PARITY OF RANKS OF ELLIPTIC CURVES WITH EQUIVALENT MOD p GALOIS REPRESENTATIONS

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ABSTRACT. Given two elliptic curves E_1 and E_2 defined over the field of rational numbers $\mathbb{Q}$ that have good and ordinary reduction at an odd prime p , and have equivalent, irreducible mod p Galois representations, we study the variation of the parity of Selmer ranks and analytic ranks of E_1 and E_2 over certain number fields.

1. INTRODUCTION

Let K be a number field. Fix an odd prime p and let $\bar{K}$ denote a fixed algebraic closure of K . For an algebraic field extension L/K and an elliptic curve E defined over K , the p -Selmer group $\text{Sel}_p(E/L)$ of E over L is defined by the exact sequence

$$(1) \quad 0 \longrightarrow \text{Sel}_p(E/L) \longrightarrow H^1(L, E[p^\infty]) \xrightarrow{\oplus_w \delta_w} \prod_w H^1(L_w, E(\overline{L}_w)).$$

Here, w varies over the set of primes of L , $\overline{L}_w$ denotes a fixed algebraic closure of L_w , $E(\overline{L}_w)$ denotes the $\overline{L}_w$ points of E , $E[p^\infty]$ denotes the p -power torsion points of E defined over $\bar{K}$ and $\oplus_w \delta_w$ denotes the natural restriction map.

Let E_1 and E_2 be two elliptic curves defined over the field of rational numbers $\mathbb{Q}$. We assume that E_1 and E_2 have good and ordinary reduction at the prime p . Let K be a number field containing the p -th roots of unity. For an abelian group M , let $M[p^n]$ denote the p^n torsion subgroup of M . We say that E_1 and E_2 are congruent at the prime p if $E_1[p] \cong E_2[p]$ as modules over the absolute Galois group of $\mathbb{Q}$. The aim of this article is to study the variation of the parity of the p -Selmer ranks and analytic ranks of E_1 and E_2 over K .

Let N_i denote the conductor of E_i over K and $\bar{N}_i$ denote the prime to p conductor of the Galois module $E_i[p]$ over K for $i = 1, 2$. Let Σ be the finite set of primes of K containing the primes of bad reduction of E_1 and E_2 , the infinite primes and the primes above p . Put

$$(2) \quad \Sigma_0 := \{v \in \Sigma \mid v|N_1/\bar{N}_1 \text{ or } v|N_2/\bar{N}_2\}.$$

Let S_{E_i} denote the set of primes $v \in \Sigma_0$ such that E_i has split multiplicative reduction at v . Let $s_p(E_i/K)$ denote the rank of the Pontryagin dual of $\text{Sel}_p(E_i/K)$, for $i = 1, 2$. Then we have the following result.

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Theorem 1.1. *Let E_1 and E_2 be two elliptic curves defined over the field of rational numbers $\mathbb{Q}$ with good and ordinary reduction at the prime p . We further assume that the following holds:*

- (a) $E_1[p]$ is an irreducible $\text{Gal}(\bar{K}/K)$ -module.
- (b) $K = K(\mu_p)$ where μ_p denotes the group of the p -th root of unity.
- (c) $\text{Sel}_p(E_1/K_{\text{cyc}})[p]$ is finite where K_{cyc} denotes the cyclotomic $\mathbb{Z}_p$ -extension of K .

If E_1 and E_2 are congruent at the prime p , then we have that

$$(3) \quad s_p(E_1/K) + |S_{E_1}| \equiv s_p(E_2/K) + |S_{E_2}| \pmod{2}.$$

Remark 1.2. (i) If E_1 and E_2 are congruent at the prime p , then the assumption (a) holds for E_1 if and only if the same holds for E_2 .

(ii) The assumption (c) holds for E_1 if and only if it holds for E_2 (see [Gr1, cf. Page 237] or Theorem 2.2 below).

We also mention that the assumption (c) holds for E_1 if and only if the cyclotomic Iwasawa μ -invariant of E_1 vanishes. In fact, there is no example known for which the assumption (a) holds but assumption (c) does not hold. There is a conjecture by Greenberg which asserts that if E is an elliptic curve defined over $\mathbb{Q}$ with good ordinary reduction at p , then there exists a $\mathbb{Q}$ -isogenous elliptic curve E' whose cyclotomic Iwasawa μ -invariant vanishes ([Gr, Conjecture 1.11]). In particular, if $E[p]$ is an irreducible $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ -module, then the cyclotomic μ -invariant of E should vanish. This conjecture is false over a general number field. In his thesis, Drinen found examples of elliptic curves E defined over $\mathbb{Q}$ with good ordinary reduction at p and number fields K such that every member of the K -isogeny class of E has the positive cyclotomic μ -invariant (see [Dr, Section 4] or [Dr1]). The examples found in [Dr] do not satisfy the assumption (a). We refer the reader to Remarks 3.4 and 3.7 and Section 4 for a discussion on the assumption (b).

It is well known that the rank of an elliptic curve E defined over a number field K is determined by the $\text{Gal}(\bar{K}/K)$ -module structure of $E[p^\infty]$. This is a consequence of the Tate conjecture for elliptic curves proved by Faltings (see [Gr1, Page 232]). Although the Galois module structure of $E[p]$ does not determine the rank of E over K , motivated by the conjecture of Tate, Greenberg and Vatsal ([GV]) studied the variation of various algebraic, arithmetic and analytic invariants associated to elliptic curves under congruence. Specifically, they considered the cyclotomic algebraic and analytic Iwasawa invariants associated to elliptic curves defined over $\mathbb{Q}$ and provided a precise formula for the difference of the Iwasawa λ -invariants of two congruent elliptic curves. The proof of Theorem 1.1 crucially relies on the methods and results of Greenberg and Vatsal. We use recent results on the special cases of the parity conjecture (see Theorem 4.3 below) and Theorem 1.1 to compare the analytic rank of congruent elliptic curves.

It is an interesting problem to prove a relation analogous to (3) for the analytic rank of congruent elliptic curves independent of Theorem 1.1. Moreover, this may help in producing new examples of elliptic curves for which p -parity conjecture (see Conjecture 1) is true. Given an elliptic curve E defined over $\mathbb{Q}$, Rubin and Silverberg (see [RS]) produced certain families of infinitely many elliptic curves which are congruent to E at $p = 3$ and 5 . In Section 4 of this article, we further compute the parity of p -Selmer ranks and analytic ranks of elliptic curves in such families over certain number fields using Theorem 1.1.

To prove our result, we shall need an equivalent but different definition of the Selmer groups over K_{cyc} of elliptic curves with good and ordinary reduction at p . In Section 2, we recall this equivalent definition which is different from the definition (1) of the Selmer groups of elliptic curves defined over K . We also recall the definition of the Selmer group of the residual Galois representation of an elliptic curve at p . In Section 3, we compare the parity of the λ -invariants of E_1 and E_2 . Using this comparison we prove our main result which compares the parity of the Selmer ranks and the analytic ranks of E_1 and E_2 in Section 4. To compare the parity of the λ -invariants of E_1 and E_2 we need to use the results of Greenberg in [Gr1]. Further, we use the results of Hachimori and Matsuno which describe the p -power torsion points of an elliptic curve over the cyclotomic $\mathbb{Z}_p$ -extension of a non-archimedean local field (see [HM]). In Section 4, we discuss several examples of elliptic curves to illustrate applications of our result and also to explain the assumptions in Theorem 1.1.

2. SELMER GROUP

Let E be an elliptic curve defined over the number field K and N_K be the conductor of E over K . Suppose that E has good and ordinary reduction at the primes above p and that $E[p]$ is an irreducible $\text{Gal}(\bar{K}/K)$ -module. Let Σ be a finite set of primes of K containing the primes above p , the infinite primes and primes of bad reduction of E . Let K_Σ denote the maximal algebraic extension of K unramified outside Σ .

Let K_{cyc} denote the cyclotomic $\mathbb{Z}_p$ extension of K and $\Gamma := \text{Gal}(K_{cyc}/K)$. Put $\Lambda := \varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma^{p^n}] \cong \mathbb{Z}_p[[X]]$, where the inverse limit is taken over the natural projection maps.

Let $\tilde{E}_v$ denote the reduction of the elliptic curve E at the prime $v|p$. For an abelian group M , put $M[p^\infty] := \bigcup_n M[p^n]$. Since E has good ordinary reduction at the primes dividing p , we have that the $\text{Gal}(\bar{K}_v/K_v)$ -module $\tilde{E}_v[p^\infty]$ is unramified for every prime $v|p$. For a Galois extension $L \subset K_\Sigma$ of K and a prime $v \nmid p$ of K , put

$$\mathcal{H}_v(L, E) := \prod_{w|v} H^1(L_w, E[p^\infty]).$$

If $v|p$, then put

$$\mathcal{H}_v(L, E) := \prod_{w|v} H^1(I_w, \tilde{E}_v[p^\infty])$$

where I_w denotes the inertia subgroup of $\text{Gal}(\bar{L}_w/L_w)$. Let Σ_0 be a subset of Σ not containing the primes above p and the infinite primes. The *imprimitive* Σ_0 -Selmer group $\text{Sel}_p^{\Sigma_0}(E/L)$ of E over L is defined by the following exact sequence:

$$(4) \quad 0 \longrightarrow \text{Sel}_p^{\Sigma_0}(E/L) \longrightarrow H^1(K_\Sigma/L, E[p^\infty]) \longrightarrow \prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{H}_v(L, E).$$

When Σ_0 is empty, we drop it from the notation $\text{Sel}_p^{\Sigma_0}(E/L)$ and write $\text{Sel}_p(E/L)$ (see [Gr1, section 1] and [GV, Page 42] for the equivalence of definitions (1) and (4)). We next define the Selmer group of the Galois module $E[p]$. For a prime $v \nmid p$ of K , put

$$\mathcal{H}_v(L, E[p]) := \prod_{w|v} H^1(L_w, E[p]).$$

If $v|p$, then put

$$\mathcal{H}_v(L, E[p]) := \prod_{w|v} H^1(I_w, \tilde{E}_v[p]).$$

The *imprimitive* Σ_0 -Selmer group $\text{Sel}_p^{\Sigma_0}(E[p]/L)$ of $E[p]$ over L is defined by the following exact sequence:

$$0 \longrightarrow \text{Sel}_p^{\Sigma_0}(E[p]/L) \longrightarrow H^1(K_{\Sigma}/L, E[p]) \longrightarrow \prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{H}_v(L, E[p]).$$

Let $\bar{N}_K$ denote the prime to p -conductor of the Galois module $E[p]$. For a finite prime w of K_{cyc} lying above a finite prime $v \nmid p$ of K , we consider the natural homomorphism

$$\gamma_w : H^1(K_{cyc, w}, E[p]) \longrightarrow H^1(K_{cyc, w}, E[p^{\infty}])[p]$$

and for a prime $w|v$ of K_{cyc} such that $v|p$ consider the map

$$\gamma_w : H^1(I_w, \tilde{E}_v[p]) \longrightarrow H^1(I_w, \tilde{E}_v[p^{\infty}])[p].$$

Using [EPW, Lemma 4.1.2] the next lemma can be proved as in [SS, Theorem 3.4]. We reproduce the proof for the convenience of the reader.

Lemma 2.1. *Let $v \nmid N_K/\bar{N}_K$ be a finite prime of K . Then, γ_w is an injective homomorphism for every prime $w|v$ of K_{cyc} .*

Proof. First suppose that $w|p$. In this case the kernel of γ_w is equal to $H^0(I_w, \tilde{E}_{p^{\infty}})/p$. Since $\tilde{E}_{p^{\infty}}$ is divisible and unramified, the kernel of $\gamma_w = 0$. Now consider a prime $w \nmid pN_K/\bar{N}_K$. Then by an argument similar to [EPW, Lemma 4.1.2], we get that $H^0(K_{cyc, w}, E[p^{\infty}])$ is divisible. Therefore, the kernel $H^0(K_{cyc, w}, E[p^{\infty}])/p$ of γ_w is trivial. $\square$

As in Section 1 we consider two elliptic curves E_1 and E_2 defined over $\mathbb{Q}$ and state the following assumptions for further reference in the article.

Assumption 1. (i) E_1 (resp. E_2) has good and ordinary reduction at p .

(ii) $E_1[p]$ is an irreducible $\text{Gal}(\bar{K}/K)$ -module.

(iii) $K = K(\mu_p)$.

(iv) $\text{Sel}_p(E_1/K_{cyc})[p]$ is a finite group where K_{cyc} denote the cyclotomic $\mathbb{Z}_p$ -extension of K .

If Assumption 1(iv) holds, then the Pontryagin dual $X(E/K_{cyc})$ of $\text{Sel}_p(E/K_{cyc})$ is a finitely generated $\mathbb{Z}_p$ -module. Now suppose that the Assumption 1(ii) holds. Then $E(K)[p] = 0$ and therefore [Gr, Proposition 4.14] implies that $\text{Sel}_p(E/K_{cyc})$ is divisible. We mention that [Gr, Proposition 4.14] is stated only when Σ_0 is the empty set. To prove the general case, consider the exact sequence

$$(5) \quad 0 \longrightarrow \text{Sel}(E/K_{cyc}) \longrightarrow \text{Sel}^{\Sigma_0}(E/K_{cyc}) \xrightarrow{\psi} \prod_{v \in \Sigma_0} H_v(K_{cyc}, E) \longrightarrow 0.$$

The surjectivity of ψ is a consequence of the Assumption 1(iv) which implies that $\text{Sel}(E/K_{cyc})$ is a torsion Λ -module (cf. [GV, Proposition 2.1, Corollary 2.3]; see also [HV, Theorem 7.2]). Now, it is well known that $H_v(K_{cyc}, E)$ is a divisible group for every finite prime v of K (see [Gr, Lemma 4.5 and successive paragraph]). Hence, $\text{Sel}^{\Sigma_0}(E/K_{cyc})$ is a divisible group. From [Gr2, Proposition 2] and using the fact

that there are finitely many primes of K_{cyc} lying above a finite prime of K , we see that the Pontryagin dual of $\mathcal{H}_v(K_{cyc}, E)$ is a finitely generated $\mathbb{Z}_p$ -module. Put

$$\lambda_E^{\Sigma_0} := \text{rank}_{\mathbb{Z}_p} X^{\Sigma_0}(E/K_{cyc}).$$

Since $\text{Sel}_p^{\Sigma_0}(E/K_{cyc})$ is divisible, $X^{\Sigma_0}(E/K_{cyc})$ is p -torsion free and

$$\lambda_E^{\Sigma_0} = \text{rank}_{\mathbb{Z}_p/p} X^{\Sigma_0}(E/K_{cyc})/p.$$

Next we compare the Selmer group of an elliptic curve E with the Selmer group of $E[p]$ over K under Assumption 1. For this we use the method of Greenberg ([Gr1]) and Greenberg-Vatsal ([GV]). Greenberg and Vatsal considered the case $K = \mathbb{Q}$ and Σ_0 contains all the primes of bad reduction of E_i , $i = 1, 2$. A similar comparison has been made in [SS, Theorem 3.4] where we study the simultaneous vanishing of imprimitive Selmer groups of congruent elliptic curves under certain stronger conditions on the set Σ_0 .

We briefly explain below how the result of Greenberg and Vatsal can be extended over more general number fields and for any set Σ_0 containing primes dividing $N_K/\bar{N}_K$. Let $\Sigma_0 \subset \Sigma \setminus p$ be a finite set of primes of K containing the prime divisors of $N_K/\bar{N}_K$. Consider the commutative diagram

$$(6) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{Sel}_p^{\Sigma_0}(E[p]/K_{cyc}) & \longrightarrow & H^1(K_{\Sigma}/K_{cyc}, E[p]) & \longrightarrow & \prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{H}_v(K_{cyc}, E[p]) \\ & & \downarrow f & & \downarrow g & & \downarrow h \\ 0 & \longrightarrow & \text{Sel}_p^{\Sigma_0}(E/K_{cyc})[p] & \longrightarrow & H^1(K_{\Sigma}/K_{cyc}, E[p^{\infty}])[p] & \longrightarrow & \prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{H}_v(K_{cyc}, E)[p]. \end{array}$$

Assumption 1(ii) implies that $E(K)[p] = 0$. From the Nakayama lemma we get that $E(K_{cyc})[p] = 0$. This implies that $H^0(K_{\Sigma}/K_{cyc}, E[p^{\infty}])/p = 0$ and therefore g is injective. From Lemma 2.1 h is injective. Now an application of the snake lemma implies that the map

$$(7) \quad f : \text{Sel}_p^{\Sigma_0}(E[p]/K_{cyc}) \longrightarrow \text{Sel}_p^{\Sigma_0}(E/K_{cyc})[p]$$

is an isomorphism. Therefore, $\text{Sel}_p^{\Sigma_0}(E/K_{cyc})[p]$ is finite if and only if $\text{Sel}_p^{\Sigma_0}(E[p]/K_{cyc})$ is finite and

$$(8) \quad \lambda^{\Sigma_0} = \text{rank}_{\mathbb{Z}_p/p} \text{Sel}_p^{\Sigma_0}(E[p]/K_{cyc}).$$

We mention that here we have used the fact that the Pontryagin dual of $\text{Sel}_p^{\Sigma_0}(E/K_{cyc})[p]$ is isomorphic to $X^{\Sigma_0}(E/K_{cyc})/p$.

Let E_1 and E_2 be two elliptic curves defined over the field of rational numbers $\mathbb{Q}$ and let $\tilde{E}_i$ denote the reduced curve at p for $i = 1, 2$. Suppose that $E_1[p] \cong E_2[p]$ as $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -modules. Let Σ be the set of primes of K containing the primes of bad reduction of E_1 and E_2 , the primes above p and infinite primes. Now consider the set

$$(9) \quad \Sigma_0 := \{v \in \Sigma \mid v|N_1/\bar{N}_1 \text{ or } v|N_2/\bar{N}_2\}$$

as defined in (2). Suppose that $E_1[p] \cong E_2[p]$ as $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -modules. Since E_i has good ordinary reduction at p , $\tilde{E}_{i,p}[p]$ is characterised as the maximal unramified quotient of $E_i[p]$, for $i = 1, 2$ (see [Gr1, Section 2]). In particular, the $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -module isomorphism $E_1[p] \cong E_2[p]$ induces the isomorphism $\tilde{E}_{1,p}[p] \cong \tilde{E}_{2,p}[p]$ of $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ -modules. Therefore we have an isomorphism $\tilde{E}_{1,p}[p] \cong \tilde{E}_{2,p}[p]$ of $\text{Gal}(\bar{K}_v/K_v)$ -modules via the restriction of the Galois action. Since E_1 and E_2 have

good ordinary reduction at p , both E_1 and E_2 have good ordinary reduction at every prime $v|p$ of K and $\tilde{E}_{i,v}[p] = \tilde{E}_{i,p}[p]$ for $i = 1, 2$. Therefore, $\tilde{E}_{1,v}[p] \cong \tilde{E}_{2,v}[p]$ as $\text{Gal}(\bar{K}_v/K_v)$ -modules for every prime $v|p$ of K . This implies that

$$(10) \quad \text{Sel}_p^{\Sigma_0}(E_1[p]/K_{cyc}) \cong \text{Sel}_p^{\Sigma_0}(E_2[p]/K_{cyc}).$$

From equation (8) and the isomorphisms (7) and (10) we get the following

Theorem 2.2. *Suppose that Assumption 1 (i), (ii) and (iii) hold and let Σ_0 be the finite set of primes of K as defined in (2). Then the assumption 1(iv) holds for E_1 if and only if it holds for E_2 . Further, if the assumption 1(iv) holds for E_1 , then $\lambda_{E_1}^{\Sigma_0} = \lambda_{E_2}^{\Sigma_0}$. $\square$*

3. PARITY OF λ -INVARIANTS OF CONGRUENT ELLIPTIC CURVE

Let $v \nmid p$ be a finite prime of K . Then for every prime $w|v$ of K_{cyc} and an elliptic curve E defined over K , the Pontryagin dual of $H^1(K_{cyc,w}, E[p^\infty])$ is a finitely generated $\mathbb{Z}_p$ -module (see for example [Gr2, Proposition 2]). Let σ_E^w denote the $\mathbb{Z}_p$ -corank of $H^1(K_{cyc,w}, E[p^\infty])$. Using the fact that there are finitely many primes of K_{cyc} above v , we get that the Pontryagin dual of $\mathcal{H}_v(K_{cyc}, E)$ is a finitely generated $\mathbb{Z}_p$ -module. Let τ_E^v denote the $\mathbb{Z}_p$ -rank of the Pontryagin dual of $\mathcal{H}_v(K_{cyc}, E)$. By an argument similar to [Gr1, Page 237] (using the exact sequence (5)) we have the following

Lemma 3.1. *Suppose that Assumption 1(i) and 1(iv) hold. Let Σ_0 be a finite set of finite primes of K not containing the primes above p . Then $\lambda_{E_i}^{\Sigma_0} = \lambda_{E_i} + \sum_{v \in \Sigma_0} \tau_{E_i}^v$ for $i = 1, 2$. $\square$*

We mention that as in [Gr1] to prove the above lemma, the assumption that the Pontryagin dual of $\text{Sel}(E/K_{cyc})$ is a torsion Λ -module is needed. The Assumption 1(iv) implies that the Pontryagin dual of $\text{Sel}(E/K_{cyc})$ is a finitely generated $\mathbb{Z}_p$ -module and therefore it is also a torsion Λ -module.

We also mention that in [GV], σ_E^v is used to denote the $\mathbb{Z}_p$ -corank of $\mathcal{H}_v(\mathbb{Q}_{cyc}, E)$ for every prime v of $\mathbb{Q}$ not equal to p . Whereas, we use the notation τ_E^v to denote the $\mathbb{Z}_p$ -corank of $\mathcal{H}_v(K_{cyc}, E)$ for every prime v of the number field K not dividing p and σ_E^w to denote the $\mathbb{Z}_p$ -corank of $H^1(K_{cyc,w}, E[p^\infty])$ for every prime $w|v$ of K_{cyc} . Next, we show that if $v \nmid p$ is a finite prime of K and $w|v$ and $w'|v$ are primes of K_{cyc} , then $\sigma_E^w = \sigma_E^{w'}$.

For every finite prime v of K , let s_v denote the number of primes of K_{cyc} lying above v . Note that s_v is a power of p . In particular s_v is odd. Let w and w' be two primes of K_{cyc} lying above v and let $\eta \in \Gamma = \text{Gal}(K_{cyc}/K)$ such that $\eta(w) = w'$. Then η induces an isomorphism between $H^1(K_{cyc,w}, E[p^\infty])$ and $H^1(K_{cyc,w'}, E[p^\infty])$. Therefore $\sigma_E^w = \sigma_E^{w'}$. This implies that $\tau_E^v = s_v \sigma_E^w$ for every prime $w|p$. Since s_v is odd we have the following

Lemma 3.2. *Let $v \nmid p$ be a finite prime of K and let $w|v$ be a prime of K_{cyc} . Then we have that $\tau_E^v = \sigma_E^w \pmod{2}$. $\square$*

Lemma 3.3. *Let $v \nmid p$ be a finite prime of K and $w|v$ be a prime of K_{cyc} . If Assumption 1(iii) holds and E has split multiplicative reduction at v , then*

$$H^1(K_{cyc,w}, E[p^\infty]) = \mathbb{Q}_p/\mathbb{Z}_p.$$

In particular, if E has split multiplicative reduction at v , then $\sigma_E^w = 1$.

Proof. Let J be the extension $K_v(E[p^\infty])$ of K_v obtained by adjoining the coordinates of points in $E[p^\infty]$. We have that $H^1(J, E[p^\infty]) = 0$ (see [OV, Lemma 5.4(ii)] and [CH, Lemma 5.1(ii)]). From the Hochschild-Serre spectral sequence we get that $H^1(\text{Gal}(J/K_{\text{cyc},w}), E[p^\infty]) \cong H^1(K_{\text{cyc},w}, E[p^\infty])$. Now, the lemma follows from [CH, Lemma 5.13(i)]. $\square$

Remark 3.4. Consider the elliptic curve

$$E : y^2 = x^3 - 584x + 5444$$

with conductor 364. The curve E has good and ordinary reduction at the prime $p = 5$ and split multiplicative reduction at 7. It is shown in [GV, Page 44] that for every prime $w|7$ of $\mathbb{Q}_{\text{cyc}}$, $\sigma_E^w = 0$. This shows that the Assumption 1(iii) is necessary in the above lemma.

Lemma 3.5. *Let $v \nmid p$ be a finite prime of K and $w|v$ be a prime of K_{cyc} . Suppose that E has either additive reduction at v or non-split, multiplicative reduction at v . If Assumption 1(iii) holds, then $H^1(K_{\text{cyc},w}, E[p^\infty]) = 0$. In particular, if E has non-split, multiplicative reduction at v or additive reduction at v , then $\sigma_E^w = 0$.*

Proof. Let $T_p E$ denote the Tate module of E and put $E^*[p^\infty] := \text{Hom}_{\mathbb{Z}_p}(T_p E, \mu_{p^\infty})$. Then from [Gr2, Proposition 2], we have that $H^1(K_{\text{cyc},w}, E[p^\infty])$ is pseudo-isomorphic to $H^0(K_{\text{cyc},w}, E^*[p^\infty])$. Since E is self dual, $E^*[p^\infty] \cong E[p^\infty]$. Therefore, $H^0(K_{\text{cyc},w}, E^*[p^\infty]) \cong H^0(K_{\text{cyc},w}, E[p^\infty])$. Now, using the assumption of the lemma and [HM, Proposition 5.1] we get that $H^0(K_{\text{cyc},w}, E[p^\infty])$ is finite. Therefore $H^1(K_{\text{cyc},w}, E[p^\infty])$ is finite. Now, it is well known that $H^1(K_{\text{cyc},w}, E[p^\infty])$ is divisible (see [Gr, Lemma 4.5 and the successive paragraph]). This implies that $H^1(K_{\text{cyc},w}, E[p^\infty]) = 0$. $\square$

Lemma 3.6. *Let $v \nmid p$ be a finite prime of K and $w|v$ be a prime of K_{cyc} . Suppose that Assumption 1(iii) holds. If E has good reduction at the prime v , then σ_E^w is even.*

Proof. First we suppose that $H^0(K_v, E[p]) = 0$. Then from [HM, Proposition 5.1] we get that $H^0(K_{\text{cyc},w}, E[p^\infty]) = 0$. By an argument similar to the proof of Lemma 3.5 we have that $H^1(K_{\text{cyc},w}, E[p^\infty]) = 0$. Now suppose that $H^0(K_v, E[p]) \neq 0$. From [HM, Proposition 5.1] we get that $H^0(K_{\text{cyc},w}, E[p^\infty]) = E[p^\infty]$. Therefore $H^1(K_{\text{cyc},w}, E[p^\infty]) \cong H^1(K_{\text{cyc},w}, (\mathbb{Q}_p/\mathbb{Z}_p)^2)$. This implies that σ_E^w is even. $\square$

Remark 3.7. Consider the elliptic curve

$$E : y^2 = x^3 + x - 10$$

with conductor 52 over $\mathbb{Q}$. The curve E has good and ordinary reduction at $p = 5$ and good reduction at the prime 7. It is shown in [GV, Page 44] that for every prime $w|7$ of $\mathbb{Q}_{\text{cyc}}$, σ_E^w is 1. This shows that the Assumption 1(iii) is necessary in the above lemma.

Consider a set of finite primes $\Sigma_0 \subset \Sigma \setminus p$. Let S_{E_i} be the set of primes in Σ_0 where E_i has split multiplicative reduction for $i = 1, 2$. Using Lemma 3.1, Lemma 3.3, Lemma 3.5 and Lemma 3.6 we have the following

Lemma 3.8. *Suppose that Assumption 1 holds. Let $\Sigma_0 \subset \Sigma \setminus p$ be a finite set of finite primes of K . Then $\lambda_{E_i}^{\Sigma_0} = \lambda_{E_i} + |S_{E_i}| \pmod{2}$, where $|S_{E_i}|$ denotes the cardinality of S_{E_i} for $i = 1, 2$.* $\square$

As before we consider two elliptic curves E_1 and E_2 defined over $\mathbb{Q}$ such that $E_1[p] \cong E_2[p]$ and consider the set Σ_0 as defined in (2). Let S_{E_1} (resp. S_{E_2}) denote the set of primes in Σ_0 where E_1 (resp. E_2) has split multiplicative reduction.

Theorem 3.9. *Suppose that Assumption 1 holds. Let Σ_0 be the finite set of finite primes of K as defined in (2). If $E_1[p] \cong E_2[p]$ as $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -modules, then $\lambda_{E_1} + |S_{E_1}| = \lambda_{E_2} + |S_{E_2}| \pmod{2}$.*

Proof. The theorem follows from Theorem 2.2 and Lemma 3.8. $\square$

4. PARITY OF THE RANK OF CONGRUENT ELLIPTIC CURVES

For an elliptic curve E defined over K , let $s_p(E/K)$ denote the $\mathbb{Z}_p$ -rank of the Pontryagin dual of $\text{Sel}_p(E/K)$. Let $r(E/K)$ denote the rank of E over K . As in the previous sections, let E_1 and E_2 be two elliptic curves defined over $\mathbb{Q}$ such that $E_1[p] \cong E_2[p]$ as $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -module. Throughout this section, we shall assume that Σ_0 is the finite set of finite primes of K as defined in (2) and S_{E_1} (resp. S_{E_2}) denotes the set of primes in Σ_0 where E_1 (resp. E_2) has split multiplicative reduction.

Theorem 4.1. *Let E be an elliptic curve defined over K and suppose that the Pontryagin dual of $\text{Sel}_p(E/K_{\text{cyc}})$ is Λ -torsion. Then*

$$s_p(E/K) \equiv \lambda_E \pmod{2}.$$

Proof [Gr, Proposition 3.10]. $\square$

Theorem 4.2. *Let E_1 and E_2 be two elliptic curves defined over $\mathbb{Q}$ such that $E_1[p] \cong E_2[p]$ as $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -modules and Σ_0 be the finite set of finite primes of K as defined in (2). Suppose that Assumption 1 holds for E_1 and E_2 . Then we have that*

$$s_p(E_1/K) + |S_{E_1}| = s_p(E_2/K) + |S_{E_2}| \pmod{2}.$$

Proof. The theorem follows from Theorem 3.9 and Theorem 4.1. $\square$

For an elliptic curve E defined over the number field K , let $L(E/K, s)$ denote the Hasse-Weil L -function of E over K . Let E be an elliptic curve defined over $\mathbb{Q}$ and K be a solvable Galois extension of $\mathbb{Q}$. It is a consequence of the Modularity theorem proved by Wiles et al. and the theory of base-change for automorphic representations of $GL(n)$ that the Hasse-Weil L -function $L(E/K, s)$ has an analytic continuation to the entire complex plane (cf. [C],[W],[BCDT]). For a solvable extension $K/\mathbb{Q}$, let $a(E/K)$ denote the order of zero of $L(E/K, s)$ at 1.

Conjecture 1 (p -parity conjecture). $s_p(E/K) = a(E/K) \pmod{2}$.

Theorem 4.3 ([DD, Theorem 6.13], [DD, Theorem 6.15], [DD, Example 6.16]). *Let E be an elliptic curve defined over $\mathbb{Q}$. The p -parity conjecture holds for E over Galois extension $K/\mathbb{Q}$ if $K \subset F$ where one of the following holds:*

- (i) $F = \mathbb{Q}(\mu_{p^n}, m^{1/p^n})$ for $m, n \geq 1$, integer.
- (ii) $p \equiv 3 \pmod{4}$ and F is a Galois p -extension of an abelian extension of $\mathbb{Q}$, Galois over $\mathbb{Q}$.
- (iii) $p = 3$ and $F \subset \mathbb{Q}(E[3^\infty])$.

Corollary 4.4. *Let $K \subset F$ be a Galois extension of $\mathbb{Q}$ where one of the following holds:*

- (i) $F = \mathbb{Q}(\mu_{p^n}, m^{1/p^n})$ for $m, n \geq 1$, integer.

(ii) $p \equiv 3 \pmod{4}$ and F is a Galois p -extension of an abelian extension of $\mathbb{Q}$, Galois over $\mathbb{Q}$.

(iii) $p = 3$ and $F \subset \mathbb{Q}(E_1[3^\infty]) \cap \mathbb{Q}(E_2[3^\infty])$.

Let Σ_0 be the finite set of primes of K as defined in (2). Further suppose that Assumption 1 holds and that $E_1[p] \cong E_2[p]$ as $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -modules. Then,

$$a(E_1/K) + |S_{E_1}| = a(E_2/K) + |S_{E_2}| \pmod{2}.$$

Finally, we mention that if the 2-primary and 3-primary subgroup of the Tate-Shafarevich group $\text{III}(E/K(E[2]))$ of an elliptic curve E is finite over the field $K(E[2])$ satisfying the assumptions of Theorem 4.3, then it is known that the parity of the rank of E over K is the same as the parity of $s_p(E/K)$ (see [DD, Theorem 7.1]). Using this fact, we can also determine the variation of the parity of ranks of elliptic curves under congruence.

We now illustrate our result with an example. Consider the pair of elliptic curves E_1 and E_2 defined by

$$E_1 = 11A1 : y^2 + y = x^3 - x^2 - 10x - 20$$

and

$$E_2 = 121C1 : y^2 + xy = x^3 + x^2 - 2x - 7.$$

We have that $E_1[3] \cong E_2[3]$ as $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -module. Both the curves satisfy Assumption 1 (i) and (ii) for $K = \mathbb{Q}(\mu_p)$. The curve 11A3 in the Cremona table is 5-isogenous to the curve 11A1. From [DD1, Table B.1.] it follows that the Iwasawa μ -invariant of 11A3 vanishes over K_{cyc} . Since 11A3 does not have any 3 isogeny defined over K , the μ -invariant of 11A3 and 11A1 are the same. Therefore Assumption 1(iv) holds for E_1 . The prime to 3-conductor of $E_1[3]$ is 11 over K . Therefore $11 = N_2/\bar{N}_2$. Since E_1 has split multiplicative reduction at 11, $S_{E_1} = \{11\}$. The curve E_2 has additive reduction at 11 and therefore S_{E_2} is empty. Using Sage we get that the analytic rank of E_1 is zero over K and the analytic rank of E_2 is 1 over K . Thus, using Theorem 4.3 we get that $s_3(E_1/K) + |S_{E_1}| = 0 + 1 \pmod{2}$ and $s_3(E_2/K) + |S_{E_2}| = 1 + 0 \pmod{2}$. In particular, $s_3(E_1/K) + |S_{E_1}| = s_3(E_2/K) + |S_{E_2}| \pmod{2}$.

In Remark 3.4 (resp. Remark 3.7) we found that Lemma 3.3 (resp. Lemma 3.6) fails if $K \neq K(\mu_p)$. Both these lemmas were crucially used in Theorem 1.1. We do not have a counterexample of this theorem when $K \neq K(\mu_p)$. We shall next discuss a few examples which in fact suggest that Theorem 1.1 could still be true without the assumption that $\mu_p \subset K$. Consider the pair of elliptic curves E_1 and E_2 , where $E_1 = 235A1$ and $E_2 = 235B1$ from the Cremona table with Weierstrass equation

$$E_1 : y^2 + xy + y = x^3 + x^2 - 3551x - 82926$$

and

$$E_2 : y^2 + xy + y = x^3 + x^2 - 5x,$$

respectively. Then, using Sage we get that $E_1[3] \cong E_2[3]$ as $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -module. Further both E_1 and E_2 have good and ordinary reduction at 3 and $E_1[3]$ is an irreducible $\text{Gal}(\bar{\mathbb{Q}}/K)$ -module. The curve E_1 has analytic rank 0 whereas the curve E_2 has analytic rank 1 over $\mathbb{Q}$. The prime to 3 conductor of $E_i[3]$ over $\mathbb{Q}$ is 47 and $5 = N_i/\bar{N}_i$ for $i = 1, 2$. Over $\mathbb{Q}$ the curve E_1 has non-split multiplicative reduction and E_2 has split multiplicative reduction at 5. In particular, $S_{E_1}(\mathbb{Q}) = 0$

and $S_{E_2}(\mathbb{Q}) = 1$. Therefore, $a(E_1/\mathbb{Q}) + S_{E_1}(\mathbb{Q}) = 0$ and $a(E_2/\mathbb{Q}) + S_{E_2}(\mathbb{Q}) = 2$. Thus, using Theorem 4.3 we see that the conclusion of Theorem 1.1

$$s_3(E_1/\mathbb{Q}) + S_{E_1}(\mathbb{Q}) = s_3(E_2/\mathbb{Q}) + S_{E_2}(\mathbb{Q}) \pmod{2}$$

holds over $\mathbb{Q}$. Both curves have split multiplicative reduction at 5 over $K = \mathbb{Q}(\mu_3)$. In particular, $|S_{E_1}| = |S_{E_2}| = 1$ over K . Using Sage, we also get that both curves have analytic rank 1 over K . Thus, using Theorem 4.3 we see that the conclusion of Theorem 1.1

$$s_p(E_1/K) + |S_{E_1}| = s_p(E_2/K) + |S_{E_2}| \pmod{2}$$

also holds over K . The elliptic curves

$$E_1 : y^2 = x^3 + x - 10$$

and

$$E_2 : y^2 = x^3 - 584x + 5444.$$

in Remarks 3.4 and 3.7 are congruent mod 5 (see [GV]). The curve E_1 has conductor 52 and E_2 has conductor 364. Both these curves satisfy the Assumptions 1 (i), (ii) and (iv). The prime to 5 conductor of $E_1[5]$ is equal to 52, Σ_0 contains 7 and E_2 has split multiplicative reduction at 7. Therefore, $|S_2(\mathbb{Q})| = 1$ and $|S_1(K)| = 0$. The analytic rank of E_1 over $\mathbb{Q}$ is 0 and the analytic rank of E_2 over $\mathbb{Q}$ is 1. From this we get that the conclusion of Theorem 1.1 holds for E_1 and E_2 . Further, it can be easily verified that the elliptic curves E_1 and E_2 satisfy the relation (3) over $\mathbb{Q}(\mu_5)$.

Next we discuss some examples of a family of elliptic curves that are congruent at the prime p and explain how our result can be used to determine the parity of their ranks.

Theorem 4.5 ([RS, Theorem 4.1]). *Fix an elliptic curve E over $\mathbb{Q}$,*

$$y^2 = x^3 + ax + b,$$

and let $J = 4a^3/(4a^3 + 27b^2)$. Define

$$\begin{aligned} a(t) &= ((-11J^2 + 28J - 16)t^4 - 4(2J^2 - 7J + 4)t^3 + 6Jt^2 + 4t + 1)a, \\ b(t) &= ((-8J^4 + 35J^3 - 4J^2 - 80J + 64)t^6 + 12(J^3 + 6J^2 - 20J + 16)t^5 \\ &\quad + 15(5J^2 - 16J + 16)t^4 + 20(J^2 - 5J + 8)t^3 - 15(J - 4)t^2 + 12t + 1)b \end{aligned}$$

and define E_t by

$$y^2 = x^3 + a(t)x + b(t).$$

Then for every $t \in \mathbb{Q}$ such that E_t is non-singular, $E_t[3]$ is isomorphic as a $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -module to $E[3]$. $\square$

We consider the elliptic curve E of conductor 11 defined by the equation

$$y^2 = x^3 - 432x + 8208.$$

The curve E has good and ordinary reduction at the prime 3. This implies that $E[3]$ is a reducible $\text{Gal}(\bar{\mathbb{Q}}_3/\mathbb{Q}_3)$ -module and therefore every elliptic curve congruent mod 3 to E has ordinary reduction at 3. From [DD1, Table 3-11A3] we get that $s_3(E/K) = a(E/K) = 0$ for $K = \mathbb{Q}(\mu_3)$. The $\text{Gal}(\bar{K}/K)$ -module $E[3]$ is irreducible and the conductor of E is equal to the prime to 3-conductor $\bar{N}$ of $E[3]$ (see [SS, Section 5]). From [DD, Table B.1.] we get that Assumption 1(iv) holds. If the conductor of E_t is square free at 11, then $11 \notin \Sigma_0$ and S_E is empty. In this case

S_{E_t} denotes the set of primes $v \nmid \bar{N}$ of K where E_t has split multiplicative reduction. Theorem 4.2 and Corollary 4.4 implies that

$$s_3(E_t/\mathbb{Q}(\mu_3)) = a(E_t/\mathbb{Q}(\mu_3)) \equiv |S_{E_t}| \pmod{2}.$$

If the conductor of E_t is not square free, then Σ_0 contains 11. In this case $|S_E| = 1$ and we have

$$s_3(E_t/\mathbb{Q}(\mu_3)) = a(E_t/\mathbb{Q}(\mu_3)) \equiv |S_{E_t}| + 1 \pmod{2}.$$

Note that since E has good reduction at 3, there are infinitely many rational numbers t such that E_t has good reduction at 3.

Next consider the elliptic curve $E=158C1$ from the Cremona table. The residual representation at $p = 3$ is again irreducible with conductor equal to 158. It can be easily checked using Sage that the curve E satisfies the Assumption 1 (i) and (ii) over $K = \mathbb{Q}(\mu_3)$. Further the 3-adic L -value of E is a 3-adic unit over K (see [SS, Section 5] for the precise statement). Using this fact and the Iwasawa main conjecture of elliptic curves over cyclotomic $\mathbb{Z}_p$ -extensions, it can be shown that the p -Selmer group of E over K_{cyc} vanishes. In particular, the Assumption 1(iv) holds. The rank of E is zero over K and using Theorem 4.5 we get a family of elliptic curves E_t such that $E_t[3] \cong E[3]$ as $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -module. The curve E has split multiplicative reduction at 2 and non-split multiplicative reduction at the prime factors of 79 in K . If the conductor of E_t is square free at 2, then $2 \notin \Sigma_0$. Thus, once again we have that

$$s_3(E_t/\mathbb{Q}(\mu_3)) = a(E_t/\mathbb{Q}(\mu_3)) \equiv |S_{E_t}| \pmod{2}.$$

If the conductor of E_t is not square free at 2, then $2 \in \Sigma_0$. Therefore $|S_E| = 1$. Therefore as in the previous example we have

$$s_3(E_t/\mathbb{Q}(\mu_3)) = a(E_t/\mathbb{Q}(\mu_3)) \equiv |S_{E_t}| + 1 \pmod{2}.$$

Given an elliptic curve E defined over $\mathbb{Q}$, Rubin and Silverberg also produce a family of elliptic curves E_t where $t \in \mathbb{Q}$, such that $E_t[5] \cong E[5]$ for every t (see [RS, Theorem 5.1] for the precise statement). We can also compute the parity of the ranks of elliptic curves in such families using our result.

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