamical zeta function $\zeta_{A,c}(s)$ with potential function $c = l - k$ such that*

$$\zeta_{A,c}(s) = \exp\{\sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in \mathrm{Per}_n(X_A)} \exp(-s \sum_{i=0}^{n-1} c(\sigma_A^i(x)))\}. \tag{1.4}$$

In [2], Boyle–Handelman have proved that the set of zeta functions of homeomorphisms flow equivalent to the two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$ is a complete invariant for flow equivalence of $(\bar{X}_A, \bar{\sigma}_A)$. If (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent via a homeomorphism $h : X_A \to X_B$ with continuous functions $k_1, l_1 : X_A \to \mathbb{Z}_+$ and $k_2, l_2 : X_B \to \mathbb{Z}_+$ satisfying (1.1) and (1.2), we may define homomorphisms $\Psi_h : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z})$ by

$$\Psi_h(f)(x) = \sum_{i=0}^{l_1(x)-1} f(\sigma_B^i(h(x))) - \sum_{j=0}^{k_1(x)-1} f(\sigma_B^j(h(\sigma_A(x)))) \tag{1.5}$$

for $f \in C(X_B, \mathbb{Z})$, $x \in X_A$ and similarly $\Psi_{h^{-1}} : C(X_A, \mathbb{Z}) \to C(X_B, \mathbb{Z})$ for $h^{-1} : X_B \to X_A$. In [10], it has been proved that $\Psi_h : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z})$ induces an isomorphism of their ordered cohomology groups (H^B, H^B_+) and (H^A, H^A_+). If (X_A, σ_A) and (X_B, σ_B) are one-sided flow equivalent, they are continuously orbit equivalent, so that we may define the map $\Psi_h : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z})$ as above. Inspired by [2], we will show the following.

Theorem 1.4 (Theorem 4.6)**.** *Assume that matrices A and B are irreducible and not permutation matrices. Suppose that (X_A, σ_A) and (X_B, σ_B) are one-sided flow equivalent via a homeomorphism $h : X_A \to X_B$. For $f \in C(X_B, \mathbb{Z}), g \in C(X_A, \mathbb{Z})$ such that their classes $[f] \in H^B$, $[g] \in H^A$ are order units of $(H^B, H^B_+), (H^A, H^A_+)$, respectively, the following formulae hold:*

$$\zeta_{A,\Psi_h(f)}(s) = \zeta_{B,f}(s), \qquad \zeta_{B,\Psi_{h^{-1}}(g)}(s) = \zeta_{A,g}(s).$$

This theorem shows that the set $Z(X_A, \sigma_A)$ of dynamical zeta functions of (X_A, σ_A) whose potential functions are order units in the ordered cohomology group (H^A, H^A_+) is invariant under one-sided flow equivalence (Corollary 4.7).

Throughout the paper, we denote by $\mathbb{R}_+$, by $\mathbb{Z}_+$ and by $\mathbb{N}$ the set of nonnegative real numbers, the set of nonnegative integers and the set of positive integers, respectively.

2. One-sided suspensions

In what follows, we assume that $A = [A(i,j)]_{i,j=1}^N$ is an $N \times N$ irreducible matrix with entries in $\{0,1\}$ and $1 < N \in \mathbb{N}$. We further assume that A is not any permutation matrix. This further assumption is equivalent to the condition (I) in the sense of [4] so that the space X_A is homeomorphic to a Cantor discontinuum. We denote by $C(X_A, \mathbb{R})$ (resp. $C(X_A, \mathbb{R}_+)$) the set of real (resp. nonnegative real) valued continuous functions on X_A. The set $C(X_A, \mathbb{Z})$ of integer valued continuous functions on X_A has a natural structure of abelian group by pointwise sums. Let us denote by H^A the quotient group of the abelian group $C(X_A, \mathbb{Z})$ by the subgroup $\{g - g \circ \sigma_A \mid g \in C(X_A, \mathbb{Z})\}$. The positive cone H^A_+ consists of the classes $[f] \in H^A$ of nonnegative integer valued continuous functions $f \in C(X_A, \mathbb{Z}_+)$. The ordered group (H^A, H^A_+) is called the ordered cohomology group for (X_A, σ_A) (cf. [2], [9], [15]). For $f \in C(X_A, \mathbb{Z})$ and $m \in \mathbb{N}$, we set

$$f^m(x) = \sum_{i=0}^{m-1} f(\sigma_A^i(x)), \qquad x \in X_A.$$

An element $[f]$ in H^A_+ is called an order unit if for any $[g] \in H^A$, there exists $n \in \mathbb{N}$ such that $n[f] - [g] \in H^A_+$. We see that $[f] \in H^A_+$ is an order unit if and only if there exists $m \in \mathbb{N}$ such that f^m is strictly positive ([2, p. 175]).

We will first define a one-sided suspension for a one-sided topological Markov shift. The triplet (l, k, b) for real valued continuous functions $l, k \in C(X_A, \mathbb{R}_+)$ and $b \in C(X_A, \mathbb{R})$ are called *a suspension triplet* for (X_A, σ_A) if they satisfy the following two conditons:

(1) The difference $c = l - k$ belongs to $C(X_A, \mathbb{Z})$ and the class $[c] \in H^A_+$ is an order unit of (H^A, H^A_+).
(2) The differences $l - b$, $k - b \circ \sigma_A$ belong to $C(X_A, \mathbb{Z}_+)$.

The triplet $(1, 0, 0)$ is called the standard suspension triplet.

We fix a suspension triplet (l, k, b) for (X_A, σ_A) for a while. We set $X^{\mathbb{R}}_{A,b} = \{(x, r) \in X_A \times \mathbb{R} \mid r \geq b(x)\}$ and define an equivalence relation $\underset{l,k}{\sim}$ in $X^{\mathbb{R}}_{A,b}$ generated by the relations

$$(x, r) \underset{l,k}{\sim} (\sigma_A(x), r - c(x)) \qquad \text{for } r \geq l(x). \tag{2.1}$$

We note that the condition $r \geq l(x)$ implies $(x, r) \in X^{\mathbb{R}}_{A,b}$ and $r - c(x) = r - l(x) + k(x) \geq k(x) \geq b(\sigma_A(x))$ so that $(\sigma_A(x), r - c(x)) \in X^{\mathbb{R}}_{A,b}$. If there exists $n \in \mathbb{Z}_+$ such that $r \geq c^m(x) + l(\sigma_A^m(x))$ for all $m \in \mathbb{Z}_+$ with $0 \leq m < n$, then

$$(x, r) \underset{l,k}{\sim} (\sigma_A(x), r - c(x)) \underset{l,k}{\sim} \cdots \underset{l,k}{\sim} (\sigma_A^n(x), r - c^n(x)).$$

Hence we have

Lemma 2.1. *For $(x, r), (x', r') \in X^{\mathbb{R}}_{A,b}$, we have $(x, r) \underset{l,k}{\sim} (x', r')$ if and only if there exist $n, n' \in \mathbb{Z}_+$ such that*

$$\sigma_A^n(x) = \sigma_A^{n'}(x'), \quad r - c^n(x) = r' - c^{n'}(x') \quad \textit{and}$$
$$r - c^m(x) \geq l(\sigma_A^m(x)), \quad r - c^{m'}(x') \geq l(\sigma_A^{m'}(x')) \quad \textit{for} \quad 0 \leq m < n, 0 \leq m' < n'.$$

Proof. It suffices to prove the only if part. For $(x, r) \in X^{\mathbb{R}}_{A,b}$ with $r \geq l(x)$, we write a directed edge from (x, r) to $(\sigma_A(x), r - c(x))$. We then have a directed graph with vertex set $X^{\mathbb{R}}_{A,b}$. Suppose $(x, r) \underset{l,k}{\sim} (x', r')$. There exists a finite sequence $(x_i, r_i) \in X^{\mathbb{R}}_{A,b}, i = 0, 1, \dots, L$ such that $(x_0, r_0) = (x, r)$ and $(x_L, r_L) = (x', r')$, and there exists a directed edge from (x_{i-1}, r_{i-1}) to (x_i, r_i) or from (x_i, r_i) to (x_{i-1}, r_{i-1}) for each $i = 1, 2, \dots, L$. Since each vertex (x_i, r_i) emits at most one directed edge, we may find n with $0 \le n \le L$ such that there exist directed edges from (x_{i-1}, r_{i-1}) to (x_i, r_i) for $i = 1, 2, \dots, n$ and from (x_i, r_i) to (x_{i-1}, r_{i-1}) for $i = n+1, \dots, L$. By putting $n' = L - n$, we see that n and n' satisfy the desired conditions for (x, r) and (x', r'). $\square$

Define a topological space

$$S_{A,b}^{l,k} = X_{A,b}^{\mathbb{R}} / \underset{l,k}{\sim} \tag{2.2}$$

as the quotient topological space of $X_{A,b}^{\mathbb{R}}$ by the equivalence relation $\underset{l,k}{\sim}$. We denote by $[x, r]$ the class of $(x, r) \in X_{A,b}^{\mathbb{R}}$ in the quotient space $S_{A,b}^{l,k}$. We will show that $S_{A,b}^{l,k}$ is a compact Hausdorff space. We note the following lemma.

Lemma 2.2. *For any $m \in \mathbb{N}$, there exists $n_m \in \mathbb{N}$ such that $c^{n_m}(x) \ge m$ for all $x \in X_A$.*

Proof. Since $[c] \in H_+^A$ is an order unit, one may take $p \in \mathbb{N}$ such that c^p is a strictly positive function so that $c^p(x) \ge 1$ for all $x \in X_A$. By the identity $c^{mp}(x) = c^{(m-1)p}(x) + c^p(\sigma_A^{(m-1)p}(x))$ for $m \in \mathbb{N}, x \in X_A$, one obtains that $c^{mp}(x) \ge m$ for all $x \in X_A$. By putting $n_m = mp$, we see the desired assertion. $\square$

We set

$$\Omega_{A,b}^{l} = \{(x, r) \in X_{A,b}^{\mathbb{R}} \mid b(x) \le r \le l(x)\},$$
$$\Omega_{A,b}^{l\circ} = \{(x, r) \in X_{A,b}^{\mathbb{R}} \mid b(x) \le r < l(x)\}.$$

Lemma 2.3. *For $(x, r) \in X_{A,b}^{\mathbb{R}}$ with $r \ge l(x)$, there exists $(z, s) \in \Omega_{A,b}^{l\circ}$ such that $(x, r) \underset{l,k}{\sim} (z, s)$.*

Proof. For $(x, r) \in X_{A,b}^{\mathbb{R}}$ with $r \ge l(x)$, by Lemma 2.2 one may take a minimum number $n \in \mathbb{N}$ satisfying $r < c^{n+1}(x) + k(\sigma_A^n(x))$, so that we have

$$c^m(x) + k(\sigma_A^{m-1}(x)) \le r < c^{n+1}(x) + k(\sigma_A^n(x)) \quad \text{for all} \quad m \le n.$$

In particular, we have

$$c^n(x) + k(\sigma_A^{n-1}(x)) \le r < c^{n+1}(x) + k(\sigma_A^n(x)).$$

As $c^{n+1}(x) = c^n(x) + l(\sigma_A^n(x)) - k(\sigma_A^n(x))$ and $b(\sigma_A^n(x)) \le k(\sigma_A^{n-1}(x))$, we get

$$b(\sigma_A^n(x)) \le r - c^n(x) < l(\sigma_A^n(x)).$$

Since $r - c^m(x) \ge l(\sigma_A^m(x))$ for all $m \in \mathbb{Z}_+$ with $m < n$, we see that

$$(x, r) \underset{l,k}{\sim} (\sigma_A^m(x), r - c^m(x)) \quad \text{for all } m \in \mathbb{Z}_+ \text{ with } m \le n.$$

By putting $z = \sigma_A^n(x), s = r - c^n(x)$, we have $(x, r) \underset{l,k}{\sim} (z, s)$ and $(z, s) \in \Omega_{A,b}^{l\circ}$. $\square$

Proposition 2.4. *$S_{A,b}^{l,k}$ is a compact Hausdorff space.*

Proof. We may restrict the equivalence relation $\underset{l,k}{\sim}$ to $\Omega^l_{A,b}$. Let $q_\Omega : \Omega^l_{A,b} \to \Omega^l_{A,b}/\underset{l,k}{\sim}$ and $q_X : X^{\mathbb{R}}_{A,b} \to S^{l,k}_{A,b}$ be the quotient maps, which are continuous. Since $\Omega^l_{A,b}$ is compact, so is $\Omega^l_{A,b}/\underset{l,k}{\sim}$. By Lemma 2.3, an element of $\Omega^l_{A,b}/\underset{l,k}{\sim}$ is represented by $\Omega^{l\circ}_{A,b}/\underset{l,k}{\sim}$. This implies that $\Omega^l_{A,b}/\underset{l,k}{\sim}$ is Hausdorff, because $\Omega^{l\circ}_{A,b}$ is a fundamental domain of the quotient space $\Omega^l_{A,b}/\underset{l,k}{\sim}$. For $(x,r) \in X^{\mathbb{R}}_{A,b}$, take a minimum number $n \in \mathbb{Z}_+$ such that $r < c^{n+1}(x)+k(\sigma^n_A(x))$, and define a continuous map $\varphi : X^{\mathbb{R}}_{A,b} \to \Omega^l_{A,b}$ by setting $\varphi((x,r)) = (\sigma^n_A(x), r - c^n(x))$. By the proof of Lemma 2.3, it induces a map $\tilde{\varphi} : S^{l,k}_{A,b} \to \Omega^l_{A,b}/\underset{l,k}{\sim}$. We will see that $\tilde{\varphi}$ is a homeomorphism, so that $S^{l,k}_{A,b}$ is a compact Hausdorff space. As the inclusion map $\iota : (x,r) \in \Omega^l_{A,b} \to (x,r) \in X^{\mathbb{R}}_{A,b}$ induces a map

$$\tilde{\iota} : [x,r] \in \Omega^l_{A,b}/\underset{l,k}{\sim} \quad \to \quad [x,r] \in S^{l,k}_{A,b}$$

which satisfies $\tilde{\varphi} \circ \tilde{\iota} = \mathrm{id}$, $\tilde{\iota} \circ \tilde{\varphi} = \mathrm{id}$, the map $\tilde{\varphi}$ is bijective. We have commutative diagrams:

$$\begin{array}{ccc} X^{\mathbb{R}}_{A,b} & \xrightarrow{\varphi} & \Omega^l_{A,b} \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_\Omega} \\ S^{l,k}_{A,b} & \xrightarrow{\tilde{\varphi}} & \Omega^l_{A,b}/\underset{l,k}{\sim} \end{array}, \qquad \begin{array}{ccc} X^{\mathbb{R}}_{A,b} & \xleftarrow{\iota} & \Omega^l_{A,b} \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_\Omega} \\ S^{l,k}_{A,b} & \xleftarrow{\tilde{\iota}} & \Omega^l_{A,b}/\underset{l,k}{\sim} \end{array}.$$

Since both the maps $\varphi : X^{\mathbb{R}}_{A,b} \to \Omega^l_{A,b}$ and $\iota : \Omega^l_{A,b} \to X^{\mathbb{R}}_{A,b}$ are continuous, the commutativity of the diagrams imply the continuity of the maps $\tilde{\varphi} : S^{l,k}_{A,b} \to \Omega^l_{A,b}/\underset{l,k}{\sim}$ and $\tilde{\iota} : \Omega^l_{A,b}/\underset{l,k}{\sim} \to S^{l,k}_{A,b}$. Hence $\tilde{\varphi} : S^{l,k}_{A,b} \to \Omega^l_{A,b}/\underset{l,k}{\sim}$ is a homeomorphism. □

We will define the flow $\phi_{A,t}, t \in \mathbb{R}_+$ on $S^{l,k}_{A,b}$ by $\phi_{A,t}([x,r]) = [x, r+t]$ for $t \in \mathbb{R}_+$. As in the discussions above, for $(x,r) \in X^{\mathbb{R}}_{A,b}$ and $t \in \mathbb{R}_+$, there exists $n \in \mathbb{Z}_+$ such that

$$b(\sigma^n_A(x)) \le t + r - c^n(x) < l(\sigma^n_A(x))$$

and

$$\phi_{A,t}([x,r]) = [\sigma^n_A(x), t + r - c^n(x)].$$

Hence the flow $\phi_{A,t}([x,r]), t \in \mathbb{R}_+$ are defined in $S^{l,k}_{A,b}$. We call the flow space $(S^{l,k}_{A,b}, \phi_A)$ the (l,k,b)-suspension of one-sided topological Markov shift (X_A, σ_A). It is simply called the *one-sided suspension* of (X_A, σ_A). The map $b_A : X_A \to S^{l,k}_{A,b}$ defined by $b_A(x) = [x, b(x)]$ is called the base map. The base map for $b \equiv 0$ is written $s_A(x) = [x, 0]$ and called the standard base map. If all the functions l, k, b are valued in integers, (l,k,b)-suspension for $X^{\mathbb{Z}}_{A,b}$ is called (l,k,b)-discrete suspension. For $l \equiv 1, k \equiv 0, b \equiv 0$, the $(1,0,0)$-suspension $(S^{1,0}_{A,0}, \phi_A)$ is called the standard one-sided suspension. The $(l,k,0)$-suspension space $S^{l,k}_{A,0}$ and the $(l,0,0)$-suspension space $S^{l,0}_{A,0}$ are denoted by $S^{l,k}_A$ and S^l_A, respectively. The standard one-sided suspension $(S^{1,0}_{A,0}, \phi_A)$ is denoted by (S^1_A, ϕ_A).

Lemma 2.5. *The standard base map $s_A : X_A \to S^1_A$ for the standard suspension is an injective continuous map.*

Proof. It suffices to show the injectivity of the map s_A. Suppose that $s_A(x) = s_A(z)$ in S^1_A for some $x, z \in X_A$. There exists a finite sequence $(x_i, r_i) \in X^{\mathbb{R}}_{A,0}$ such that

$$(x,0) \underset{1,0}{\sim} (x_1, r_1) \underset{1,0}{\sim} \cdots \underset{1,0}{\sim} (x_n, r_n) \underset{1,0}{\sim} (z, 0)$$

where $r_i \in \mathbb{Z}_+, i = 1, \dots, n$. If $r_i = r_{i+1}$, we may take $x_i = x_{i+1}$. Let $K = \mathrm{Max}\{r_i \mid i = 1, \dots, n\}$. Take $i_K \in \{1, \dots, n\}$ such that $r_{i_K} = K$. It then follows that $\sigma^K_A(x_{i_K}) = x$ and $\sigma^K_A(x_{i_K}) = z$, so that $x = z$. ☐

Lemma 2.6. *For a suspension triplet (l, k, b) for (X_A, σ_A), put $c = l - k$. If $c' \in C(X_A, \mathbb{Z})$ satisfies $[c] = [c']$ in H^A, there exist a suspension triplet (l', k', b') for (X_A, σ_A) and a homeomorphism $\Phi : S^{l,k}_{A,b} \to S^{l',k'}_{A,b'}$ such that $c' = l' - k'$ and*

$$\Phi \circ b_A = b'_A, \qquad \Phi \circ \phi_{A,t} = \phi_{A,t} \circ \Phi \quad \text{for } t \in \mathbb{R}_+. \tag{2.3}$$

Hence $(S^{l,k}_{A,b}, \phi_A)$ and $(S^{l',k'}_{A,b'}, \phi_A)$ are topologically conjugate compartible to their base maps.

Proof. Since $[c] = [c']$ in H^A, there exists $d \in C(X_A, \mathbb{Z})$ such that $c - c' = d \circ \sigma_A - d$. We may assume that $d(x) \in \mathbb{Z}_+$ for all $x \in X_A$. Define

$$l'(x) = l(x) + d(x), \qquad k'(x) = k(x) + d(\sigma_A(x)), \qquad b'(x) = b(x) + d(x), \qquad x \in X_A.$$

It is easy to see that $c' = l' - k'$ and (l', k', b') is a suspension triplet for (X_A, σ_A). Define $\Phi : X^{\mathbb{R}}_{A,b} \to X^{\mathbb{R}}_{A,b'}$ by $\Phi((x, r)) = (x, r + d(x))$. We know that $(x, r) \in X^{\mathbb{R}}_{A,b}$ if and only if $(x, r + d(x)) \in X^{\mathbb{R}}_{A,b'}$. Since $\Phi((\sigma_A(x), r - c(x))) = (\sigma_A(x), r - c(x) + d(\sigma_A(x))) = (\sigma_A(x), r + d(x) - c'(x))$, we have $\Phi((x, r)) \underset{l',k'}{\sim} \Phi((\sigma_A(x), r - c(x)))$ for $r \geq l'(x)$. It is easy to see that Φ extends to a homeomorphism $S^{l,k}_{A,b} \to S^{l',k'}_{A,b'}$ which is still denoted by Φ and satisfies the equalities (2.3). ☐

3. One-sided flow equivalence

We will define one-sided flow equivalence on one-sided topological Markov shifts.

Definition 3.1. (X_A, σ_A) and (X_B, σ_B) are said to be *one-sided flow equivalent* if there exist suspension triplets (l_1, k_1, b_1) for (X_A, σ_A) and (l_2, k_2, b_2) for (X_B, σ_B), a homeomorphism $h : X_A \to X_B$, and continuous maps $\Phi_1 : S^{l_1,k_1}_{A,b_1} \to S^1_B$, $\Phi_2 : S^{l_2,k_2}_{B,b_2} \to S^1_A$ such that

$$\Phi_1 \circ \phi_{A,t} = \phi_{B,t} \circ \Phi_1 \quad \text{for } t \in \mathbb{R}_+, \qquad \Phi_1 \circ b_{1,A} = s_B \circ h, \tag{3.1}$$

$$\Phi_2 \circ \phi_{B,t} = \phi_{A,t} \circ \Phi_2 \quad \text{for } t \in \mathbb{R}_+, \qquad \Phi_2 \circ b_{2,B} = s_A \circ h^{-1}, \tag{3.2}$$

where $b_{1,A} : X_A \to S^{l_1,k_1}_{A,b_1}$ and $b_{2,B} : X_B \to S^{l_2,k_2}_{B,b_2}$ are the base maps defined by $b_{1,A}(x) = [x, b_1(x)]$ for $x \in X_A$ and $b_{2,B}(y) = [y, b_2(y)]$ for $y \in X_B$, respectively.

In this case, we say that (X_A, σ_A) and (X_B, σ_B) are one-sided flow equivalent via a homeomorphism $h : X_A \to X_B$. If there exists a homeomorphism $h : X_A \to X_B$ satisfying (3.1) (resp. (3.2)), we say that (X_B, σ_B) (resp. (X_A, σ_A)) is a *cross section* of the one-sided suspension S^{l_1,k_1}_{A,b_1} through $b_{1,A} \circ h^{-1}$ (resp. S^{l_2,k_2}_{B,b_2} through $b_{2,B} \circ h$).

Proposition 3.2. *If* (X_A, σ_A) *and* (X_B, σ_B) *are continuously orbit equivalent, then they are one-sided flow equivalent.*

Proof. Let $h : X_A \to X_B$ be a homeomorphism which gives rise to a continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B) with continuous functions $k_1, l_1 : X_A \to \mathbb{Z}_+$ and $k_2, l_2 : X_B \to \mathbb{Z}_+$ satisfying (1.1) and (1.2), respectively. Put $c_1(x) = l_1(x) - k_1(x), x \in X_A$ and $c_2(y) = l_2(y) - k_2(y), y \in X_B$. We set $b_1 \equiv 0$, $b_2 \equiv 0$. By [10, Theorem 5.11], the map $\Psi_h : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z})$ defined by (1.5) induces an isomorphism of ordered groups between (H^B, H^B_+) and (H^A, H^A_+), so that the elements $\Psi_h(1) = c_1$ and similarly $\Psi_{h^{-1}}(1) = c_2$ give rise to an order unit of (H^A, H^A_+) and of (H^B, H^B_+), respectively. Then (l_1, k_1, b_1) is a suspension triplet for (X_A, σ_A) and (l_2, k_2, b_2) is a suspension triplet for (X_B, σ_B). Define $\Phi_1 : X^{\mathbb{R}}_{A,0} \to X^{\mathbb{R}}_{B,0}$ by $\Phi_1((x, r)) = (h(x), r)$ for $x \in X_A, r \geq 0$. As S^1_B is the standard suspension, we have for $(x, r) \in X^{\mathbb{R}}_{A,0}$ with $r \geq l_1(x)$

$$(h(x), r) \underset{1,0}{\sim} (\sigma_B^{l_1(x)}(h(x)), r - l_1(x))$$

and

$$(h(\sigma_A(x)), r - c_1(x)) \underset{1,0}{\sim} (\sigma_B^{k_1(x)}(h(\sigma_A(x))), r - c_1(x) - k_1(x)).$$

Since $\sigma_B^{l_1(x)}(h(x)) = \sigma_B^{k_1(x)}(h(\sigma_A(x)))$ and $r - l_1(x) = r - c_1(x) - k_1(x)$, we have

$$(h(x), r) \underset{1,0}{\sim} (h(\sigma_A(x)), r - c_1(x)) \quad \text{in } S^1_B$$

so that the map $\Phi_1 : X^{\mathbb{R}}_{A,0} \to X^{\mathbb{R}}_{B,0}$ induces a continuous map $S_A^{l_1,k_1} \to S^1_B$ which is still denoted by Φ_1. It is clear to see that the equalities $\Phi_1 \circ \phi_{A,t} = \phi_{B,t} \circ \Phi_1$ for $t \in \mathbb{R}_+$ and $\Phi_1 \circ b_{1,A} = s_B \circ h$ hold. We similarly have a continuous map $\Phi_2 : S_B^{l_2,k_2} \to S^1_A$ defined by $\Phi_2([y, s]) = [h^{-1}(y), s]$ satisfying the equalities $\Phi_2 \circ \phi_{B,t} = \phi_{A,t} \circ \Phi_2$ for $t \in \mathbb{R}_+$ and $\Phi_2 \circ b_{2,B} = s_A \circ h^{-1}$ to prove that (X_A, σ_A) and (X_B, σ_B) are one-sided flow equivalent. □

Conversely we have

Proposition 3.3. *If* (X_A, σ_A) *and* (X_B, σ_B) *are one-sided flow equivalent, then they are continuously orbit equivalent.*

Proof. Suppose that (X_A, σ_A) and (X_B, σ_B) are one-sided flow equivalent. Take suspension triplets (l_1, k_1, b_1) for (X_A, σ_A) and (l_2, k_2, b_2) for (X_B, σ_B), a homeomorphism $h : X_A \to X_B$, and continuous maps $\Phi_1 : S^{l_1,k_1}_{A,b_1} \to S^1_B$, $\Phi_2 : S^{l_2,k_2}_{B,b_2} \to S^1_A$ satisfying the equalities (3.1) and (3.2). For $(x, r) \in X^{\mathbb{R}}_{A,b_1}$ with $r \geq l_1(x)$, we have $[x, r] = [\sigma_A(x), r - c_1(x)]$ in S^{l_1,k_1}_{A,b_1}. It follows that

$$[x, l_1(x)] = [x, (l_1(x) - b_1(x)) + b_1(x)],$$
$$[\sigma_A(x), l_1(x) - c_1(x)] = [\sigma_A(x), (k_1(x) - b_1(\sigma_A(x))) + b_1(\sigma_A(x))].$$

As $l_1(x) - b_1(x) \geq 0$ and $k_1(x) - b_1(\sigma_A(x)) \geq 0$, we have

$$\begin{aligned}
[x, l_1(x)] &= \phi_{A,l_1(x)-b_1(x)}([x, b_1(x)]) = \phi_{A,l_1(x)-b_1(x)}(b_{1,A}(x)), \\
[\sigma_A(x), l_1(x) - c_1(x)] &= \phi_{A,k_1(x)-b_1(\sigma_A(x))}([\sigma_A(x), b_1(\sigma_A(x))]) \\
&= \phi_{A,k_1(x)-b_1(\sigma_A(x))}(b_{1,A}(\sigma_A(x))).
\end{aligned}$$

Hence we have $\phi_{A,l_1(x)-b_1(x)}(b_{1,A}(x)) = \phi_{A,k_1(x)-b_1(\sigma_A(x))}(b_{1,A}(\sigma_A(x)))$ so that

$$\Phi_1(\phi_{A,l_1(x)-b_1(x)}(b_{1,A}(x))) = \Phi_1(\phi_{A,k_1(x)-b_1(\sigma_A(x))}(b_{1,A}(\sigma_A(x)))).$$

By (3.1) and (3.2), we have

$$\phi_{B,l_1(x)-b_1(x)}(s_B(h(x))) = \phi_{B,k_1(x)-b_1(\sigma_A(x))}(s_B(h(\sigma_A(x)))).$$

It follows that

$$[h(x), l_1(x) - b_1(x)] = [h(\sigma_A(x)), k_1(x) - b_1(\sigma_A(x))] \quad \text{in } S_B^1.$$

Put $l_1'(x) = l_1(x) - b_1(x)$ and $k_1'(x) = k_1(x) - b_1(\sigma_A(x))$. They are valued in nonnegative integers. Since

$$[h(x), l_1(x) - b_1(x)] = [\sigma_B^{l_1'(x)}(h(x)), 0] = s_B(\sigma_B^{l_1'(x)}(h(x))),$$

$$[h(\sigma_A(x)), k_1(x) - b_1(\sigma_A(x))] = [\sigma_B^{k_1'(x)}(h(\sigma_A(x))), 0] = s_B(\sigma_B^{k_1'(x)}(h(\sigma_A(x)))),$$

and the standard base map $s_B : X_B \to S_B^1$ is injective, we have $\sigma_B^{l_1'(x)}(h(x)) = \sigma_B^{k_1'(x)}(h(\sigma_A(x)))$ for $x \in X_A$. We similarly have continuous maps $l_2', k_2' \in C(X_B, \mathbb{Z}_+)$ such that $\sigma_A^{l_2'(y)}(h^{-1}(y)) = \sigma_A^{k_2'(y)}(h^{-1}(\sigma_B(y)))$ for $y \in X_B$. Consequently (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent. □

Therefore we may conclude the following theorem.

Theorem 3.4. *Assume that matrices A and B are irreducible and not permutation matrices. One-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are one-sided flow equivalent if and only if they are continuously orbit equivalent.*

It is well known that two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if $\mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N$ is isomorphic to $\mathbb{Z}^M/(\mathrm{id} - B)\mathbb{Z}^M$ as abelian groups and $\det(\mathrm{id} - A) = \det(\mathrm{id} - B)$, where N, M are the sizes of the matrices A, B respectively ([3], [5], [12]). By using [10, Theorem 3.6], we see the following characterization of one-sided flow equivalence which is a corollary of the above theorem.

Corollary 3.5. *Assume that matrices A and B are irreducible and not permutation matrices. One-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are one-sided flow equivalent if and only if there exists an isomorphism $\Phi : \mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N \to \mathbb{Z}^M/(\mathrm{id} - B)\mathbb{Z}^M$ of abelian groups such that $\Phi([u_A]) = [u_B]$ and $\det(\mathrm{id} - A) = \det(\mathrm{id} - B)$, where $[u_A]$ (resp. $[u_B]$) is the class of the vector $u_A = [1, \dots, 1]$ in $\mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N$ (resp. $u_B = [1, \dots, 1]$ in $\mathbb{Z}^M/(\mathrm{id} - B)\mathbb{Z}^M$).*

The statement of the following proposition is more general than that of Proposition 3.2.

Proposition 3.6. *Suppose that (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent via a homeomorphism $h : X_A \to X_B$. For $f \in C(X_B, \mathbb{Z}_+)$ and $g \in C(X_A, \mathbb{Z}_+)$ such that $[f] \in H_+^B$ and $[g] \in H_+^A$ are order units of (H^B, H_+^B) and of (H^A, H_+^A) respectively, there exist $l_f, k_f \in C(X_A, \mathbb{Z}_+)$ and $l_g, k_g \in C(X_B, \mathbb{Z}_+)$ such that $(l_f, k_f, 0)$ and $(l_g, k_g, 0)$ are suspension triplets for (X_A, σ_A) and for (X_B, σ_B) respectively and continuous maps $\Phi_f : S_A^{l_f, k_f} \to S_B^f$ and $\Phi_g : S_B^{l_g, k_g} \to S_A^g$ such that*

$$\Phi_f \circ \phi_{A,t} = \phi_{B,t} \circ \Phi_f \quad \text{for } t \in \mathbb{R}_+, \qquad \Phi_f \circ s_A = s_B \circ h,$$

$$\Phi_g \circ \phi_{B,t} = \phi_{A,t} \circ \Phi_g \quad \text{for } t \in \mathbb{R}_+, \qquad \Phi_g \circ s_B = s_A \circ h^{-1}.$$

Proof. Put $l_f(x) = f^{l_1(x)}(h(x)), k_f(x) = f^{k_1(x)}(h(\sigma_A(x)))$ for $x \in X_A$ so that $\Psi_h(f)(x) = l_f(x) - k_f(x)$. Since $[f] \in H^B_+$ is an order unit and $\Psi_h : H^B \to H^A$ preserves the orders, the class $[\Psi_h(f)]$ gives rise to an order unit of (H^A, H^A_+). As $l_f - k_f = \Psi_h(f)$, we see that $(l_f, k_f, 0)$ is a suspension triplet for (X_A, σ_A) so that we may consider the one-sided suspensions $(S_A^{l_f,k_f}, \phi_A)$ and (S_B^f, ϕ_B). We define the map $\Phi_f : X^{\mathbb{R}}_{A,0} \to X^{\mathbb{R}}_{B,0}$ by $\Phi_f((x,r)) = (h(x), r)$. For $r \geq l_f(x)$, we have

$$(h(x), r) \underset{f,0}{\sim} (\sigma_B(h(x)), r - f(h(x))) \underset{f,0}{\sim} \cdots \underset{f,0}{\sim} (\sigma_B^{l_1(x)}(h(x)), r - f^{l_1(x)}(h(x)))$$

and similarly

$$(h(\sigma_A(x)), r - \Psi_h(f)(x)) \underset{f,0}{\sim} (\sigma_B^{k_1(x)}(h(\sigma_A(x))), r - \Psi_h(f)(x) - f^{k_1(x)}(h(\sigma_A(x)))).$$

As the equalities $\sigma_B^{l_1(x)}(h(x)) = \sigma_B^{k_1(x)}(h(\sigma_A(x)))$ and $f^{l_1(x)}(h(x)) = \Psi_h(f)(x) + f^{k_1(x)}(h(\sigma_A(x)))$ hold, we have

$$\Phi_f((x,r)) \underset{f,0}{\sim} \Phi_f((\sigma_A(x), r - \Psi_h(f)(x))).$$

Hence $\Phi_f : X^{\mathbb{R}}_{A,0} \to X^{\mathbb{R}}_{B,0}$ induces a continuous map $S_A^{l_f,k_f} \to S_B^f$ which is still denoted by Φ_f. It is easy to see that the map satisfies the desired properties. We similarly have a map $\Phi_g : S_B^{l_g,k_g} \to S_A^g$ satisfying the desired properties. □

We give an example of one-sided flow equivalent topological Markov shifts. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The one-sided topological Markov shift (X_A, σ_A) is called the full 2-shift, and the other one (X_B, σ_B) is called the golden mean shift whose shift space X_B consists of sequences $(y_n)_{n \in \mathbb{N}}$ of $1, 2$ such that the word $(2, 2)$ is forbidden (cf. [7]). They are continuously orbit equivalent as in [8, Section 5] through the homeomorphism $h : X_A \longrightarrow X_B$ defined by substituting the word $(2, 1)$ for the symbol 2 from the leftmost of a sequence $(x_n)_{n \in \mathbb{N}}$ in order, so that they are one-sided flow equivalent. Put for $i = 1, 2$

$$U_{A,i} = \{(x_n)_{n \in \mathbb{N}} \in X_A \mid x_1 = i\}, \qquad U_{B,i} = \{(y_n)_{n \in \mathbb{N}} \in X_B \mid y_1 = i\}.$$

By setting

$$\begin{cases} k_1(x) = 0,\ l_1(x) = 1 & \text{for } x \in U_{A,1}, \\ k_1(x) = 0,\ l_1(x) = 2 & \text{for } x \in U_{A,2}, \end{cases} \qquad \begin{cases} k_2(y) = 0,\ l_2(y) = 1 & \text{for } y \in U_{B,1}, \\ k_2(y) = 1,\ l_2(y) = 1 & \text{for } y \in U_{B,2}, \end{cases}$$

the continuous functions $k_1, l_1 : X_A \to \mathbb{Z}_+$ and $k_2, l_2 : X_B \to \mathbb{Z}_+$ satisfy (1.1) and (1.2), respectively. By Proposition 3.2, the suspension flows $(S_A^{l_1}, \phi_A)$ and (S_B^1, ϕ_B), and similarly $(S_B^{l_2,k_2}, \phi_B)$ and (S_A^1, ϕ_A) give rise to flow equivalence between (X_A, σ_A) and (X_B, σ_B).

4. One-sided suspensions and zeta functions

The zeta function $\zeta_\phi(s)$ of a flow $\phi_t : S \to S$ on a compact metric space with at most countably many closed orbits is defined by the formula (1.3). In the first half of this section, we will show that the zeta function $\zeta_{\phi_A}(s)$ of the flow ϕ_A of the one-sided suspension $S_{A,b}^{l,k}$ of (X_A, σ_A) is given by the dynamical zeta function $\zeta_{A,c}(s)$ with potential function $c = l - k$. We first provide a lemma.

Lemma 4.1. *Let $(S^{l,k}_{A,b}, \phi_A)$ be a one-sided suspension of (X_A, σ_A). Then there exists a bijective correspondence between primitive periodic orbits $\tau \in P_{orb}(S^{l,k}_{A,b}, \phi_A)$ and periodic orbits $\gamma_\tau \in P_{orb}(X_A)$ of (X_A, σ_A) such that*

$$\ell(\tau) = \beta_{\gamma_\tau}(c)$$

where $\ell(\tau)$ is the primitive length of the periodic orbit τ defined by $\ell(\tau) = \min\{t \in \mathbb{R}_+ \mid \phi_{A,t}(u) = u\}$ for any point $u \in \tau$ and $\beta_{\gamma_\tau}(c) = \sum_{i=0}^{p-1} c(\sigma_A^i(x))$ for $c = l - k$ and $\gamma_\tau = \{x, \sigma_A(x), \dots, \sigma_A^{p-1}(x)\}$.

Proof. For an arbitrary point (x, r) in a primitive periodic orbit $\tau \in P_{orb}(S^{l,k}_{A,b}, \phi_A)$, one sees that $(x, r) \underset{l,k}{\sim} \phi_{A,\ell(\tau)}(x, r)$. Since $\phi_{A,\ell(\tau)}(x, r) = (x, r + \ell(\tau))$, there exists a number $p \in \mathbb{Z}_+$ such that

$$\ell(\tau) = \{l(x) - r\} + \{l(\sigma_A(x)) - k(x)\} + \cdots + \{l(\sigma_A^{p-1}(x)) - k(\sigma_A^{p-2}(x))\} + \{r - k(\sigma_A^{p-1}(x))\}$$

and $\sigma_A^p(x) = x$, so that $\ell(\tau) = \sum_{i=0}^{p-1} c(\sigma_A^i(x))$.

Conversely, for a periodic orbit $\gamma = \{x, \sigma_A(x), \dots, \sigma_A^{p-1}(x)\} \in P_{orb}(X_A)$, we have $(x, r) \underset{l,k}{\sim} \phi_{A,\beta_\gamma(c)}(x, r)$ for any $r \in \mathbb{R}$ with $b(x) \le r \le l(x)$. Hence $\tau_\gamma = \{\phi_{A,t}(x, r) \mid 0 \le t \le \beta_\gamma(c)\}$ gives a primitive periodic orbit of $(S^{l,k}_{A,b}, \phi_A)$ with length $\beta_\gamma(c)$. $\square$

Let us denote by $\mathrm{Per}_n(X_A)$ the set $\{x \in X_A \mid \sigma_A^n(x) = x\}$ of n-periodic points. For a Hölder continuous function f on X_A, the dynamical zeta function $\zeta_{A,f}(s)$ is defined by

(4.1)
$$\zeta_{A,f}(s) = \exp\{\sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in \mathrm{Per}_n(X_A)} \exp(-s \sum_{k=0}^{n-1} f(\sigma_A^k(x)))\} \quad \text{(see [17], [19], [11], etc.)}$$

where the right hand side makes sense for a complex number $s \in \mathbb{C}$ with $\mathrm{Re}(s) > h$ for some positive constant $h > 0$. The function $\zeta_{A,f}(s)$ is called the dynamical zeta function on X_A with potential function f. We may especially define the zeta function $\zeta_{A,c}(s)$ for an integer valued continuous function c on X_A whose class $[c]$ in H^A is an order unit of (H^A, H^A_+). By a routine argument as in [11, p.100] with Lemma 4.1, we have

Proposition 4.2. *Assume that a matrix A is irreducible and not any permutation matrix. The zeta function* (1.3) *of the flow of the one-sided suspension $(S^{l,k}_{A,b}, \phi_A)$ of (X_A, σ_A) is given by the dynamical zeta function $\zeta_{A,c}(s)$ with potential function $c = l - k$ such as*

$$\zeta_{A,c}(s) = \exp\{\sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in \mathrm{Per}_n(X_A)} \exp(-s \sum_{k=0}^{n-1} c(\sigma_A^k(x)))\}. \tag{4.2}$$

Remark 4.3. The class $[c]$ of the function c in H^A is an order unit of the ordered cohomology group (H^A, H^A_+), so that there exists $N_c \in \mathbb{N}$ such that $N_c[c] - 1 \in H^A_+$. Hence $N_c c - 1 = f + g \circ \sigma_A - g$ for some $f \in C(X_A, \mathbb{Z}_+)$ and $g \in C(X_A, \mathbb{Z})$. For a periodic point $x \in \mathrm{Per}_n(X_A)$ with $\sigma_A^n(x) = x$, we have

$$\sum_{i=0}^{n-1} \{N_c c(\sigma_A^i(x)) - 1(\sigma_A^i(x))\} = \sum_{i=0}^{n-1} f(\sigma_A^i(x)) \ge 0$$

and hence $\sum_{i=0}^{n-1} c(\sigma_A^i(x)) \geq \frac{n}{N_c}$. The ordinary zeta function $\zeta_A(t)$ of (X_A, σ_A) is written as $\zeta_A(t) = \exp\{\sum_{n=1}^{\infty} \frac{1}{n}|\mathrm{Per}_n(X_A)|t^n\}$ where $|\mathrm{Per}_n(X_A)|$ denotes the cardinality of $\mathrm{Per}_n(X_A)$, so that $\zeta_A(t) = \zeta_{A,1}(s)$ for $c = 1$ where $t = e^{-s}$. As the function $\zeta_A(t)$ is analytic in $t \in \mathbb{C}$ with $|t| < \frac{1}{r_A}$, where r_A is the maximum eigenvalue of the matrix A, we see that so is $\zeta_{A,1}(s)$ in $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \log r_A$. Similarly, for the function $c = l - k$, we have

$$|\sum_{x \in \mathrm{Per}_n(X_A)} \exp(-s \sum_{k=0}^{n-1} c(\sigma_A^k(x)))| \leq \sum_{x \in \mathrm{Per}_n(X_A)} \{\exp(-\mathrm{Re}(s))\}^{\frac{n}{N_c}}$$

so that $\zeta_{A,c}(s)$ is analytic at least in $s \in \mathbb{C}$ with $\mathrm{Re}(s) > N_c \log r_A$. We actually know that $\zeta_{A,c}(s)$ is analytic in the half plane $\mathrm{Re}(s) > h_{top}(S_{A,b}^{l,k}, \phi_A)$ the topological entropy of the flow of the suspension $(S_{A,b}^{l,k}, \phi_A)$ as seen in [1, Theorem 2.7] (cf. [6], [14], [18]).

In the second half of this section, we will study some relationships between zeta functions of one-sided flow equivalent topological Markov shifts. Suppose that (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent via a homeomorphism $h : X_A \to X_B$ with continuous functions $k_1, l_1 : X_A \to \mathbb{Z}_+$ and $k_2, l_2 : X_B \to \mathbb{Z}_+$ satisfying (1.1) and (1.2), respectively. The functions $c_1 = l_1 - k_1 \in C(X_A, \mathbb{Z})$ and $c_2 = l_2 - k_2 \in C(X_B, \mathbb{Z})$ satisfy $\Psi_h(1) = c_1$ and $\Psi_{h^{-1}}(1) = c_2$. In [10], it has been shown that the homeomorphism $h : X_A \to X_B$ induces a bijective correspondence $\xi_h : P_{orb}(X_A) \to P_{orb}(X_B)$ between their periodic orbits such that the functions c_1, c_2 measure the difference of the length of periods between $\gamma \in P_{orb}(X_A)$ and $\xi_h(\gamma) \in P_{orb}(X_B)$. As a result, the ordinary zeta functions $\zeta_A(t), \zeta_B(t)$ are written in terms of dynamical zeta functions in the following way.

Proposition 4.4 ([10]). *$\zeta_A(t) = \zeta_{B,c_2}(s)$ and $\zeta_B(t) = \zeta_{A,c_1}(s)$ for $t = e^{-s}$.*

The above formulae imply the formulae

$$\zeta_{A,1}(s) = \zeta_{B,c_2}(s), \qquad \zeta_{B,1}(s) = \zeta_{A,c_1}(s). \tag{4.3}$$

Assume that (X_A, σ_A) and (X_B, σ_B) are one-sided flow equivalent via a homeomorphism $h : X_A \to X_B$. They are continuously orbit equivalent, so that the maps $\Psi_h : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z})$ and $\Psi_{h^{-1}} : C(X_A, \mathbb{Z}) \to C(X_B, \mathbb{Z})$ are defined by (1.5). They are independent of the choice of the functions $k_1, l_1 : X_A \to \mathbb{Z}_+$ and $k_2, l_2 : X_B \to \mathbb{Z}_+$ satisfying (1.1) and (1.2), respectively ([10, Lemma 4.2]). We provide a lemma.

Lemma 4.5. *For $m \in \mathbb{Z}_+$ and $f \in C(X_B, \mathbb{Z})$, $g \in C(X_A, \mathbb{Z})$, we have*

(i) $\Psi_h(f)^m(x) = f^{l_1^m(x)}(h(x)) - f^{k_1^m(x)}(h(\sigma_A^m(x)))$ *for $x \in X_A$, so that*

$$g^m(x) = \Psi_{h^{-1}}(g)^{l_1^m(x)}(h(x)) - \Psi_{h^{-1}}(g)^{k_1^m(x)}(h(\sigma_A^m(x))).$$

(ii) $\Psi_{h^{-1}}(g)^m(y) = g^{l_2^m(y)}(h^{-1}(y)) - g^{k_2^m(y)}(h^{-1}(\sigma_B^m(y)))$ *for $y \in X_B$, so that*

$$f^m(y) = \Psi_h(f)^{l_2^m(y)}(h^{-1}(y)) - \Psi_h(f)^{k_2^m(y)}(h^{-1}(\sigma_B^m(y))).$$

Proof. (i) As in [10, Lemma 4.3], the identity

$$\begin{aligned}
&\sum_{i=0}^{m-1}\{ \sum_{i'=0}^{l_1(\sigma_A^i(x))-1} f(\sigma_B^{i'}(h(\sigma_A^i(x)))) - \sum_{j'=0}^{k_1(\sigma_A^i(x))-1} f(\sigma_B^{j'}(h(\sigma_A^{i+1}(x))))\} \\
= &\sum_{i'=0}^{l_1^m(x)-1} f(\sigma_B^{i'}(h(x))) - \sum_{j'=0}^{k_1^m(x)-1} f(\sigma_B^{j'}(h(\sigma_A^m(x))))
\end{aligned}$$

holds so that we see

$$\sum_{i=0}^{m-1} \Psi_h(f)(\sigma_A^i(x)) = f^{l_1^m(x)}(h(x)) - f^{k_1^m(x)}(h(\sigma_A^m(x))).$$

As $\Psi_{h^{-1}} = (\Psi_h)^{-1}$([10, Proposition 4.5]), the desired identities hold. (ii) is similarly shown. □

We generalize Proposition 4.4 such as in the following theorem.

Theorem 4.6. *Assume that matrices A and B are irreducible and not permutation matrices. Suppose that (X_A, σ_A) and (X_B, σ_B) are one-sided flow equivalent via a homeomorphism $h : X_A \to X_B$. For $f \in C(X_B, \mathbb{Z}), g \in C(X_A, \mathbb{Z})$ such that their classes $[f] \in H^B, [g] \in H^A$ are order units of $(H^B, H^B_+), (H^A, H^A_+)$, respectively, the following formulae hold:*

$$\zeta_{A,g}(s) = \zeta_{B,\Psi_{h^{-1}}(g)}(s), \qquad \zeta_{B,f}(s) = \zeta_{A,\Psi_h(f)}(s).$$

Proof. We may assume that (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent via a homeomorphism $h : X_A \to X_B$. For $f \in C(X_B, \mathbb{Z})$ such that the class $[f]$ is an order unit of (H^A, H^A_+), we will prove the equality $\zeta_{B,f}(s) = \zeta_{A,\Psi_h(f)}(s)$. A routine argument as in [11, p. 100] shows

$$\zeta_{A,\Psi_h(f)}(s) = \prod_{\gamma \in P_{orb}(X_A)} (1 - t^{\beta_\gamma(\Psi_h(f))})^{-1} \quad \text{where} \quad t = e^{-s}$$

and $\beta_\gamma(\Psi_h(f)) = \sum_{i=0}^{p-1} \Psi_h(f)(\sigma_A^i(x))$ for a periodic orbit

$$\gamma = \{x, \sigma_A(x), \dots, \sigma_A^{p-1}(x)\} \in P_{orb}(X_A).$$

We see the following formula by Lemma 4.5:

$$\sum_{i=0}^{p-1} \Psi_h(f)(\sigma_A^i(x)) = f^{l_1^p(x)}(h(x)) - f^{k_1^p(x)}(h(\sigma_A^p(x))), \quad x \in X_A.$$

As $\gamma = \{x, \sigma_A(x), \dots, \sigma_A^{p-1}(x)\} \in P_{orb}(X_A)$ and $\sigma_A^p(x) = x$, we have

$$\sum_{i=0}^{p-1} \Psi_h(f)(\sigma_A^i(x)) = f^{l_1^p(x)}(h(x)) - f^{k_1^p(x)}(h(x)). \tag{4.4}$$

Since we may identify the periodic orbits $P_{orb}(X_A)$ of the one-sided topological Markov shift (X_A, σ_A) with the periodic orbits $P_{orb}(\bar{X}_A)$ of the two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$, an argument in [10, Section 6] shows that there exists a bijective correspondence $\xi_h : P_{orb}(X_A) \to P_{orb}(X_B)$. By [10, Lemma 6.5], we see that $\xi_h(\gamma)$ has its period $l_1^p(x) - k_1^p(x)$ and hence $\xi_h(\gamma) = \{\sigma_B^i(h(x)) \mid k_1^p(x) \le i \le l_1^p(x) - 1\}$ so that

$$\sum_{i=k_1^p(x)}^{l_1^p(x)-1} f(\sigma_B^i(h(x))) = \beta_{\xi_h(\gamma)}(f).$$

By using (4.4), we have

$$\beta_\gamma(\Psi_h(f)) = \sum_{i=0}^{p-1} \Psi_h(f)(\sigma_A^i(x)) = \sum_{i=k_1^p(x)}^{l_1^p(x)-1} f(\sigma_B^i(h(x))) = \beta_{\xi_h(\gamma)}(f).$$

Since $\xi_h : P_{orb}(X_A) \to P_{orb}(X_B)$ is bijective, one sees that

$$\zeta_{A,\Psi_h(f)}(s) = \prod_{\eta \in P_{orb}(X_B)} (1 - t^{\beta_\eta(f)})^{-1} = \zeta_{B,f}(s).$$

The other equality $\zeta_{B,\Psi_{h^{-1}}(g)}(s) = \zeta_{A,g}(s)$ is similarly shown. □

Corollary 4.7. *Assume that a matrix A is irreducible and not any permutation matrix. The set $Z(X_A, \sigma_A)$ of dynamical zeta functions of (X_A, σ_A) whose potential functions are order units of the ordered cohomology group (H^A, H_+^A) is invariant under one-sided flow equivalence.*

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