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SOME COUNTEREXAMPLES RELATED TO THE STATIONARY KIRCHHOFF EQUATION

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ABSTRACT. In this note we consider the stationary Kirchhoff equation

$$\begin{cases} -M(\|u\|^2)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where M is a continuous positive function and $\|\cdot\|$ is the standard norm in $H_0^1(\Omega)$. We show that the equation does not enjoy the usual comparison principles (both weak or strong) nor the sub and supersolutions method.

1. INTRODUCTION

The purpose of this paper is to gain some insight into basic properties of the nonlocal elliptic equation

$$\begin{cases} -M(\|u\|^2)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a domain of $\mathbb{R}^N$, M is a continuous, positive function and f a given nonlinearity. The norm $\|\cdot\|$ is the standard norm in $H_0^1(\Omega)$, that is,

$$\|u\|^2 = \int_\Omega |\nabla u(x)|^2 dx.$$

This equation is usually referred to nowadays as *Kirchhoff's equation*, because it is a generalization of the original one

$$\rho u_{tt} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L |u_x|^2 dx \right) u_{xx} = 0, \tag{1.2}$$

which was first proposed by Kirchhoff in [12] as an extension of the classical wave equation to model free vibrations of elastic strings. The parameters ρ, P_0, h, E, L involved in (1.2) are all positive constants related to the model.

A great deal of papers have been devoted to the study of (1.1) with different hypotheses on M and f. To note just a few, we refer the reader to the recent ones [3], [8], [9] (see also the references therein).

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Most of the above papers rely on the use of variational methods to obtain solutions of (1.1). But there is another group of papers which use the standard method of sub and supersolutions, both for problem (1.1) and for its immediate generalizations substituting the Laplacian by the $p-$Laplacian or even the $p(x)-$Laplacian (cf. [11], [13], [1], [6]). With some unclear proofs (in our opinion), the authors claim that the comparison principle and the method of sub and supersolutions hold only assuming some monotonicity in the function M. To be more precise, the comparison principle is claimed to hold if M is nondecreasing and bounded from below in Theorem 2.2 of [11], Theorem 3.2 of [13], Lemma 2.1 of [1] and Lemma 2.1 in [6]. On the other hand, the method of sub and supersolutions for (1.1) when both M and f are nondecreasing is considered in Theorems 2.3 and 2.4 of [11], Theorem 3.3 of [13] and Theorems 2.2 and 2.3 of [1] (also in Proposition 2.2 of [6] for some related systems). The proofs are not the same in all cases, but they are not correct in light of the results of the present paper.

As a matter of fact, our main intention in this note is to show that actually comparison principles (both weak and strong) and the sub and supersolutions method are in general not valid for problem (1.1). The only hypotheses we need to assume are $N \geq 3$ and that M, aside from being positive and continuous, verifies:

$$\text{(H)} \qquad \text{there exist } t_2 > t_1 > 0 \text{ such that } \frac{M(t_2)}{t_2^{\frac{2}{N-2}}} > \frac{M(t_1)}{t_1^{\frac{2}{N-2}}}.$$

This hypothesis holds for instance when $M(t)t^{-\frac{2}{N-2}}$ is not nondecreasing, and it is compatible with the usual hypotheses assumed in previous works (in particular, in [11], [13], [1], [6]) that M is nondecreasing and bounded from below. The case $N = 1$ can also be considered, but the inequality above has to be reversed. However, we assume for simplicity that $N \geq 3$ throughout.

Let us proceed to the statement of our results. It is important to mention that we will always be dealing with solutions, subsolutions and supersolutions in the weak sense in $H_0^1(\Omega)$. We begin with the statements regarding the weak and strong comparison principles. We denote $L(u) = -M(\|u\|^2)\Delta u$ for brevity.

Theorem 1. *Assume $N \geq 3$ and M is continuous and positive and verifies* (H). *Then there exist a domain Ω and functions $\underline{u}, \bar{u} \in H_0^1(\Omega) \cap C(\overline{\Omega})$ such that*

$$\begin{cases} L(\underline{u}) \leq L(\bar{u}) & \text{in } \Omega, \\ \underline{u} = \bar{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

but $\underline{u} > \bar{u}$ somewhere in Ω.

Theorem 2. *Assume $N \geq 3$ and M is continuous and positive and verifies* (H). *Then there exist a domain Ω and functions $\underline{u}, \bar{u} \in H_0^1(\Omega) \cap C(\overline{\Omega})$ such that*

$$\begin{cases} L(\underline{u}) \leq L(\bar{u}) & \text{in } \Omega, \\ \underline{u} \leq \bar{u} & \text{in } \Omega, \\ \underline{u} \not\equiv \bar{u} & \text{in } \Omega, \end{cases}$$

but $\underline{u} = \bar{u}$ somewhere in Ω.

Next, we consider the nonvalidity of the method of sub and supersolutions in its standard form. However, it is worth mentioning that a "strengthened" method of sub and supersolutions has been introduced in [2] (see also [4]) for the study of (1.1) and related problems.

For $\theta > 0$ we restrict oursleves to the "linear" problem:

$$\begin{cases} L(u) = \theta u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\theta > 0$. Then we have:

Theorem 3. *Assume $N \geq 3$ and M is continuous and positive and verifies* (H). *Then there exist a real number $\theta > 0$, a domain Ω and functions $\underline{u}, \bar{u} \in H_0^1(\Omega) \cap C(\overline{\Omega})$ such that $\underline{u}$ is a subsolution of* (1.3)*, $\bar{u}$ is a supersolution of* (1.3)*, $\underline{u} \leq \bar{u}$ in Ω, but problem* (1.3) *does not admit any solution in the order interval $[\underline{u}, \bar{u}]$.*

It is also worthy to remark that the comparison principle does not hold for instance if the function $M(t)t^{1/2}$ is not strictly monotone, as a consequence of Theorem 1 in [3]. On the other hand, when M is nondecreasing (resp. nonincreasing) and $M(t)t$ is nonincreasing (resp. nondecreasing), the comparison principle holds (cf. Proposition 2 in [7]). For completeness, we also include an interesting result about the relation between (restricted forms of) the method of sub and supersolutions and the comparison principle.

Theorem 4. *Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \geq 1$, and assume M is continuous and bounded away from zero in $[0, +\infty)$, with $M(t)t^{1/2}$ increasing. Then the following two assertions are equivalent:*

(a) *The method of sub and supersolutions is valid for* (1.1) *for bounded Carathéodory functions f which are nonnegative and nondecreasing in the second variable, provided both the sub and the supersolutions are bounded.*
(b) *The comparison principle is valid in the following form: if $\underline{u}, \bar{u} \in H^1(\Omega) \cap L^\infty(\Omega)$ verify $L\underline{u} \leq L\bar{u}$ in Ω with $\underline{u} \leq \bar{u} = 0$ on $\partial\Omega$ and $L\bar{u} \in L^\infty(\Omega)$, $L\bar{u} \geq 0$ in Ω, then $\underline{u} \leq \bar{u}$ in Ω.*

Remark 1. Since we are assuming that the sub and supersolutions are bounded, the boundedness hypothesis on f in (a) only concerns its x-dependence. It is to be stressed, however, that more general versions of the method could be considered.

Corollary 5. *Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \geq 1$, and assume M is continuous, nonincreasing and bounded away from zero in $[0, +\infty)$, with $M(t)t^{1/2}$ increasing. Then the comparison principle and the method of sub and supersolutions hold for problem* (1.1) *as in (a) and (b) above.*

Let us finally mention that similar results can be obtained when the H_0^1−norm in (1.1) is replaced by another similar term, such as

$$\int_\Omega u(x)^q dx,$$

for some $q > 0$, with the corresponding changes in hypothesis (H). The resulting equation appears in the study of the population of bacteria subject to spreading when $q = 1$ (see [5]), while it reduces to the well-known Carrier's equation appearing in the study of nonlinear deflection of beams for $q = 2$ (cf. [10]).

The proofs will be carried out in the next section.

2. Proofs

In this section we will prove Theorems 1, 2, 3 and 4. The proofs of the first three all follow by constructing suitable counterexamples which involve the principal eigenfunction of the Laplacian in balls.

Denote by B_R the ball of radius $R > 0$ centered at the origin and $B = B_1$. Let $\phi \in C^\infty(\overline{B})$ be the positive eigenfunction associated to the first eigenvalue λ_1 of $-\Delta$ in B, normalized by $\phi(0) = 1$. Then ϕ is radially symmetric and radially decreasing.

For $R > 0$, let

$$\phi_R(x) = \phi\left(\frac{x}{R}\right), \quad x \in B_R.$$

Using that ϕ is radially decreasing it follows that $\phi_{R_2} \geq \phi_{R_1}$ in B_{R_1} whenever $R_2 > R_1$. Moreover, it is clear that $-\Delta\phi_R = \frac{\lambda_1}{R^2}\phi_R$ in B_R, $\phi_R(0) = 1$ while

$$\|\phi_R\|^2 = \int_{B_R} |\nabla\phi_R(x)|^2 dx = R^{N-2}\int_B |\nabla\phi(x)|^2 dx =: AR^{N-2}.$$

For t_1, t_2 as in hypothesis (H), we denote $R_1 = (t_1/A)^{\frac{1}{N-2}}$, $R_2 = (t_2/A)^{\frac{1}{N-2}}$. Then it follows that $R_1 < R_2$ and

$$\frac{M(AR_1^{N-2})}{R_1^2} < \frac{M(AR_2^{N-2})}{R_2^2}. \tag{2.1}$$

The counterexamples which prove our theorems are based on the eigenfunctions ϕ_R when R takes the particular values R_1 and R_2.

Proof of Theorem 1. By (2.1) and the continuity of M, we can choose $\eta > 1$ such that

$$\frac{M(A\eta^2 R_1^{N-2})}{R_1^2}\eta < \frac{M(AR_2^{N-2})}{R_2^2}.$$

Let $\Omega = B_{R_2}$ and set $\bar{u} = \phi_{R_2}$. Define

$$\underline{u} = \begin{cases} \eta\phi_{R_1} & \text{in } B_{R_1}, \\ 0 & \text{in } B_{R_2} \setminus B_{R_1}. \end{cases}$$

It is clear that

$$L(\bar{u}) = M(\|\phi_{R_2}\|^2)(-\Delta\phi_{R_2}) = \lambda_1\frac{M(AR_2^{N-2})}{R_2^2}\phi_{R_2}$$

in B_{R_2}. Similarly, in B_{R_1} we see that

$$L(\underline{u}) = \lambda_1\frac{M(A\eta^2 R_1^{N-2})}{R_1^2}\eta\phi_{R_1} \leq \lambda_1\frac{M(AR_2^{N-2})}{R_2^2}\phi_{R_2}.$$

In $B_{R_2} \setminus B_{R_1}$ it is clear that $L(\underline{u}) = 0 \leq L(\bar{u})$. It follows in a standard way that $L(\underline{u}) \leq L(\bar{u})$ in B_{R_2} in the weak sense. Then the functions $\underline{u}$ and $\bar{u}$ verify the properties in the statement of the theorem. Moreover, $\underline{u}(0) = \eta > 1 = \bar{u}(0)$, so that the comparison principle does not hold. □

Proof of Theorem 2. We choose $\Omega = B_{R_2}$ and $\underline{u}$, $\bar{u}$ as above with $\eta = 1$. Then clearly $L(\underline{u}) \leq L(\bar{u})$ in B_{R_2}, while $\bar{u} = \phi_{R_2} \geq \phi_{R_1} = \underline{u}$ in B_{R_1} (in $B_{R_2} \setminus B_{R_1}$ the inequality $\bar{u} > \underline{u}$ is evident). Hence $\overline{u} \geq \underline{u}$ with $\overline{u} \not\equiv \underline{u}$. Moreover, $\overline{u}(0) = \underline{u}(0) = 1$, so the proof is concluded. □

Proof of Theorem 3. As before, we choose $\Omega = B_{R_2}$ and take

$$\theta = \lambda_1\frac{M(AR_1^{N-2})}{R_1^2} \tag{2.2}$$

in (1.3). Consider the same functions $\bar{u}$ and $\underline{u}$ as in the proof of the previous theorems with $\eta = 1$. Then by (2.1) we obtain

$$L(\bar{u}) = \lambda_1 \frac{M(AR_2^{N-2})}{R_2^2}\phi_{R_2} \geq \theta\bar{u}$$

in B_{R_2}, while $L(\underline{u}) \leq \theta\underline{u}$ in B_{R_2} in the weak sense. Moreover, $\underline{u} \leq \bar{u}$ in B_{R_2}.

But we claim that the problem (1.3) does not admit any solution $u \in H_0^1(\Omega)$ verifying $\underline{u} \leq u \leq \bar{u}$. To show the claim, assume u is any such solution. Then $u = 0$ on $\partial\Omega$ in the sense of traces, while

$$-\Delta u = \frac{\theta}{M(\|u\|^2)}u \qquad \text{in } B_{R_2}.$$

It follows by the uniqueness and simplicity of the principal eigenvalue of $-\Delta$ that, for some $\mu > 0$:

$$\frac{\theta}{M(\|u\|^2)} = \frac{\lambda_1}{R_2^2} \quad \text{and} \quad u = \mu\phi_{R_2}. \tag{2.3}$$

Since $\underline{u} \leq \mu\phi_{R_2} \leq \bar{u}$, we see in particular that $1 = \underline{u}(0) = \bar{u}(0) = \mu$. Then from (2.3) we have

$$\theta = \lambda_1 \frac{M(AR_2^{N-2})}{R_2^2},$$

which contradicts the definition (2.2) of θ, thanks to (2.1). This concludes the proof. □

Proof of Theorem 4. Assume (a) holds. Let $h = L\bar{u} \in L^\infty(\Omega)$, and consider the problem

$$\begin{cases} -M(\|w\|^2)\Delta w = h(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Observe that we may assume $h \not\equiv 0$, otherwise the comparison principle is a simple consequence of the maximum principle for the Laplacian by the positivity of M.

We claim that $\bar{w} = K\bar{u}$ is a supersolution of (2.4) if $K > 1$ is large enough. Indeed, by our assumption on M we have

$$M\left(K^2\int_\Omega |\nabla\bar{u}(x)|^2dx\right)K \geq M\left(\int_\Omega |\nabla\bar{u}(x)|^2dx\right).$$

Since $-\Delta\bar{u} \geq 0$ in Ω we deduce $L\bar{w} \geq h$ in Ω.

We remark that, as a consequence of standard regularity theory, $\bar{u} \in C^{1,\alpha}(\overline{\Omega})$ for every $\alpha \in (0, 1)$. By the strong maximum principle, and since h is not trivial, we have $\bar{u} > 0$ in Ω with $\frac{\partial\bar{u}}{\partial\nu} < 0$ on $\partial\Omega$, where ν stands for the outward unit normal field. Since $\underline{u}$ is bounded in Ω, it follows that $\underline{u} \leq K\bar{u}$ in Ω if K is taken large enough.

Thus the method of sub and supersolutions, which we assume to be valid, can be applied, and we obtain a solution w of (2.4) verifying $\underline{u} \leq w \leq M\bar{u}$. However, problem (2.4) admits a unique solution, which is necessarily $\bar{u}$ by Theorem 1 in [3]. Therefore $w = \bar{u}$ and $\underline{u} \leq \bar{u}$ is shown.

Conversely, assume (b) holds. If we have a subsolution $\underline{u}$ and a supersolution $\bar{u}$ of (1.1) verifying $\underline{u} \leq \overline{u}$ in Ω, we may consider, in a standard fashion, the problem:

$$\begin{cases} -M(\|u\|^2)\Delta u = f(x, \underline{u}) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

By Theorem 1 in [3], there exists a unique solution of (2.5), which will be denoted by u_1. Since f is nondecreasing, the comparison principle gives that $u_1 \geq \underline{u}$ in Ω. Using also the monotonicity of f in the second variable and the comparison principle we get $u_1 \leq \bar{u}$ in Ω.

Now define recursively the sequence $\{u_n\}$, where each function u_n is obtained as the solution of (2.5) with $\underline{u}$ replaced by u_{n-1}. It follows as before that $\underline{u} \leq u_n \leq \bar{u}$ in Ω, and that $\{u_n\}$ is an increasing sequence.

The boundedness of $\underline{u}$ and $\bar{u}$ implies that the sequence $\{u_n\}$ is bounded. Therefore, since M is bounded away from zero, we have $|\Delta u_n|$ bounded, which automatically gives $C^{1,\alpha}(\overline{\Omega})$ bounds for $\{u_n\}$ for every $\alpha \in (0,1)$. Thus $u_n \to u$ in $C^1(\overline{\Omega})$, where $u := \sup_{n\in\mathbb{N}} u_n$. Passing to the limit in the equation verified by u_n, we see that u is a solution of (1.1) which verifies $\underline{u} \leq u \leq \bar{u}$. The proof is concluded. $\square$

Proof of Corollary 5. It is immediate from Theorem 4, since the comparison principle holds by Proposition 2 in [7]. $\square$

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