ON FLOW EQUIVALENCE OF ONE-SIDED TOPOLOGICAL MARKOV SHIFTS

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ABSTRACT. We introduce notions of suspension and flow equivalence on one-sided topological Markov shifts, which we call one-sided suspension and one-sided flow equivalence, respectively. We prove that one-sided flow equivalence is equivalent to continuous orbit equivalence on one-sided topological Markov shifts. We also show that the zeta function of the flow on a one-sided suspension is a dynamical zeta function with some potential function and that the set of certain dynamical zeta functions is invariant under one-sided flow equivalence of topological Markov shifts.

1. INTRODUCTION

Flow equivalence relation on two-sided topological Markov shifts is one of the most important and interesting equivalence relations on symbolic dynamical systems. It has a close relationship to classifications of not only continuous time dynamical systems but also associated $C^*$-algebras. For an irreducible square matrix $A = [A(i, j)]_{i,j=1}^N$ with its entries in $\{0, 1\}$, the two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$ is defined as a compact Hausdorff space $\bar{X}_A$ consisting of bi-infinite sequences $(\bar{x}_n)_{n \in \mathbb{Z}}$ of $\bar{x}_n \in \{1, 2, \ldots, N\}$ such that $A(\bar{x}_n, \bar{x}_{n+1}) = 1, n \in \mathbb{Z}$ with shift homeomorphism $\bar{\sigma}_A$ defined by $\bar{\sigma}_A((\bar{x}_n)_{n \in \mathbb{Z}}) = (\bar{x}_{n+1})_{n \in \mathbb{Z}}$. For a positive continuous function $g$ on $\bar{X}_A$, let us denote by $S_A^g$ the compact Hausdorff space obtained from $\{(\bar{x}, r) \in \bar{X}_A \times \mathbb{R} \mid \bar{x} \in \bar{X}_A, 0 \leq r \leq g(\bar{x})\}$ by identifying $((\bar{x}, g(\bar{x})))$ with $(\bar{\sigma}_A(\bar{x}), 0)$ for each $\bar{x} \in \bar{X}_A$. Let $\bar{\phi}_{A,t}, t \in \mathbb{R}$ be the flow on $\bar{S}_A^g$ defined by $\bar{\phi}_{A,t}([(\bar{x}, r)]) = [(\bar{x}, r + t)]$ for $[(\bar{x}, r)] \in \bar{S}_A^g$. The dynamical system $(\bar{S}_A^g, \bar{\phi}_A)$ is called the suspension of $(\bar{X}_A, \bar{\sigma}_A)$ by a ceiling function $g$. Two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are said to be flow equivalent if there exists a positive continuous function $g$ on $\bar{X}_A$ such that $(\bar{S}_A^g, \bar{\phi}_A)$ is topologically conjugate to $(\bar{S}_B^g, \bar{\phi}_B)$. It is well known that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if $\mathbb{Z}^N/(id - A)\mathbb{Z}^N$ is isomorphic to $\mathbb{Z}^M/(id - B)\mathbb{Z}^M$ as abelian groups and $\det(id - A) = \det(id - B)$, where $N, M$ are the sizes of the matrices $A, B$, respectively ([3], [5], [12]). Let us denote by $\mathcal{K}$ the $C^*$-algebra of compact operators on the separable infinite dimensional Hilbert space $\ell^2(\mathbb{N})$ and $\mathcal{C}$ its maximal abelian $C^*$-subalgebra consisting of diagonal elements on $\ell^2(\mathbb{N})$. Let us denote by $O_A$ the Cuntz–Krieger algebra and by $D_A$ its canonical maximal abelian $C^*$-subalgebra. Since the group $\mathbb{Z}^N/(id - A)\mathbb{Z}^N$ is a complete invariant of the isomorphism class of the tensor product $C^*$-algebra $O_A \otimes \mathcal{K}$ ([15]), the $C^*$-algebra $O_A \otimes \mathcal{K}$ with $\det(id - A)$ is a complete invariant for flow equivalence of the two-sided topological
Markov shift \( (\bar{X}_A, \bar{\sigma}_A) \). It has been recently shown in [9] that the isomorphism class of the pair \((O_A \otimes K, D_A \otimes C)\) is a complete invariant for flow equivalence class of \((\bar{X}_A, \bar{\sigma}_A)\).

One-sided topological Markov shifts \((X_A, \sigma_A)\) are also an important and interesting class of dynamical systems. The space \(X_A\) is defined as a compact Hausdorff space consisting of right infinite sequences \((x_n)_{n \in \mathbb{N}}\) of \(x_n \in \{1, 2, \ldots, N\}\) such that \(A(x_n, x_{n+1}) = 1, n \in \mathbb{N}\) with continuous map \(\sigma_A\) defined by \(\sigma_A((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}\). In [8], the author has introduced a notion of continuous orbit equivalence between one-sided topological Markov shifts. One-sided topological Markov shifts \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are said to be continuously orbit equivalent if there exists a homeomorphism \(h : X_A \to X_B\) and continuous functions \(k_1, l_1 : X_A \to \mathbb{Z}_+\) and \(k_2, l_2 : X_B \to \mathbb{Z}_+\) such that
\[
\begin{align*}
\sigma_B^{k_1(x)}(h(\sigma_A(x))) &= \sigma_B^{l_1(x)}(h(x)) \quad \text{for} \quad x \in X_A, \\
\sigma_A^{k_2(y)}(h^{-1}(\sigma_B(y))) &= \sigma_A^{l_2(y)}(h^{-1}(y)) \quad \text{for} \quad y \in X_B.
\end{align*}
\]

In [9], it has been proved that the isomorphism class of \(O_A\) with \(\det(id - A)\) is a complete invariant for continuous orbit equivalence of the one-sided topological Markov shift \((X_A, \sigma_A)\). We have already known in [8] that the isomorphism class of the pair \((O_A, D_A)\) is a complete invariant for continuous orbit equivalence of \((X_A, \sigma_A)\). Hence we may regard continuous orbit equivalence of one-sided topological Markov shifts as a one-sided counterpart of flow equivalence of two-sided topological Markov shifts.

In this paper, we will introduce a notion of flow equivalence on one-sided topological Markov shifts \((X_A, \sigma_A)\). We will first introduce one-sided suspension \(S^{l,k}_{A,b}\) with a flow \(\phi_A\) associated to three real valued continuous functions \(l, k, b \in C(X_A, \mathbb{R})\) on \(X_A\) for a one-sided topological Markov shift \((X_A, \sigma_A)\). The space \(S^{l,k}_{A,b}\) is determined by a base map \(b : X_A \to \mathbb{R}\) and a ceiling function \(l : X_A \to \mathbb{R}_+\) by identifying \((x, r)\) with \((\sigma_A(x), r - (l - k))\) for \(r \geq l(x)\). By using the one-sided suspension, we will define one-sided flow equivalence on one-sided topological Markov shifts in Definition 3.1.

As a main result of the paper, we will prove the following theorem.

**Theorem 1.1 (Theorem 3.4).** Assume that matrices \(A\) and \(B\) are irreducible and not permutation matrices. One-sided topological Markov shifts \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are one-sided flow equivalent if and only if they are continuously orbit equivalent.

By using [10] Theorem 3.6, we see the following characterization of one-sided flow equivalence which is a corollary of the above theorem.

**Corollary 1.2 (Corollary 3.5).** Assume that matrices \(A\) and \(B\) are irreducible and not permutation matrices. One-sided topological Markov shifts \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are one-sided flow equivalent if and only if there exists an isomorphism \(\Phi : \mathbb{Z}^N / (id - A)\mathbb{Z}^N \to \mathbb{Z}^M / (id - B)\mathbb{Z}^M\) of abelian groups such that \(\Phi([u_A]) = [u_B]\) and \(\det(id - A) = \det(id - B)\), where \([u_A]\) (resp. \([u_B]\)) is the class of the vector \(u_A = [1, \ldots, 1]\) in \(\mathbb{Z}^N / (id - A)\mathbb{Z}^N\) (resp. \(u_B = [1, \ldots, 1]\) in \(\mathbb{Z}^M / (id - B)\mathbb{Z}^M\)).

The zeta function \(\zeta_\phi(s)\) of a flow \(\phi_t : S \to S\) on a compact metric space \(S\) with at most countably many closed orbits is defined by
\[
(1.3) \quad \zeta_\phi(s) = \prod_{\tau \in P_{orb}(S, \phi)} (1 - e^{-s\ell(\tau)})^{-1} \quad \text{(see [11, 17, 19], etc.)}
\]
where \( P_{\operatorname{orb}}(S, \phi) \) denotes the set of primitive periodic orbits of the flow \( \phi_t : S \to S \) and \( \ell(\tau) \) is the primitive length of the closed orbit defined by \( \ell(\tau) = \min \{ t \in \mathbb{R}_+ \mid \phi_t(u) = u \} \) for any point \( u \in \tau \). For a one-sided suspension \( S_{A,b}^{l,k} \) of \((X_A, \sigma_A)\), there exists a bijective correspondence between primitive periodic orbits \( \tau \in P_{\operatorname{orb}}(S_{A,b}^{l,k}, \phi_A) \) and periodic orbits \( \gamma_\tau \in P_{\operatorname{orb}}(X_A) \) of \((X_A, \sigma_A)\) such that the length \( \ell(\tau) \) of the orbit \( \tau \) is \( \sum_{i=0}^{p-1} c(\sigma_A^i(x)) \) for \( c = l - k \) and \( \gamma_\tau = \{ x, \sigma_A(x), \ldots, \sigma_A^{p-1}(x) \} \). Therefore we have

**Proposition 1.3 (Proposition 4.2).** Assume that a matrix \( A \) is irreducible and not any permutation matrix. The zeta function \( \zeta_{\phi_A}(s) \) of the flow \( \phi_A \) of the one-sided suspension \( S_{A,b}^{l,k} \) is given by the dynamical zeta function \( \zeta_{A,c}(s) \) with potential function \( c = l - k \) such that

\[
\zeta_{A,c}(s) = \exp\left( \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in \operatorname{Per}_n(X_A)} \exp(-s \sum_{i=0}^{n-1} c(\sigma_A^i(x))) \right).
\]

In [2], Boyle–Handelman have proved that the set of zeta functions of homeomorphisms flow equivalent to the two-sided topological Markov shift \((\bar{X}_A, \bar{\sigma}_A)\) is a complete invariant for flow equivalence of \((\bar{X}_A, \bar{\sigma}_A)\). If \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are continuously orbit equivalent via a homeomorphism \( h : X_A \to X_B \) with continuous functions \( k_1, l_1 : X_A \to \mathbb{Z}_+ \) and \( k_2, l_2 : X_B \to \mathbb{Z}_+ \) satisfying (1.1) and (1.2), we may define homomorphisms \( \Psi_h : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z}) \) by

\[
\Psi_h(f)(x) = \sum_{j=0}^{l_2(x)-1} f(\sigma_B^j(h(x))) - \sum_{i=0}^{l_1(x)-1} f(\sigma_B^i(h(\sigma_A(x))))
\]

for \( f \in C(X_B, \mathbb{Z}) \) and similarly \( \Psi_{h^{-1}} : C(X_A, \mathbb{Z}) \to C(X_B, \mathbb{Z}) \) for \( h^{-1} : X_B \to X_A \). In [4], it has been proved that \( \Psi_h : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z}) \) induces an isomorphism of their ordered cohomology groups \((H^B, H_B)\) and \((H^A, H_A)\). If \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are one-sided flow equivalent, they are continuously orbit equivalent, so that we may define the map \( \Psi_h : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z}) \) as above. Inspired by [2], we will show the following.

**Theorem 1.4 (Theorem 4.6).** Assume that matrices \( A \) and \( B \) are irreducible and not permutation matrices. Suppose that \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are one-sided flow equivalent via a homeomorphism \( h : X_A \to X_B \). For \( f \in C(X_B, \mathbb{Z}) \), \( g \in C(X_A, \mathbb{Z}) \) such that their classes \([f] \in H_B^+, [g] \in H_A^+\) are order units of \((H^B, H_B^+)\) and \((H^A, H_A^+)\), respectively, the following formulae hold:

\[
\zeta_{A,\Psi_h(f)}(s) = \zeta_{B,f}(s), \quad \zeta_{B,\Psi_{h^{-1}}(g)}(s) = \zeta_{A,g}(s).
\]

This theorem shows that the set \( Z(X_A, \sigma_A) \) of dynamical zeta functions of \((X_A, \sigma_A)\) whose potential functions are order units in the ordered cohomology group \((H^A, H_A^+)\) is invariant under one-sided flow equivalence (Corollary 4.7).

Throughout the paper, we denote by \( \mathbb{R}_+ \), by \( \mathbb{Z}_+ \) and by \( \mathbb{N} \) the set of nonnegative real numbers, the set of nonnegative integers and the set of positive integers, respectively.

### 2. One-sided suspensions

In what follows, we assume that \( A = [A(i,j)]_{i,j=1}^N \) is an \( N \times N \) irreducible matrix with entries in \( \{0, 1\} \) and \( 1 < N \in \mathbb{N} \). We further assume that \( A \) is not any
We note that the condition graph with vertex set denoted by \([4]\) so that the space \(X_A\) is homeomorphic to a Cantor discontinuum. We denote by \(C(X_A, \mathbb{R})\) (resp. \(C(X_A, \mathbb{R}^+))\) the set of real (resp. nonnegative real) valued continuous functions on \(X_A\). The set \(C(X_A, \mathbb{Z})\) of integer valued continuous functions on \(X_A\) has a natural structure of abelian group by pointwise sums. Let us denote by \(H^A\) the quotient group of the abelian group \(C(X_A, \mathbb{Z})\) by the subgroup \(\{g - g \circ \sigma_A \mid g \in C(X_A, \mathbb{Z})\}\). The positive cone \(H^A_+\) consists of the classes \([f] \in H^A\) of nonnegative integer valued continuous functions \(f \in C(X_A, \mathbb{Z}^+\)). The ordered group \((H^A, H^A_+)\) is called the ordered cohomology group for \((X_A, \sigma_A)\) (cf. \([2], [9], [15]\)). For \(f \in C(X_A, \mathbb{Z})\) and \(m \in \mathbb{N}\), we set

\[
f^m(x) = \sum_{i=0}^{m-1} f(\sigma_A^i(x)), \quad x \in X_A.
\]

An element \([f]\) in \(H^A_+\) is called an order unit if for any \([g] \in H^A\), there exists \(n \in \mathbb{N}\) such that \(n[f] - [g] \in H^A_+\). We see that \([f] \in H^A_+\) is an order unit if and only if there exists \(m \in \mathbb{N}\) such that \(f^m\) is strictly positive ([2], p. 175).

We will first define a one-sided suspension for a one-sided topological Markov shift. The triplet \((l, k, b)\) for real valued continuous functions \(l, k \in C(X_A, \mathbb{R}^+)\) and \(b \in C(X_A, \mathbb{R})\) are called a suspension triplet for \((X_A, \sigma_A)\) if they satisfy the following two conditions:

1. The difference \(c = l - k\) belongs to \(C(X_A, \mathbb{Z})\) and the class \([c] \in H^A_+\) is an order unit of \((H^A, H^A_+)\).
2. The differences \(l - b, k - b \circ \sigma_A\) belong to \(C(X_A, \mathbb{Z}^+)\).

The triplet \((1, 0, 0)\) is called the standard suspension triplet.

We fix a suspension triplet \((l, k, b)\) for \((X_A, \sigma_A)\) for a while. We set \(X^R_{A,b} = \{(x, r) \in X_A \times \mathbb{R} \mid r \geq b(x)\}\) and define an equivalence relation \(\sim\) in \(X^R_{A,b}\) generated by the relations

\[
(x, r) \sim_{l,k} (\sigma_A(x), r - c(x)) \quad \text{for} \quad r \geq l(x).
\]

We note that the condition \(r \geq l(x)\) implies \((x, r) \in X^R_{A,b}\) and \(r - c(x) = r - l(x) + k(x) \geq k(x) \geq b(\sigma_A(x))\) so that \((\sigma_A(x), r - c(x)) \in X^R_{A,b}\). If there exists \(n \in \mathbb{Z}^+\) such that \(r \geq c^n(x) + l(\sigma_A^m(x))\) for all \(m \in \mathbb{Z}^+\) with \(0 \leq m < n\), then

\[
(x, r) \sim_{l,k} (\sigma_A(x), r - c(x)) \sim_{l,k} \cdots \sim_{l,k} (\sigma_A^n(x), r - c^n(x)).
\]

Hence we have

**Lemma 2.1.** For \((x, r), (x', r') \in X^R_{A,b}\), we have \((x, r) \sim_{l,k} (x', r')\) if and only if there exist \(n, n' \in \mathbb{Z}^+\) such that

\[
\sigma_A^n(x) = \sigma_A^n(x'), \quad r - c^n(x) = r' - c^n(x') \quad \text{and}
\]

\[r - c^n(x) \geq l(\sigma_A^n(x)), \quad r - c^n(x) \geq l(\sigma_A^m(x')) \quad \text{for} \quad 0 \leq m < n, 0 \leq m' < n'.\]

**Proof.** It suffices to prove the only if part. For \((x, r) \in X^R_{A,b}\) with \(r \geq l(x)\), we write a directed edge from \((x, r)\) to \((\sigma_A(x), r - c(x))\). We then have a directed graph with vertex set \(X^R_{A,b}\). Suppose \((x, r) \sim_{l,k} (x', r')\). There exists a finite sequence \((x_i, r_i) \in X^R_{A,b}, i = 0, 1, \ldots, L\) such that \((x_0, r_0) = (x, r)\) and \((x_L, r_L) = (x', r')\).
and there exists a directed edge from $(x_{i-1}, r_{i-1})$ to $(x_i, r_i)$ or from $(x_i, r_i)$ to $(x_{i-1}, r_{i-1})$ for each $i = 1, 2, \ldots, L$. Since each vertex $(x_i, r_i)$ emits at most one directed edge, we may find $n$ with $0 \leq n \leq L$ such that there exist directed edges from $(x_{i-1}, r_{i-1})$ to $(x_i, r_i)$ for $i = 1, 2, \ldots, n$ and from $(x_i, r_i)$ to $(x_{i-1}, r_{i-1})$ for $i = n + 1, \ldots, L$. By putting $n' = L - n$, we see that $n$ and $n'$ satisfy the desired conditions for $(x, r)$ and $(x', r')$.

Define a topological space

$$
S_{A,b}^{l,k} = X_{A,b}^R / \sim_{l,k}
$$

as the quotient topological space of $X_{A,b}^R$ by the equivalence relation $\sim_{l,k}$. We denote by $[x, r]$ the class of $(x, r) \in X_{A,b}^R$ in the quotient space $S_{A,b}^{l,k}$. We will show that $S_{A,b}^{l,k}$ is a compact Hausdorff space. We note the following lemma.

Lemma 2.2. For any $m \in \mathbb{N}$, there exists $n_m \in \mathbb{N}$ such that $e^{n_m}(x) \geq m$ for all $x \in X_A$.

Proof. Since $[c] \in H_A^1$ is an order unit, one may take $p \in \mathbb{N}$ such that $e^p$ is a strictly positive function so that $e^p(x) \geq 1$ for all $x \in X_A$. By the identity $e^{mp}(x) = e^{(m-1)p}(x) + e^p(\sigma_A^{(m-1)}(x))$ for $m \in \mathbb{N}, x \in X_A$, one obtains that $e^{mp}(x) \geq m$ for all $x \in X_A$. By putting $n_m = mp$, we see the desired assertion.

We set

$$
\Omega_{A,b}^l = \{(x, r) \in X_{A,b}^R \mid b(x) \leq r \leq l(x)\},
\Omega_{A,b}^l = \{(x, r) \in X_{A,b}^R \mid b(x) \leq r < l(x)\}.
$$

Lemma 2.3. For $(x, r) \in X_{A,b}^R$ with $r \geq l(x)$, there exists $(z, s) \in \Omega_{A,b}^l$ such that $(x, r) \sim_{l,k} (z, s)$.

Proof. For $(x, r) \in X_{A,b}^R$ with $r \geq l(x)$, by Lemma 2.2 one may take a minimum number $n \in \mathbb{N}$ satisfying $r < e^{n+1}(x) + k(\sigma_A^n(x))$, so that we have

$$
e^{n+1}(x) + k(\sigma_A^n(x)) \leq r < e^{n+1}(x) + k(\sigma_A^n(x)) \quad \text{for all } m \leq n.
$$

In particular, we have

$$
e^n(x) + k(\sigma_A^{n-1}(x)) \leq r < e^{n+1}(x) + k(\sigma_A^n(x)).
$$

As $e^{n+1}(x) = e^n(x) + l(\sigma_A^n(x)) - k(\sigma_A^n(x))$ and $b(\sigma_A^n(x)) \leq k(\sigma_A^{n-1}(x))$, we get

$$b(\sigma_A^n(x)) \leq r - e^n(x) < l(\sigma_A^n(x)).
$$

Since $r - e^m(x) \geq l(\sigma_A^m(x))$ for all $m \in \mathbb{Z}_+$ with $m < n$, we see that

$$(x, r) \sim_{l,k} (\sigma_A^m(x), r - e^m(x)) \quad \text{for all } m \in \mathbb{Z}_+ \text{ with } m \leq n.
$$

By putting $z = \sigma_A^n(x), s = r - e^n(x)$, we have $(x, r) \sim_{l,k} (z, s)$ and $(z, s) \in \Omega_{A,b}^l$.

Proposition 2.4. $S_{A,b}^{l,k}$ is a compact Hausdorff space.
Proof. We may restrict the equivalence relation \( \sim \) to \( \Omega_{i,k}^{l,b} \). Let \( q_{\Omega} : \Omega_{i,k}^{l,b} \to \Omega_{i,k}^{l,b} / \sim \) and \( q_{\Omega} : X_{i,k}^{l,b} \to S_{i,k}^{l,b} \) be the quotient maps, which are continuous. Since \( \Omega_{i,k}^{l,b} \) is compact, so is \( \Omega_{i,k}^{l,b} / \sim \). By Lemma 2.3, an element of \( \Omega_{i,k}^{l,b} / \sim \) is represented by \( \Omega_{i,k}^{l,b} / \sim \). This implies that \( \Omega_{i,k}^{l,b} / \sim \) is Hausdorff, because \( \Omega_{i,k}^{l,b} \) is a fundamental domain of the quotient space \( \Omega_{i,k}^{l,b} / \sim \). For \( (x,r) \in X_{i,k}^{l,b} \), take a minimum number \( n \in \mathbb{Z}_+ \) such that \( r < c^{n+1}(x) + k(\sigma_{l,k}^n(x)) \), and define a continuous map \( \varphi : X_{i,k}^{l,b} \to \Omega_{i,k}^{l,b} \) by setting \( \varphi((x,r)) = (\sigma_{l,k}^n(x), r - c^n(x)) \). By the proof of Lemma 2.3, it induces a map \( \tilde{\varphi} : S_{i,k}^{l,b} \to \Omega_{i,k}^{l,b} / \sim \). We will see that \( \tilde{\varphi} \) is a homeomorphism, so that \( S_{i,k}^{l,b} \) is a compact Hausdorff space. As the inclusion map \( \iota : (x,r) \in \Omega_{i,k}^{l,b} \to (x,r) \in X_{i,k}^{l,b} \) induces a map \( \tilde{\iota} : [x,r] \in \Omega_{i,k}^{l,b} / \sim \to [x,r] \in S_{i,k}^{l,b} \) which satisfies \( \tilde{\varphi} \circ \tilde{\iota} = \text{id} \), \( \tilde{\iota} \circ \tilde{\varphi} = \text{id} \), the map \( \tilde{\varphi} \) is bijective. We have commutative diagrams:

\[
\begin{array}{ccc}
X_{i,k}^{l,b} & \xrightarrow{\varphi} & \Omega_{i,k}^{l,b} \\
\downarrow q_{X} & & \downarrow q_{\Omega} \\
S_{i,k}^{l,b} & \xrightarrow{\tilde{\varphi}} & \Omega_{i,k}^{l,b} / \sim \\
\end{array}
\quad
\begin{array}{ccc}
X_{i,k}^{l,b} & \xleftarrow{\iota} & \Omega_{i,k}^{l,b} \\
\downarrow q_{X} & & \downarrow q_{\Omega} \\
S_{i,k}^{l,b} & \xleftarrow{\tilde{\iota}} & \Omega_{i,k}^{l,b} / \sim \\
\end{array}
\]

Since both the maps \( \varphi : X_{i,k}^{l,b} \to \Omega_{i,k}^{l,b} \) and \( \iota : \Omega_{i,k}^{l,b} \to X_{i,k}^{l,b} \) are continuous, the commutativity of the diagrams imply the continuity of the maps \( \tilde{\varphi} : S_{i,k}^{l,b} \to \Omega_{i,k}^{l,b} / \sim \) and \( \tilde{\iota} : \Omega_{i,k}^{l,b} / \sim \to S_{i,k}^{l,b} \). Hence \( \tilde{\varphi} : S_{i,k}^{l,b} \to \Omega_{i,k}^{l,b} / \sim \) is a homeomorphism.

We will define the flow \( \phi_{A,t}, t \in \mathbb{R}_+ \) on \( S_{A,b}^{l,b} \) by \( \phi_{A,t}([x,r]) = [x,r+t] \) for \( t \in \mathbb{R}_+ \). As in the discussions above, for \( (x,r) \in X_{A,b}^{l,b} \) and \( t \in \mathbb{R}_+ \), there exists \( n \in \mathbb{Z}_+ \) such that

\[
b(\sigma_{A}^n(x)) \leq t + r - c^n(x) < l(\sigma_{A}^n(x))
\]

and

\[
\phi_{A,t}([x,r]) = [\sigma_{A}^n(x), t + r - c^n(x)].
\]

Hence the flow \( \phi_{A,t}([x,r]), t \in \mathbb{R}_+ \) are defined on \( S_{A,b}^{l,b} \). We call the flow space \( (S_{A,b}^{l,b}, \phi_{A}) \) the \( (l,b) \)-suspension of one-sided topological Markov shift \( (X_{A,b}, \sigma_{A}) \). It is simply called the one-sided suspension of \( (X_{A,b}, \sigma_{A}) \). The map \( b_{A} : X_{A,b} \to S_{A,b}^{l,b} \) defined by \( b_{A}(x) = [x,b(x)] \) is called the base map. The base map for \( b \equiv 0 \) is written \( s_{A}(x) = [x,0] \) and called the standard base map. If all the functions \( l,b \) are valued in integers, \( (l,b) \)-suspension for \( X_{A,b}^{l,b} \) is called \( (l,b) \)-discrete suspension. For \( l \equiv 1, k \equiv 0, b \equiv 0 \), the \( (1,0,0) \)-suspension \( (S_{A,b}^{1,0}, \phi_{A}) \) is called the standard one-sided suspension. The \( (l,b) \)-suspension space \( S_{A,b}^{l,b} \) and the \( (l,0) \)-suspension space \( S_{A,b}^{l,0} \) are denoted by \( S_{A,b}^{l,b} \) and \( S_{A,b}^{l,0} \), respectively. The standard one-sided suspension \( (S_{A,b}^{1,0}, \phi_{A}) \) is denoted by \( (S_{A,b}^{1}, \phi_{A}) \).
Lemma 2.5. The standard base map $s_A : X_A \to S^1_A$ for the standard suspension is an injective continuous map.

Proof. It suffices to show the injectivity of the map $s_A$. Suppose that $s_A(x) = s_A(z)$ in $S^1_A$ for some $x, z \in X_A$. There exists a finite sequence $(x_i, r_i) \in X_{A,0}^R$ such that

\[(x, 0) \sim (x_1, r_1) \sim \cdots \sim (x_n, r_n) \sim (z, 0)\]

where $r_i \in \mathbb{Z}_+$, $i = 1, \ldots, n$. If $r_i = r_{i+1}$, we may take $x_i = x_{i+1}$. Let $K = \max\{r_i \mid i = 1, \ldots, n\}$. Take $i_K \in \{1, \ldots, n\}$ such that $r_{i_K} = K$. It then follows that

$$\sigma^K(x_{i_K}) = x$$

and $\sigma^K(x_{i_K}) = z$, so that $x = z$. \hfill $\square$

Lemma 2.6. For a suspension triplet $(l, k, b)$ for $(X_A, \sigma_A)$, put $c = l - k$. If $c' \in C(X_A, \mathbb{Z})$ satisfies $[c] = [c']$ in $H^A$, there exists a suspension triplet $(l', k', b')$ for $(X_A, \sigma_A)$ and a homeomorphism $\Phi : S_{A,b}^{l,k} \to S_{A,b'}^{l',k'}$ such that $c' = l' - k'$ and

\[(2.3) \quad \Phi \circ b_A = b'_A, \quad \Phi \circ \phi_{A,t} = \phi_{A,t} \circ \Phi \quad \text{for } t \in \mathbb{R}_+.
\]

Hence $(S_{A,b}^{l,k}, \Phi_A)$ and $(S_{A,b'}^{l',k'}, \Phi_A)$ are topologically conjugate compatible to their base maps.

Proof. Since $[c] = [c']$ in $H^A$, there exists $d \in C(X_A, \mathbb{Z})$ such that $c - c' = d \circ \sigma_A - d$. We may assume that $d(x) \in \mathbb{Z}_+$ for all $x \in X_A$. Define

$$l'(x) = l(x) + d(x), \quad k'(x) = k(x) + d(\sigma_A(x)), \quad b'(x) = b(x) + d(x), \quad x \in X_A.$$  

It is easy to see that $c' = l' - k'$ and $(l', k', b')$ is a suspension triplet for $(X_A, \sigma_A)$. Define $\Phi : X_{A,b}^R \to X_{A,b'}^R$ by $\Phi((x, r)) = (x, r + d(x))$. We know that $(x, r) \in X_{A,b}^R$ if and only if $(x, r + d(x)) \in X_{A,b'}^R$. Since $\Phi((\sigma_A(x), r - c(x))) = (b_A(x), x, r - c(x) + d(\sigma_A(x))) = (\sigma_A(x), r + d(x) - c'(x))$, we have $\Phi((x, r)) \sim _{(l', k')} \Phi((\sigma_A(x), r - c(x)))$ for $r \geq l'(x)$. It is easy to see that $\Phi$ extends to a homeomorphism $S_{A,b}^{l,k} \to S_{A,b'}^{l',k'}$ which is still denoted by $\Phi$ and satisfies the equalities (2.3). \hfill $\square$

3. One-sided flow equivalence

We will define one-sided flow equivalence on one-sided topological Markov shifts.

Definition 3.1. $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are said to be one-sided flow equivalent if there exist suspension triplets $(l_1, k_1, b_1)$ for $(X_A, \sigma_A)$ and $(l_2, k_2, b_2)$ for $(X_B, \sigma_B)$, a homeomorphism $h : X_A \to X_B$, and continuous maps $\Phi_1 : S_{A,b_1}^{l_1,k_1} \to S_{B,b_2}^{l_2,k_2}$, $\Phi_2 : S_{B,b_2}^{l_2,k_2} \to S_{A,b_1}^{l_1,k_1}$ such that

\[(3.1) \quad \Phi_1 \circ \phi_{A,t} = \phi_{B,t} \circ \Phi_1 \quad \text{for } t \in \mathbb{R}_+, \quad \Phi_1 \circ b_{1,A} = s_B \circ h,
\]

\[(3.2) \quad \Phi_2 \circ \phi_{B,t} = \phi_{A,t} \circ \Phi_2 \quad \text{for } t \in \mathbb{R}_+, \quad \Phi_2 \circ b_{2,B} = s_A \circ h^{-1},
\]

where $b_{1,A} : X_A \to S_{A,b_1}^{l_1,k_1}$ and $b_{2,B} : X_B \to S_{B,b_2}^{l_2,k_2}$ are the base maps defined by $b_{1,A}(x) = [x, b_1(x)]$ for $x \in X_A$ and $b_{2,B}(y) = [y, b_2(y)]$ for $y \in X_B$, respectively.

In this case, we say that $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are one-sided flow equivalent via a homeomorphism $h : X_A \to X_B$. If there exists a homeomorphism $h : X_A \to X_B$ satisfying (3.1) (resp. (3.2)), we say that $(X_B, \sigma_B)$ (resp. $(X_A, \sigma_A)$) is a cross section of the one-sided suspension $S_{A,b_1}^{l_1,k_1}$ through $b_{1,A} \circ h^{-1}$ (resp. $S_{B,b_2}^{l_2,k_2}$ through $b_{2,B} \circ h$).
Proposition 3.2. If \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are continuously orbit equivalent, then they are one-sided flow equivalent.

Proof. Let \(h : X_A \to X_B\) be a homeomorphism which gives rise to a continuous orbit equivalence between \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) with continuous functions \(k_1, l_1 : X_A \to \mathbb{Z}_+\) and \(k_2, l_2 : X_B \to \mathbb{Z}_+\) satisfying (1.1) and (1.2), respectively. Put \(c_1(x) = l_1(x) - k_1(x), x \in X_A\) and \(c_2(y) = l_2(y) - k_2(y), y \in X_B\). We set \(b_1 \equiv 0, b_2 \equiv 0\). By [10] Theorem 5.11, the map \(\Psi_h : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z})\) defined by (1.5) induces an isomorphism of ordered groups between \((H^B, H^B_+)^1\) and \((H^A, H^A_+)^1\), so that the elements \(\Psi_h(1) = c_1\) and similarly \(\Psi_h^{-1}(1) = c_2\) give rise to an order unit of \((H^A, H^A_+)\) and of \((H^B, H^B_+)\), respectively. Then \((l_1, k_1, b_1)\) is a suspension triplet for \((X_A, \sigma_A)\) and \((l_2, k_2, b_2)\) is a suspension triplet for \((X_B, \sigma_B)\). Define \(\Phi_1 : X_{A,0}^R \to X_{B,0}^R\) by \(\Phi_1((x, r)) = (h(x), r)\) for \(x \in X_A, r \geq 0\). As \(S_B^I\) is the standard suspension, we have for \((x, r) \in X_{A,0}^R\) with \(r \geq l_1(x)\)

\[
(h(x), r) \sim_{t,0} (\sigma_{B}^{k_1}(h(x)), r - l_1(x))
\]

and

\[
(h(\sigma_A(x)), r - c_1(x)) \sim_{t,0} (\sigma_{B}^{k_1}(h(\sigma_A(x))), r - c_1(x) - k_1(x)).
\]

Since \(\sigma_B^{k_1}(h(x)) = \sigma_B^{k_1}(h(\sigma_A(x)))\) and \(r - l_1(x) = r - c_1(x) - k_1(x)\), we have

\[
(h(\sigma_A(x)), r - c_1(x)) \sim_{t,0} (\sigma_{A}(x), r - c_1(x)) \quad \text{in } S_B^I
\]

so that the map \(\Phi_1 : X_{A,0}^R \to X_{B,0}^R\) induces a continuous map \(S_{A,1}^{l_1, k_1} \to S_B^I\) which is still denoted by \(\Phi_1\). It is clear to see that the equalities \(\Phi_1 \circ \phi_{A,t} = \phi_{B,t} \circ \Phi_1\) for \(t \in \mathbb{R}_+\) and \(\Phi_1 \circ b_{A,1} = s_B \circ h\) hold. We similarly have a continuous map \(\Phi_2 : S_{B,1}^{l_2, k_2} \to S_A^I\) defined by \(\Phi_2([y, s]) = [h^{-1}(y), s]\) satisfying the equalities \(\Phi_2 \circ \phi_{B,t} = \phi_{A,t} \circ \Phi_2\) for \(t \in \mathbb{R}_+\) and \(\Phi_2 \circ b_{2, B} = s_A \circ h^{-1}\) to prove that \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are one-sided flow equivalent.

Conversely we have

Proposition 3.3. If \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are one-sided flow equivalent, then they are continuously orbit equivalent.

Proof. Suppose that \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are one-sided flow equivalent. Take suspension triplets \((l_1, k_1, b_1)\) for \((X_A, \sigma_A)\) and \((l_2, k_2, b_2)\) for \((X_B, \sigma_B)\), a homeomorphism \(h : X_A \to X_B\), and continuous maps \(\Phi_1 : S_{A,b_1}^{l_1, k_1} \to S_B^I, \Phi_2 : S_{B,b_2}^{l_2, k_2} \to S_A^I\) satisfying the equalities (3.1) and (3.2). For \((x, r) \in X_{A,b_1}^R\) with \(r \geq l_1(x)\), we have

\[
[x, r] = [\sigma_A(x), r - c_1(x)] \quad \text{in } S_{A,b_1}^{l_1, k_1}
\]

It follows that

\[
[x, l_1(x)] = [x, (l_1(x) - b_1(x)) + b_1(x)],
\]

\[
[\sigma_A(x), l_1(x) - c_1(x)] = [\sigma_A(x), (k_1(x) - b_1(\sigma_A(x))) + b_1(\sigma_A(x))].
\]

As \(l_1(x) - b_1(x) \geq 0\) and \(k_1(x) - b_1(\sigma_A(x)) \geq 0\), we have

\[
[x, l_1(x)] = \phi_{A,l_1(x)-b_1(x)}([x, b_1(x)]) = \phi_{A,l_1(x)-b_1(x)}(b_1(A,x)),
\]

\[
[\sigma_A(x), l_1(x) - c_1(x)] = \phi_{A,k_1(x)-b_1(\sigma_A(x))}([\sigma_A(x), b_1(\sigma_A(x))])
\]

\[
= \phi_{A,k_1(x)-b_1(\sigma_A(x))}(b_1(A,\sigma_A(x))).
\]
Hence we have $\phi_{A,t_1(x) - b_1(x)}(b_1, A(x)) = \phi_{A,k_1(x) - b_1, A(x)}(b_1, A(x))$ so that

$$\Phi_1(\phi_{A,t_1(x) - b_1(x)}(b_1, A(x))) = \Phi_1(\phi_{A,k_1(x) - b_1, A(x)}(b_1, A(x))).$$

By (3.1) and (3.2), we have

$$\phi_{B,t_1(x) - b_1(x)}(s_B(h(x))) = \phi_{B,k_1(x) - b_1, A(x)}(s_B(h(A(x))).$$

It follows that

$$[h(x), l_1(x) - b_1(x)] = [h(A(x)), k_1(x) - b_1(\sigma_A(x))] \text{ in } S_B.$$ Put $l_1'(x) = l_1(x) - b_1(x)$ and $k_1'(x) = k_1(x) - b_1(\sigma_A(x))$. They are valued in nonnegative integers. Since

$$[h(x), l_1'(x), b_1(x)] = [\sigma_B^{l_1'(x)}(h(x)), 0] = s_B^{l_1'(x)}(h(x)),$$

$$[h(A(x)), k_1(x) - b_1(\sigma_A(x))] = [\sigma_A^{k_1'(x)}(h(A(x))), 0] = s_B^{\sigma_A^{k_1'(x)}(h(A(x)))},$$

and the standard base map $s_B : X_B \to S_B$ is injective, we have $\sigma_B^{l_1'(x)}(h(x)) = \sigma_B^{k_1'(x)}(h(A(x)))$ for $x \in X_A$. We similarly have continuous maps $l_2', k_2' \in C(X_B, \mathbb{Z}_+)$ such that $\sigma_B^{l_2'(y)}(h^{-1}(y)) = \sigma_B^{k_2'(y)}(h^{-1}(\sigma_B(y)))$ for $y \in X_B$. Consequently $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuously orbit equivalent.

Therefore we may conclude the following theorem.

**Theorem 3.4.** Assume that matrices $A$ and $B$ are irreducible and not permutation matrices. One-sided topological Markov shifts $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are one-sided flow equivalent if and only if they are continuously orbit equivalent.

It is well known that two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if both $\mathbb{Z}^N / (id - A)\mathbb{Z}^N$ is isomorphic to $\mathbb{Z}^M / (id - B)\mathbb{Z}^M$ as abelian groups and $\det(id - A) = \det(id - B)$, where $N, M$ are the sizes of the matrices $A, B$ respectively ([3, 5, 12]). By using [10] Theorem 3.6, we see the following characterization of one-sided flow equivalence which is a corollary of the above theorem.

**Corollary 3.5.** Assume that matrices $A$ and $B$ are irreducible and not permutation matrices. One-sided topological Markov shifts $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are one-sided flow equivalent if and only if there exists an isomorphism $\Phi : \mathbb{Z}^N / (id - A)\mathbb{Z}^N \to \mathbb{Z}^M / (id - B)\mathbb{Z}^M$ of abelian groups such that $\Phi([u_A]) = [u_B]$ and $\det(id - A) = \det(id - B)$, where $[u_A]$ (resp. $[u_B]$) is the class of the vector $u_A = [1, \ldots, 1]$ in $\mathbb{Z}^N / (id - A)\mathbb{Z}^N$ (resp. $u_B = [1, \ldots, 1]$ in $\mathbb{Z}^M / (id - B)\mathbb{Z}^M$).

The statement of the following proposition is more general than that of Proposition 3.2.

**Proposition 3.6.** Suppose that $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuously orbit equivalent via a homeomorphism $h : X_A \to X_B$. For $f \in C(X_B, \mathbb{Z}_+)$ and $g \in C(X_A, \mathbb{Z}_+)$ such that $[f] \in H^+_A$ and $[g] \in H^+_B$ are order units of $(H^+_A, H^+_B)$ and of $(H^+_A, H^+_A)$ respectively, there exist $l_f, k_f \in C(X_A, \mathbb{Z}_+)$ and $l_g, k_g \in C(X_B, \mathbb{Z}_+)$ such that $(l_f, k_f, 0)$ and $(l_g, k_g, 0)$ are suspension triplets for $(X_A, \sigma_A)$ and for $(X_B, \sigma_B)$ respectively and continuous maps $\Phi_f : \sigma_A^{l_f, k_f} \to S_B$ and $\Phi_g : \sigma_B^{l_g, k_g} \to S_A$ such that

$$\Phi_f \circ \phi_{A,t} = \phi_{B,t} \circ \Phi_f \quad \text{for } t \in \mathbb{R}_+,$$

$$\Phi_g \circ \phi_{B,t} = \phi_{A,t} \circ \Phi_g \quad \text{for } t \in \mathbb{R}_+.$$
Proof. Put \( l_f(x) = f_1(x)(h(x)), k_f(x) = f_k(x)(h(\sigma_A(x))) \) for \( x \in X_A \) so that \( \Psi_h(f)(x) = l_f(x) - k_f(x) \). Since \( [f] \in H^B \) is an order unit and \( \Psi_h : H^B \to H^A \) preserves the orders, the class \([\Psi_h(f)]\) gives rise to an order unit of \((H^A, H^A_+). \) As \( l_f - k_f = \Psi_h(f) \), we see that \((l_f, k_f, 0)\) is a suspension triplet for \((X_A, \sigma_A)\) so that we may consider the one-sided suspensions \((S_A^{l_f, k_f}, \phi_A)\) and \((S_B^{l_f, k_f}, \phi_B)\). We define the map \( \Phi_f : X^R_{A,0} \to X^R_{B,0} \) by \( \Phi_f((x, r)) = (h(x), r) \). For \( r \geq l_f(x) \), we have

\[
(h(x), r) \sim_{f_0} (\sigma_B(h(x)), r - f(h(x))) \sim_{f_0} \cdots \sim_{f_0} (\sigma_B^{l_f(x)}(h(x)), r - f^{l_f(x)}(h(x)))
\]

and similarly

\[
(h(\sigma_A(x)), r - \Psi_h(f)(x)) \sim_{f_0} (\sigma_B^{k_f(x)}(h(\sigma_A(x))), r - \Psi_h(f)(x) - f^{k_f(x)}(h(\sigma_A(x))))
\]

As the equalities \( \sigma_B^{l_f(x)}(h(x)) = \sigma_B^{k_f(x)}(h(\sigma_A(x))) \) and \( f^{l_f(x)}(h(x)) = \Psi_h(f)(x) + f^{k_f(x)}(h(\sigma_A(x))) \) hold, we have

\[
\Phi_f((x, r)) \sim_{f_0} \Phi_f((\sigma_A(x), r - \Psi_h(f)(x))).
\]

Hence \( \Phi_f : X^R_{A,0} \to X^R_{B,0} \) induces a continuous map \( S_A^{l_f, k_f} \to S_B^{l_f, k_f} \) which is still denoted by \( \Phi_f \). It is easy to see that the map satisfies the desired properties. We similarly have a map \( \Phi_g : S_B^{g, k_g} \to S_A^g \) satisfying the desired properties.

We give an example of one-sided flow equivalent topological Markov shifts. Let \( A = [1, 1], B = [1, 0, 1] \). The one-sided topological Markov shift \((X_A, \sigma_A)\) is called the full 2-shift, and the other one \((X_B, \sigma_B)\) is called the golden mean shift whose shift space \( X_B \) consists of sequences \((y_n)_{n \in \mathbb{N}}\) of 1, 2 such that the word \((2, 2)\) is forbidden (cf. \([7]\)). They are continuously orbit equivalent as in \([8]\) Section 5] through the homeomorphism \( h : X_A \to X_B \) defined by substituting the word \((2, 1)\) for the symbol 2 from the leftmost of a sequence \((x_n)_{n \in \mathbb{N}}\) in order, so that they are one-sided flow equivalent. Put for \( i = 1, 2 \)

\[
U_{A,i} = \{(x_n)_{n \in \mathbb{N}} \in X_A \mid x_1 = i\}, \quad U_{B,i} = \{(y_n)_{n \in \mathbb{N}} \in X_B \mid y_1 = i\}.
\]

By setting

\[
\begin{align*}
k_1(x) &= 0, l_1(x) = 1 \quad \text{for } x \in U_{A,1}, \\
k_1(x) &= 0, l_1(x) = 2 \quad \text{for } x \in U_{A,2}, \\
k_2(y) &= 0, l_2(y) = 1 \quad \text{for } y \in U_{B,1}, \\
k_2(y) &= 1, l_2(y) = 1 \quad \text{for } y \in U_{B,2},
\end{align*}
\]

the continuous functions \( k_1, l_1 : X_A \to \mathbb{Z}_+ \) and \( k_2, l_2 : X_B \to \mathbb{Z}_+ \) satisfy (1.1) and (1.2), respectively. By Proposition 3.2, the suspension flows \((S_A^{k_1, k_2}, \phi_A)\) and \((S_B^{k_1, k_2}, \phi_B)\), and similarly \((S_A^{1, 2}, \phi_A)\) and \((S_B^{1, 2}, \phi_B)\) give rise to flow equivalence between \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\).

4. **One-sided suspensions and zeta functions**

The zeta function \( \zeta_{\phi}(s) \) of a flow \( \phi : S \to S \) on a compact metric space with at most countably many closed orbits is defined by the formula (1.3). In the first half of this section, we will show that the zeta function \( \zeta_{\phi_A}(s) \) of the flow \( \phi_A \) of the one-sided suspension \( S_A^{l, k} \) of \((X_A, \sigma_A)\) is given by the dynamical zeta function \( \zeta_{A,c}(s) \) with potential function \( c = l - k \). We first provide a lemma.
Lemma 4.1. Let $(S_{A,b}^{l,k}, \phi_A)$ be a one-sided suspension of $(X_A, \sigma_A)$. Then there exists a bijective correspondence between primitive periodic orbits $\tau \in P_{orb}(S_{A,b}^{l,k}, \phi_A)$ and periodic orbits $\gamma_\tau \in P_{orb}(X_A)$ of $(X_A, \sigma_A)$ such that

$$\ell(\tau) = \beta_{\gamma_\tau}(c)$$

where $\ell(\tau)$ is the primitive length of the periodic orbit $\tau$ defined by $\ell(\tau) = \min\{t \in \mathbb{R}_+ \mid \phi_{A,t}(u) = u\}$ for any point $u \in \tau$ and $\beta_{\gamma_\tau}(c) = \sum_{i=0}^{p-1} c(\sigma_A^i(x))$ for $c = t - k$ and $\gamma_\tau = \{x, \sigma_A(x), \ldots, \sigma_A^{p-1}(x)\}$.

Proof. For an arbitrary point $(x, r)$ in a primitive periodic orbit $\tau \in P_{orb}(S_{A,b}^{l,k}, \phi_A)$, one sees that $(x, r) \sim \phi_{A,\ell(\tau)}(x, r)$. Since $\phi_{A,\ell(\tau)}(x, r) = (x, r + \ell(\tau))$, there exists a number $p \in \mathbb{Z}_+$ such that

$$\ell(\tau) = \{l(x) - r\} + \{l(\sigma_A(x)) - k(x)\} + \cdots + \{l(\sigma_A^{p-1}(x)) - k(\sigma_A^{p-2}(x))\} + \{r - k(\sigma_A^{p-1}(x))\}$$

and $\sigma_A^p(x) = x$, so that $\ell(\tau) = \sum_{i=0}^{p-1} c(\sigma_A^i(x))$.

Conversely, for a periodic orbit $\gamma = \{x, \sigma_A(x), \ldots, \sigma_A^{p-1}(x)\} \in P_{orb}(X_A)$, we have $(x, r) \sim \phi_{A,\beta_i(c)}(x, r)$ for any $r \in \mathbb{R}$ with $b(x) \leq r \leq l(x)$. Hence $\tau_\gamma = \{\phi_{A,\ell}(x, r) \mid 0 \leq t \leq \beta_i(c)\}$ gives a primitive periodic orbit of $(S_{A,b}^{l,k}, \phi_A)$ with length $\beta_i(c)$. \hfill \blacksquare

Let us denote by $\text{Per}_n(X_A)$ the set $\{x \in X_A \mid \sigma_A^n(x) = x\}$ of $n$-periodic points. For a H"older continuous function $f$ on $X_A$, the dynamical zeta function $\zeta_{A,f}(s)$ is defined by

$$\zeta_{A,f}(s) = \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in \text{Per}_n(X_A)} \exp(-s \sum_{k=0}^{n-1} f(\sigma_A^k(x)))\right\} \text{ (see [17], [19], etc.)}$$

where the right hand side makes sense for a complex number $s \in \mathbb{C}$ with $\text{Re}(s) > h$ for some positive constant $h > 0$. The function $\zeta_{A,f}(s)$ is called the dynamical zeta function on $X_A$ with potential function $f$. We may especially define the zeta function $\zeta_{A,c}(s)$ for an integer valued continuous function $c$ on $X_A$ whose class $[c]$ in $H_A$ is an order unit of $(H_A, H_A^+)$. By a routine argument as in [11] p.100] with Lemma 4.1, we have

Proposition 4.2. Assume that a matrix $A$ is irreducible and not any permutation matrix. The zeta function [13] of the flow of the one-sided suspension $(S_{A,b}^{l,k}, \phi_A)$ of $(X_A, \sigma_A)$ is given by the dynamical zeta function $\zeta_{A,c}(s)$ with potential function $c = l - k$ such as

$$\zeta_{A,c}(s) = \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in \text{Per}_n(X_A)} \exp(-s \sum_{k=0}^{n-1} c(\sigma_A^k(x)))\right\}.$$  

(4.2)

Remark 4.3. The class $[c]$ of the function $c$ in $H_A$ is an order unit of the ordered cohomology group $(H_A, H_A^+)$, so that there exists $N_c \in \mathbb{N}$ such that $N_c[c] - 1 \in H_A$. Hence $N_c c - 1 = f + g \circ \sigma_A - g$ for some $f \in C(X_A, \mathbb{Z}_+)$ and $g \in C(X_A, \mathbb{Z})$. For a periodic point $x \in \text{Per}_n(X_A)$ with $\sigma_A^n(x) = x$, we have

$$\sum_{i=0}^{n-1} \{N_c c(\sigma_A^i(x)) - (1(\sigma_A^i(x)))\} = \sum_{i=0}^{n-1} f(\sigma_A^i(x)) \geq 0$$

where $\zeta_{A,c}(s)$ with $c = l - k$ such as

$$\zeta_{A,c}(s) = \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in \text{Per}_n(X_A)} \exp(-s \sum_{k=0}^{n-1} c(\sigma_A^k(x)))\right\}.$$  

(4.2)
and hence \( \sum_{i=0}^{n-1} c(\sigma_A^i(x)) \geq \frac{n}{N} \). The ordinary zeta function \( \zeta_A(t) \) of \((X_A, \sigma_A)\) is written as \( \zeta_A(t) = \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{n} |\text{Per}_n(X_A)|t^n \right\} \) where \(|\text{Per}_n(X_A)|\) denotes the cardinality of \(\text{Per}_n(X_A)\), so that \(\zeta_A(t) = \zeta_{A,1}(s)\) for \(c = 1\) when \(t = e^{-s}\). As the function \(\zeta_A(t)\) is analytic in \(t \in \mathbb{C} \) with \(|t| < \frac{1}{r_A}\), where \(r_A\) is the maximum eigenvalue of the matrix \(A\), we see that so is \(\zeta_{A,1}(s)\) in \(s \in \mathbb{C} \) with \(\text{Re}(s) > \log r_A\). Similarly, for the function \(c = l - k\), we have

\[
| \sum_{x \in \text{Per}_n(X_A)} \exp(-s \sum_{k=0}^{n-1} c(\sigma_A^k(x))) | \leq \sum_{x \in \text{Per}_n(X_A)} \{ \exp(-\text{Re}(s)) \}^{\frac{1}{r_A}}
\]

so that \(\zeta_{A,c}(s)\) is analytic at least in \(s \in \mathbb{C} \) with \(\text{Re}(s) > N_c \log r_A\). We actually know that \(\zeta_{A,c}(s)\) is analytic in the half plane \(\text{Re}(s) > h_{\text{top}}(S_{A,B}^{l,k}, \phi_A)\) the topological entropy of the flow of the suspension \((S_{A,B}^{l,k}, \phi_A)\) as seen in [1, Theorem 2.7] (cf. [6], [14], [18]).

In the second half of this section, we will study some relationships between zeta functions of one-sided flow equivalent topological Markov shifts. Suppose that \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are continuously orbit equivalent via a homeomorphism \(h : X_A \to X_B\) with continuous functions \(k_1, l_1 : X_A \to \mathbb{Z}_+\) and \(k_2, l_2 : X_B \to \mathbb{Z}_+\) satisfying (1.1) and (1.2), respectively. The functions \(c_1 = l_1 - k_1 \in C(X_A, \mathbb{Z})\) and \(c_2 = l_2 - k_2 \in C(X_B, \mathbb{Z})\) satisfy \(\Psi_h(1) = c_1\) and \(\Psi_h^{-1}(1) = c_2\). In [10], it has been shown that the homeomorphism \(h : X_A \to X_B\) induces a bijective correspondence \(\xi_h : P_{\text{orb}}(X_A) \to P_{\text{orb}}(X_B)\) between their periodic orbits such that the functions \(c_1, c_2\) measure the difference of the length of periods between \(\gamma \in P_{\text{orb}}(X_A)\) and \(\xi_h(\gamma) \in P_{\text{orb}}(X_B)\). As a result, the ordinary zeta functions \(\zeta_A(t), \zeta_B(t)\) are written in terms of dynamical zeta functions in the following way.

**Proposition 4.4 ([10]).** \(\zeta_A(t) = \zeta_{B,c_2}(s)\) and \(\zeta_B(t) = \zeta_{A,c_1}(s)\) for \(t = e^{-s}\).

The above formulae imply the formulae

(4.3) \(\zeta_{A,1}(s) = \zeta_{B,c_2}(s), \quad \zeta_{B,1}(s) = \zeta_{A,c_1}(s)\).

Assume that \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are one-sided flow equivalent via a homeomorphism \(h : X_A \to X_B\). They are continuously orbit equivalent, so that the maps \(\Psi_h : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z})\) and \(\Psi_h^{-1} : C(X_A, \mathbb{Z}) \to C(X_B, \mathbb{Z})\) are defined by (1.5). They are independent of the choice of the functions \(k_1, l_1 : X_A \to \mathbb{Z}_+\) and \(k_2, l_2 : X_B \to \mathbb{Z}_+\) satisfying (1.1) and (1.2), respectively ([10 Lemma 4.2]). We provide a lemma.

**Lemma 4.5.** For \(m \in \mathbb{Z}_+\) and \(f \in C(X_B, \mathbb{Z})\), \(g \in C(X_A, \mathbb{Z})\), we have

(i) \(\Psi_h(f)^m(x) = f^{l_1}_h(x)(h(x)) - f^{k_1}_h(x)(h(\sigma_A^m(x)))\) for \(x \in X_A\), so that
\[g^m(x) = \Psi_{h^{-1}}(g)_h^{l_1}(x)(h(x)) - \Psi_{h^{-1}}(g)_h^{k_1}(x)(h(\sigma_A^m(x))).\]

(ii) \(\Psi_{h^{-1}}(g)^m(y) = g_2^{l_2}(y)(h^{-1}(y)) - g_2^{k_2}(y)(h^{-1}(\sigma_B^m(y)))\) for \(y \in X_B\), so that
\[f^m(y) = \Psi_h(f)_h^{l_2}(y)(h^{-1}(y)) - \Psi_h(f)_h^{k_2}(y)(h^{-1}(\sigma_B^m(y))).\]
Proof. (i) As in [10, Lemma 4.3], the identity
\[
\sum_{i=0}^{m-1} \sum_{i'=0}^{l_1(x)-1} f(\sigma_A^i(h(\sigma_A^j(x)))) - \sum_{j'=0}^{k_1(x)-1} f(\sigma_A^{j+1}(x)))
\]
holds so that we see
\[
\sum_{i=0}^{m-1} \Psi_h(f)(\sigma_A(x)) = f^{l_1(x)}(h(x)) - f^{k_1(x)}(h(\sigma_A^m(x))).
\]

As \(\Psi_{h^{-1}} = (\Psi_h)^{-1}([10\text{ Proposition 4.5}])\), the desired identities hold. (ii) is similarly shown. \(\square\)

We generalize Proposition 4.4 such as in the following theorem.

**Theorem 4.6.** Assume that matrices \(A\) and \(B\) are irreducible and not permutation matrices. Suppose that \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are one-sided flow equivalent via a homeomorphism \(h : X_A \to X_B\). For \(f \in C(X_B, \mathbb{Z})\), \(g \in C(X_A, \mathbb{Z})\) such that their classes \([f] \in H^B, [g] \in H^A\) are order units of \((H^B, H^B_+), (H^A, H^A_+)\), respectively, the following formulae hold:
\[
\zeta_A.g(s) = \zeta_B,\Psi_{h^{-1}}(g)(s), \quad \zeta_B.f(s) = \zeta_A,\Psi_h(f)(s).
\]

**Proof.** We may assume that \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are continuously orbit equivalent via a homeomorphism \(h : X_A \to X_B\). For \(f \in C(X_B, \mathbb{Z})\) such that the class \([f]\) is an order unit of \((H^A, H^A_+)\), we will prove the equality \(\zeta_B.f(s) = \zeta_A,\Psi_h(f)(s)\). A routine argument as in [11, p. 100] shows
\[
\zeta_A,\Psi_h(f)(s) = \prod_{\gamma \in P_{orb}(X_A)} (1 - t^{\beta_\gamma(\Psi_h(f))})^{-1} \quad \text{where} \quad t = e^{-s}
\]
and \(\beta_\gamma(\Psi_h(f)) = \sum_{i=0}^{p-1} \Psi_h(f)(\sigma_A^i(x))\) for a periodic orbit
\[
\gamma = \{x, \sigma_A(x), \ldots, \sigma_A^{p-1}(x)\} \in P_{orb}(X_A).
\]

We see the following formula by Lemma 4.5
\[
\sum_{i=0}^{p-1} \Psi_h(f)(\sigma_A^i(x)) = f^{l_1(x)}(h(x)) - f^{k_1(x)}(h(\sigma_A^m(x))}, \quad x \in X_A.
\]

As \(\gamma = \{x, \sigma_A(x), \ldots, \sigma_A^{p-1}(x)\} \in P_{orb}(X_A)\) and \(\sigma_A^p(x) = x\), we have
\[
\sum_{i=0}^{p-1} \Psi_h(f)(\sigma_A^i(x)) = f^{l_1(x)}(h(x)) - f^{k_1(x)}(h(x)).
\]

Since we may identify the periodic orbits \(P_{orb}(X_A)\) of the one-sided topological Markov shift \((X_A, \sigma_A)\) with the periodic orbits \(P_{orb}(X_A)\) of the two-sided topological Markov shift \((X_A, \sigma_A)\), an argument in [10, Section 6] shows that there exists a bijective correspondence \(\xi_h : P_{orb}(X_A) \to P_{orb}(X_B)\). By [10, Lemma 6.5], we see...
that \( \xi_h(\gamma) \) has its period \( l_1^p(x) - k_1^p(x) \) and hence \( \xi_h(\gamma) = \{ \sigma_B^i(h(x)) \mid k_1^p(x) \leq i \leq l_1^p(x) - 1 \} \) so that

\[
\sum_{i=k_1^p(x)}^{l_1^p(x)-1} f(\sigma_B^i(h(x))) = \beta_{\xi_h(\gamma)}(f).
\]

By using (4.4), we have

\[
\beta_\gamma(\Psi_h(f)) = \sum_{i=0}^{p-1} \Psi_h(f)(\sigma_A^i(x)) = \sum_{i=k_1^p(x)}^{l_1^p(x)-1} f(\sigma_B^i(h(x))) = \beta_{\xi_h(\gamma)}(f).
\]

Since \( \xi_h : P_{\text{orb}}(X_A) \to P_{\text{orb}}(X_B) \) is bijective, one sees that

\[
\zeta_{A,\Psi_h(f)}(s) = \prod_{\eta \in P_{\text{orb}}(X_B)} (1 - t^{\beta_\eta(f)})^{-1} = \zeta_{B,f}(s).
\]

The other equality \( \zeta_{B,\Psi^{-1}_h(g)}(s) = \zeta_{A,g}(s) \) is similarly shown. \( \square \)

**Corollary 4.7.** Assume that a matrix \( A \) is irreducible and not any permutation matrix. The set \( Z(X_A,\sigma_A) \) of dynamical zeta functions of \( (X_A,\sigma_A) \) whose potential functions are order units of the ordered cohomology group \( (H^A, H^+_A) \) is invariant under one-sided flow equivalence.

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**References**


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