HIGHER CHORDALITY: FROM GRAPHS TO COMPLEXES

KARIM A. ADIPRASITO, ERAN NEVO, AND JOSE A. SAMPER

(Communicated by Irena Peeva)

ABSTRACT. We generalize the fundamental graph-theoretic notion of chordality for higher dimensional simplicial complexes by putting it into a proper context within homology theory. We generalize some of the classical results of graph chordality to this generality, including the fundamental relation to the Leray property and chordality theorems of Dirac.

1. INTRODUCTION

Chordality is one of the most fundamental notions in graph theory. It finds application and stands in relation to graph colorings and perfect graphs, algorithmic graph theory, graph embeddings and Apollonian packings.

However, graph chordality only allows us to study the one-dimensional skeleton of a simplicial complex and therefore, for instance, only applies to the study of initial algebraic Betti numbers of the complex.

This motivates us to develop suitable notions of chordality for higher skeleta of simplicial complexes. In this paper we develop a chordality notion in the realm of (simplicial) homology theory and study its fundamental properties. In forthcoming papers, we will develop another chordality notion in the realm of stress spaces (from framework rigidity, e.g., [Lee96, TWW95], and McMullen’s weight algebra), relate it to the homological notion developed here, and explore its relevance in geometry (for polytopes and for simplicial nonpositive curvature) and in commutative algebra (for algebraic shifting and algebraic Betti numbers).

The classical definition states that a graph G is chordal if for every simple cycle z of length ≥ 4 in G , there exists an edge $e \in G$ that connects two nonadjacent vertices of z .

One of the fundamental theorems concerning chordal graphs is the following result:

Theorem 1.1. *Let G be a simple graph. Then the following are equivalent:*

- (1) G is chordal.
- (2) The complex of cliques of every connected induced subgraph of G is contractible.

Received by the editors May 13, 2015 and, in revised form, October 17, 2015.

2010 *Mathematics Subject Classification.* Primary 05Cxx, 05E45, 13F55.

Key words and phrases. Simplicial complex, chordal graph, Leray property, Castelnuovo-Mumford regularity.

The first author acknowledges support by an IPDE/EPDI postdoctoral fellowship, a Minerva postdoctoral fellowship of the Max Planck Society, and NSF Grant DMS 1128155.

The research of the second author was partially supported by an ISF grant 805/11.

- (3) *The complex of cliques of every connected induced subgraph of G is acyclic, over any ring of coefficients.*
- (4) *The Stanley-Reisner ideal $I = (x_i x_j : ij \notin E(G), i, j \in V(G))$ has a linear resolution. Equivalently, I has regularity 2 (over some, equivalently any, field), unless G is complete and $I = (0)$.*

Parts of this result can be traced back at least to the sixties [LB63], and has been proven in a variety of settings; cf. [Frö90], [Kal87], [KM06], [Gre98].

Moving to higher dimensional simplicial complexes, attempts to generalize Fröberg's result $(1) \Leftrightarrow (4)$ [Frö90] have been made in recent years. Several combinatorial notions of chordality were introduced, e.g., [HVT08, Emt10, Woo11], which imply the existence of a linear resolution over any field, but *not* vice versa. Indeed, in Woodrooffe's notion the Alexander dual $\Delta^\vee$ of the complex Δ is shellable. However, according to the Eagon-Reiner theorem [ER98], it is enough for $\Delta^\vee$ to be Cohen-Macaulay over the field to guarantee a linear resolution of the Stanley-Reisner ideal I_Δ . In fact, already in dimension 2, there are examples of nonshellable contractible complexes with connected vertex links, e.g., Hachimori [Hac03], and thus are Cohen-Macaulay over *any* field. Thus, for Hachimori's complex Γ , $\Delta = \Gamma^\vee$ is not chordal according to any of the above definitions but I_Δ has a linear resolution over any field (this answers [Emt10, Question 1]; see also [Olt09]).

In the other direction, Connon and Faridi found a necessary combinatorial condition for the existence of a linear resolution over all fields [CF13a], and a necessary and sufficient condition when the characteristic is 2 [CF13b]. In the latter case, the homological and combinatorial descriptions are essentially the same. Note that generally, having a linear resolution is characteristic dependent, as the 6-vertex triangulation of the projective plane demonstrates.

The above discussion demonstrates that the chordality notion is naturally best studied as a homological one. A natural formulation for an extension would be to define the following. A simplicial complex Δ is *decomposition k -chordal* if and only if every (simplicial) k -cycle z in Δ (with respect to some coefficient ring) admits a *decomposition*, i.e., it can be written as a sum of complete k -cycles, which support only vertices that are supported in z . By a complete k -cycle we mean a cycle whose faces with nonzero coefficient are precisely the k -faces of a $(k+1)$ -simplex.¹

For decomposition chordality, the following theorem seems the appropriate generalization of Theorem 1.1. It sharpens the Eagon-Reiner theorem [ER98] by providing a more “economical” criterion for deciding whether the Alexander dual of a complex is Cohen-Macaulay.

Theorem 1.2 (Propagation of chordality). *Let Δ be any simplicial complex and assume that*

(1) *Δ is decomposition ℓ -chordal for all $k \leq \ell \leq 2k-1$, and*

(2) *Δ has no missing (a.k.a. empty) faces of dimension $> k$.*

Then Δ is decomposition ℓ -chordal for all $\ell \geq k$, and is k -Leray.

For $k = 1$, this recovers the implication $(1) \Rightarrow (3)$ of Theorem 1.1 for chordal graph theory. Analogs of the implications $(3) \Rightarrow (4) \Rightarrow (1)$ for complexes also hold and follow from known results; see Theorem 5.1 below for details. Theorem

¹Decomposition k -chordal complexes slightly generalize the *strongly triangulable* complexes of Cordovil, Lemos and Linhares Sales [CLLS09], in the sense that we allow facets of dimension $< k$. More importantly, our combinatorial analog of Dirac's elimination order is significantly more general than the one in [CLLS09], yet implies decomposition chordality, thus our Proposition 6.3 implies [CLLS09, Thm.5.2].

1.2 is tight; see Proposition 5.3. In particular, decomposition k -chordality is not enough to guarantee higher decomposition chordality unless $k = 1$ (or the trivial case $k = 0$). Part (2) of Theorem 1.1 does not generalize under the assumptions of Theorem 1.2, as the homotopy groups of Δ can be nontrivial in any dimension.

In Section 2 we set notation, in Section 3 we define *resolution* chordality and relate it to decomposition chordality. In Section 4 we study fundamental properties of decomposition and resolution chordality with respect to basic operations on complexes, such as cone, join, link and generalizations of Dirac's Gluing Lemma (cf. Lemma 4.6). In Section 5 we prove the Propagation Theorem 1.2 and its consequences for regularity of the associated Stanley-Riesner ideal. In Section 6 we define a combinatorial analog of Dirac's elimination order for higher dimensions; this notion implies decomposition chordality. We also establish analogs of Dirac's Minimal Cut theorem for decomposition chordality.

2. BASIC NOTATION

Throughout, we allow any simplicial complex to be a relative simplicial complex, i.e., a pair of abstract simplicial complexes $\Psi = (\Delta, \Gamma)$, $\Gamma \subset \Delta$ (where an abstract simplicial complex is a downclosed subset of the powerset 2^S for some finite set S); cf. [Sta96], [AS14]. A k -dimensional simplicial complex is complete if it coincides with the k -skeleton of some simplex.

The deletion $\Delta - \sigma$ of a face σ of Δ is the maximal subcomplex of Δ that does not contain σ ; this is naturally extended to deletion of a collection of faces from Δ . For a subset V of the vertices of Δ , let $\Delta|_V$ denote the induced subcomplex of Δ on V .

Consider two simplicial complexes Δ_1 and Δ_2 whose vertex sets are disjoint. The join $\Delta_1 * \Delta_2$ is the simplicial complex whose faces are of the form $F_1 \cup F_2$ where $F_1 \in \Delta_1$ and $F_2 \in \Delta_2$.

Now, let $I \subset \{-1, 0, 1, 2, \dots\}$ denote any subset, and let Δ denote any simplicial complex. We then denote by $\Delta^{(I)}$ the collection of faces $\sigma \in \Delta$ with $\dim \sigma \in I$. As a special case, we obtain the k -skeleton $\text{Sk}_k \Delta := \Delta^{(\leq k)}$ of Δ , and the collection of k -faces $\mathcal{F}_k := \Delta^{(k)}$.

A k -clique is a pure simplicial complex of dimension k that contains all possible faces of dimension $\leq k$ on its vertex set. With this, one can associate to any simplicial complex Δ its k -cliques complex (which contains Δ), defined as

$$\text{Cl}_k \Delta := \{\sigma \in \mathcal{F}_0(\Delta) : \text{Sk}_k \sigma \subseteq \Delta\}.$$

Recall that the (closed) star and link of a face σ in Δ are the subcomplexes

$$\text{st}_\sigma \Delta := \bigcup_{\sigma \subseteq \tau \in K} 2^\tau \quad \text{and} \quad \text{lk}_\sigma \Delta := \{\tau \setminus \sigma : \sigma \subset \tau \in \Delta\}.$$

Notice that the previous notation is not standard, but it is convenient for the purpose of the paper. The extended link of a face $\sigma \in \Delta$ is the complex

$$\widetilde{\text{lk}}_\sigma \Delta := \text{st}_\sigma \Delta - \sigma \cong \partial \sigma * \text{lk}_\sigma \Delta$$

where $\partial \sigma$ is the complex of proper faces of σ . Note that $\text{lk}_\sigma \subseteq \widetilde{\text{lk}}_\sigma$ and if $\sigma = v$ is a vertex of Δ , then $\text{lk}_v = \widetilde{\text{lk}}_v$. In general, the dimension of $\text{lk}_\sigma \Delta$ is one less than the dimension of $\text{st}_\sigma \Delta$, while the dimension of $\text{lk}_\sigma \Delta$ is the dimension of $\text{st}_\sigma \Delta$ minus the cardinality of σ .

A nonface of Δ is, naturally, a simplex on $\Delta^{(0)}$ that is not a face of Δ . A minimal nonface, or missing face, of Δ is an inclusion minimal nonface of Δ . Equivalently, a simplex σ is a missing face of Δ if and only if $\partial\sigma \subset \Delta$, but $\sigma \notin \Delta$.

In simplicial homology, a total order on the vertex set V is fixed. We denote by $C_k(\Delta, R)$ the space of k -chains with coefficients in R , that is, the free R module spanned by *ordered* k -dimensional faces of Δ . $C_{-1}(\Delta, R)$ is free of rank 1, generated by the empty face. The vertex support of a k -chain c is denoted by $c^{(0)}$, and we say that a face $\sigma \in \Delta$ is supported in c if σ is contained in a face with nonzero coefficient in c . Furthermore, for a k -chain c and a subcomplex Γ of Δ , $c|_\Gamma$ denotes the restriction of c to the summands supported in Γ . The simplicial boundary map of $C_k(\Delta, R)$ is denoted by ∂ . A k -chain c with $\partial c = 0$ is called a k -cycle. A k -cycle z is complete if the faces with nonzero coefficients are the facets of the boundary of a $(k+1)$ -simplex.

For ordered sets $\tau \subseteq \sigma \subseteq V$, let $\text{sgn}(\tau, \sigma)$ be 1 or -1 depending on whether the permutation $\sigma \mapsto (\tau, \sigma - \tau)$ has an even or odd number of inversions, respectively. The boundary map ∂ applied to a face σ viewed as a k -chain can be expressed as $\sum_{\sigma'} \text{sgn}(\sigma', \sigma) \sigma'$, where σ' ranges over all $(k-1)$ -dimensional faces of σ .

Notice that if the vertices of the join $\Delta_1 * \Delta_2$ are ordered such that the vertices of Δ_1 come before those of Δ_2 , then the boundary operation on the join is given by $\partial(\sigma_1 * \sigma_2) = \partial(\sigma_1) * \sigma_2 + (-1)^{j+1} \sigma_1 * \partial(\sigma_2)$, where $\sigma_1 \in \Delta_1$ is j -dimensional and $\sigma_2 \in \Delta_2$, and ∂ extends linearly to all chains.

We say that a simplicial complex is k -Leray if the reduced homology $\tilde{H}_j(\Delta|_V) = 0$ for every $j \geq k$ and every subset V of vertices of Δ .

Let W denote the vertex set of Δ and $\mathbb{F}$ a field. Consider the polynomial ring $S := \mathbb{F}[x_w \mid w \in W]$. Let I_Δ be the ideal generated by the square free monomials indexed by the subsets of W that are not faces of Δ . The Stanley-Reisner ring of Δ is the ring $\mathbb{F}[\Delta] := S/I_\Delta$. All resolutions of $\mathbb{F}[\Delta]$ considered in this paper are $\mathbb{Z}$ -graded free resolutions of S -modules.

3. RESOLUTION AND DECOMPOSITION OF CYCLES

Let G denote a chordal graph, and let $\tilde{G} = \text{Cl}_1 G$ denote the complex induced by its 1-cliques. Then, if z is any 1-cycle in $\tilde{G}$, there exists a 2-chain c with $\partial c = z$ and $c^{(0)} = z^{(0)}$. Equivalently, z can be written as a sum of 1-cycles of length 3 that contain no vertices that are not already vertices of z . This gives rise to two simple notions of higher chordality:

We say that a $(k+1)$ -chain $c \in C_{k+1}(\Delta)$ is a resolution of a k -cycle $z \in Z_k(\Delta)$ if $c^{(0)} = z^{(0)}$ and $\partial c = z$. We say that a (relative) complex in which every k -cycle admits a resolution is ρ_k -chordal, or resolution k -chordal. We say that Δ is δ_k -chordal, or decomposition k -chordal, if every k -cycle z can be written as a sum of complete k -cycles (z_i) in such a way that $z_i^{(0)} \subset z^{(0)}$ for all z_i . The k -th reduced homology group is denoted by $\tilde{H}_k(\Delta, R)$.

Let us observe some straightforward properties of resolution and decomposition chordality:

Facts 3.1 (Resolution and decomposition chordality).

- (1) A graph is δ_1 -chordal if and only if it is chordal in the classical sense (regardless of the ring of coefficients).
- (2) A complex Δ is δ_k -chordal if and only if $\text{Cl}_k \Delta$ is ρ_k -chordal.

- (3) Any ρ_k -chordal complex is δ_k -chordal.
- (4) A simplicial complex Δ is resolution k -chordal if and only if for every subset of vertices $V \subset \Delta^{(0)}$, we have in reduced homology $\tilde{H}_k(\Delta|_V) = 0$. (Reduced homology is used to include the case when $k = 0$.)
- (5) In particular, the simplex is resolution k -chordal for all k .
- (6) Resolution k -chordality only depends on the faces of dimension k and $(k + 1)$ of a simplicial complex, i.e., for two simplicial complexes with the same set of k - and $(k + 1)$ -faces, either both are ρ_k -chordal or both are not ρ_k -chordal.

In higher dimension, we find it more useful to discuss resolution k -chordality than decomposition k -chordality, however, as we have seen, they are equivalent for k -clique complexes.

4. BASIC OPERATIONS AND CHORDALITY OF COMPLEXES

An important observation is the following cone lemma. Note that on the level of simplicial chains, we have a natural map for any vertex $v \in \Delta^{(0)}$

$$\begin{aligned} \ell_v : C_k(\Delta) &\longrightarrow C_{k-1}(\text{lk}_v \Delta) \\ \sum_{\sigma \in \Delta^{(k)}} g_\sigma \cdot \sigma &\longmapsto \sum_{\sigma \in \Delta^{(k)}} g_\sigma \cdot \text{sgn}(v, \sigma) \cdot \text{lk}_v \sigma = \sum_{\sigma \in \Delta^{(k)} : v \in \sigma} g_\sigma \cdot \text{sgn}(v, \sigma) \cdot (\sigma - v) \end{aligned}$$

that descends to a quasiisomorphism of chain complexes

$$(1) \quad C_\bullet(\text{st}_v \Delta, \text{st}_v \Delta - v) \cong C_{\bullet-1}(\text{lk}_v \Delta).$$

In particular, we have:

Lemma 4.1 (Cone lemma). *A relative simplicial complex $(\text{st}_v \Delta, \text{st}_v \Delta - v)$ is resolution k -chordal if and only if $\text{lk}_v \Delta$ is resolution $(k - 1)$ -chordal.* $\square$

On the other hand, as we shall see, this simple lemma is the main ingredient for any simplicial homology theory to satisfy the propagation of chordality property.

More generally, for any face $\tau \in \Delta$, the extended-link map

$$\begin{aligned} \ell_\tau : C_k(\Delta) &\longrightarrow C_{k-1}(\tilde{\text{lk}}_\tau \Delta) \\ \sum_{\sigma \in \Delta^{(k)}} g_\sigma \cdot \sigma &\longmapsto \sum_{\sigma \in \Delta^{(k)}} g_\sigma \cdot \tilde{\text{lk}}_\tau \sigma = \sum_{\sigma \in \Delta^{(k)} : \tau \subseteq \sigma} g_\sigma \cdot \text{sgn}(\tau, \sigma) \cdot (\partial \tau * (\sigma - \tau)) \end{aligned}$$

descends to an isomorphism in homology

$$H_k(\text{st}_\tau \Delta, \partial \text{st}_\tau \Delta) \cong H_{k-1}(\tilde{\text{lk}}_\tau \Delta),$$

and thus,

Lemma 4.2 (Extended cone lemma). *The relative complex $(\text{st}_\tau \Delta, \partial \text{st}_\tau \Delta)$ is resolution k -chordal if and only if $\tilde{\text{lk}}_\tau \Delta$ is resolution $(k - 1)$ -chordal.*

We conclude with the following result for the links:

Lemma 4.3 (Links lemma). *Let Δ be any simplicial complex, and let F denote any ℓ -dimensional face of Δ . If Δ is resolution k -, $(k - 1)$ -, $\dots$ $(k - \ell - 1)$ -chordal, then $\text{lk}_F \Delta$ is resolution $(k - \ell - 1)$ -chordal.*

Proof. Recall that if $F = \{v_1, \dots, v_k\}$, then $\text{lk}_F(\Delta) = \text{lk}_{v_1}(\text{lk}_{v_2} \dots \text{lk}_{v_k}(\Delta) \dots)$, and so it suffices to treat the case when F is a vertex v . Let z denote any $(k-1)$ -cycle in $\text{lk}_v \Delta$; consider a preimage $a = \ell_v^{-1} z$ supported in $\text{st}_v \Delta$, and let b denote a resolution of z in Δ (which exists by the assumed $(k-1)$ -chordality). Consider the k -cycle $\widehat{z} := a - b$. By k -chordality of Δ , $\widehat{z}$ admits a resolution $\widehat{c}$ in Δ ; the k -chain $c := \ell_v \widehat{c}$ of $\text{lk}_v \Delta$ is the desired resolution of z . $\square$

The chordality conditions in the above lemma are tight:

Example 4.4. Let Δ be the cone with apex v over a square boundary C . Then Δ is resolution 2-chordal but C is not decomposition 1-chordal.

The next result characterizes the behavior of chordality under joins:

Lemma 4.5 (Join lemma). *Let Δ denote any simplicial complex, and let c denote any d -cycle whose vertex support is disjoint from $\Delta^{(0)}$. Let $\Delta * c$ denote the complex whose faces are $\sigma \cup \tau$ for $\sigma \in \Delta$ and τ supported in c .*

If $\Delta * c$ is resolution k -chordal, then Δ is resolution $(k-d-1)$ -chordal.

Proof. Let z denote any $(k-d-1)$ -cycle in Δ . Then $\widehat{z} = z * c$ is a k -cycle in $\Delta * c$, and therefore admits a resolution $\widehat{c}$. Note that $\widehat{c}$ must have the form $\widehat{c} = \tilde{c} * c$ and that $\tilde{c}$ resolves z . $\square$

Let us finally establish a generalization of Dirac's Gluing Theorem [Dir61, Theorem 2]:

Lemma 4.6 (Dirac's Gluing Lemma). *Let Δ, Γ denote any two simplicial complexes and let $\mathcal{M} := \Delta \cup \Gamma$.*

- (i) *If Δ and Γ are resolution k -chordal complexes, and $\Delta \cap \Gamma$ is resolution $(k-1)$ -chordal, then $\mathcal{M}$ is resolution k -chordal.*
- (ii) *Similarly, if Δ and $(\Gamma, \Gamma \cap \Delta)$ are resolution k -chordal, then so is $\mathcal{M}$.*
- (iii) *If $\Delta, \mathcal{M}$ and $\Delta \cap \Gamma$ are resolution k -chordal, then Γ is resolution k -chordal.*

Proof. (i) Let z denote any k -cycle in $\mathcal{M}$, and consider the chains $a := z|_{\Delta}$ and $b = z - a$. Consider now $\tilde{z} := \partial a = -\partial b \subset \Delta \cap \Gamma$. There exists a resolution $\tilde{c}$ of $\tilde{z}$ by $(k-1)$ -chordality of $\Delta \cap \Gamma$, and there exist resolutions for $a - \tilde{c}$ and $b + \tilde{c}$ by k -chordality of Δ resp. Γ . The sum of the resulting resolutions gives the desired resolution of z .

(ii) Let z, a, b be as before. As $(\Gamma, \Gamma \cap \Delta)$ is ρ_k -chordal, it has a resolution b' of b , so in $\mathcal{M}$ we have $\partial b' = b + c_b$ where the chain c_b is supported in $\Delta \cap \Gamma$ and its vertex support is contained in the vertex support of b . Further, $\partial c_b = \partial a$. As Δ is ρ_k -chordal, it has a resolution a' of $a - c_b$, so in $\mathcal{M}$ we get that $a' + b'$ resolves z .

(iii) Now, for a k -cycle z in Γ with vertex set $V = z^{(0)}$ consider the Mayer-Vietoris sequence

$$\begin{aligned} \cdots \longrightarrow \widetilde{H}_k(\Delta|_V \cap \Gamma|_V) &\longrightarrow \widetilde{H}_k(\Delta|_V) \oplus \widetilde{H}_k(\Gamma|_V) \longrightarrow \widetilde{H}_k(\Delta|_V \cup \Gamma|_V) \\ &\longrightarrow \widetilde{H}_{k-1}(\Delta|_V \cap \Gamma|_V) \longrightarrow \cdots \end{aligned}$$

By resolution k -chordality of $\Delta, \Delta \cup \Gamma$ and $\Delta \cap \Gamma$ we conclude $\widetilde{H}_k(\Gamma|_V) = 0$, proving that Γ is resolution k -chordal as well. $\square$

5. PROPAGATION OF CHORDALITY

One of the fundamental theorems of chordal graph theory is the classical Theorem 1.1, asserting that for a chordal graph its (1-)clique complex is 1-Leray, and equivalently, that its Stanley-Reisner ideal has a linear resolution. For homological chordality, the following more detailed version of Theorem 1.2 seems the appropriate generalization:

Theorem 5.1. *Let Δ denote any (abstract) simplicial complex without missing faces of dimension $> k$, and fix any field of coefficients. The following are equivalent:*

- (1) Δ is resolution i -chordal for $i \in [k, 2k - 1]$.
- (2) Δ is resolution i -chordal for $i \geq k$.
- (3) Δ is k -Leray.
- (4) The Stanley-Reisner ring of Δ is of (Castelnuovo-Mumford) regularity at most k .

If Δ has no missing faces of dimension $< k$, then this is additionally equivalent to

- (5) I_Δ , the Stanley-Reisner ideal of Δ , admits a linear resolution.
- (6) Δ^* , the combinatorial Alexander dual of Δ , is Cohen-Macaulay.

Proof. The equivalence (2) $\iff$ (3) follows from Fact 3.1(4). The equivalence (3) $\iff$ (4) was settled in [KM06] and (under the additional assumptions) (4) $\iff$ (5) $\iff$ (6) are part of a classical characterization; cf. [Frö90], [ER98] with the last part of the equivalence known as the Eagon-Reiner theorem. It remains to show that (1) implies (2) (the converse implication is trivial).

Let γ denote any j -cycle of Δ , $j \geq 2k$, and assume by induction that Δ is known to be resolution i -chordal for all $k \leq i < j$, and that every j -cycle supported on less vertices than γ admits a resolution. (The base case for the last assumption is that the simplex is resolution l -chordal for any l ; see Fact 3.1(5).)

Define $\mu(\gamma) = (M, m)$ to denote the pair with M equal to the maximum dimension of a missing face σ supported in γ (namely, σ is a missing face of Δ all of whose facets are supported in γ), and $m > 0$ equal to the number of missing M -faces supported in γ . Using the lexicographic order on $\mathbb{N}^2$, and adding a minimum $\hat{0}$ for the $(j+1)$ -simplex, we may assume by induction that every j -cycle γ' with $\mu(\gamma') <_{\text{lex}} \mu(\gamma)$ admits a resolution.

Now, let σ be any minimal nonface supported in γ of maximal dimension, let v denote any of its vertices and let $\tau = \sigma \setminus \{v\}$. Since Δ is resolution $(j - |\tau|)$ -chordal, the cycle $\text{lk}_\tau \gamma$ (defined by iteratively applying the coning isomorphism of Lemma 4.1 to $\gamma|_{(\text{st}_\tau \Delta, \partial \text{st}_\tau \Delta)}$) admits a resolution r .

Consider $\gamma' := \gamma - \gamma|_{(\text{st}_\tau \Delta, \partial \text{st}_\tau \Delta)} + (-1)^{j-1} \partial \tau * r$; it is a j -cycle. We shall see that either $\gamma' = 0$ or $\mu(\gamma') < \mu(\gamma)$.

In the first case ($\gamma' = 0$), $\gamma = \gamma|_{\text{st}_\tau \Delta} + (-1)^j \partial \tau * r = \tau * r' + (-1)^j \partial \tau * r$ for some chain r' . Applying the differential to both sides we get $r' = \partial r$, hence $\gamma = \partial(\tau * r)$, and so $\tau * r$ resolves γ . In the second case ($\gamma' \neq 0$), note that σ is not supported on γ' , and that the missing faces of γ' and not of γ all have dimension $< \dim(\sigma)$, so $\mu(\gamma') < \mu(\gamma)$ and hence γ' has a resolution, denote it by s .

Now, for any $(j - |\tau|)$ -face σ' in the support of r , we have $\tau * \sigma' \in \Delta$ as Δ has no missing faces of dimension $> k$, so $\tau * r$ is a chain of Δ . The sum $c := s + (-1)^j \tau * r$ gives the desired resolution of γ : indeed, $\partial c = \partial s + (-1)^{2j} \tau * \partial r + (-1)^j \partial \tau * r = \gamma' + \gamma|_{\text{st}_v \Delta} + (-1)^j \partial \tau * r = \gamma$. $\square$

Remark 5.2. Notice that Theorem 5.1 is a strengthening of Theorem 1.2. A complex Δ with no missing faces of dimension larger than k satisfies $\Delta = \text{Cl}_k(\Delta)$; thus, resolution and decomposition ℓ -chordality are equivalent for $\ell \geq k$ in this setting.

This theorem is tight:

Proposition 5.3. *For every $k \geq 2$, there is a simplicial complex $\mathfrak{J}_k$ that is resolution i -chordal for $i \in [k, 2k - 2]$, and has no missing face of dimension $> k$, but that is not k -Leray.*

Proof. Let Δ, Δ' be two disjoint copies of the boundary complex of the k -simplex. Then the join $\mathfrak{J}_k = \Delta * \Delta'$ has missing faces exclusively in dimension k , and it satisfies the desired chordality property since every induced subcomplex has homology only in dimension $k - 1$ (precisely if this subcomplex coincides with Δ or Δ') or $2k - 1$ (if it is the entire complex $\mathfrak{J}_k$). $\square$

Remark 5.4. The implication (1) $\Rightarrow$ (2) in Theorem 5.1 follows also from the recent algebraic result of Herzog and Srinivasan [HS13]: let $\beta_{a,j}(\Delta)$ denote the algebraic Betti numbers of the Stanley-Reisner ring of Δ , where $a, j \in \mathbb{N}$, and define

$$t_a(\Delta) := \max\{j : \beta_{a,j}(\Delta) \neq 0\} \\ = \max\{j : \exists W \subseteq \Delta^{(0)}, |W| = j, \tilde{H}_{j-a-1}(\Delta|_W; \mathbb{F}) \neq 0\},$$

where the second equality holds by Hochster's formula, and $\mathbb{F}$ is the field fixed in Theorem 5.1. It is shown in [HS13, Cor. 4] that for any $a \geq 1$, $t_a(\Delta) \leq t_{a-1}(\Delta) + t_1(\Delta)$.

Proof of (1) $\Rightarrow$ (2): What we need to show is that for any $a \geq 1$, $t_a(\Delta) \leq a + k$. For $a = 1$ this is obvious, as $t_1(\Delta)$ is the size of a maximal missing face of Δ , and by assumption Δ has no missing faces of $\dim > k$. For $a > 1$, by induction on a , and the result [HS13, Cor. 4] recalled above, $t_a(\Delta) \leq t_{a-1}(\Delta) + t_1(\Delta) \leq (a - 1) + k + (1 + k) = a + 2k$. However, by the assumption on vanishing homology in (1), $t_a(\Delta) \notin [a + k + 1, a + 2k]$, so $t_a(\Delta) \leq a + k$. $\square$

Axiomatics. We now notice that our theory remains valid if we consider slightly different homology theories. Consider any functor $\mathcal{F}$ from the category of simplicial complexes to the category of chain complexes over a ring R such that the k -th module of $\mathcal{F}(\Delta)_\bullet$ is $C_k(\Delta, R)$, perhaps with a different differential. A simplicial homology theory is any homology theory arising in this way. We say that such a theory is “resolving” if it satisfies the extended cone lemma, i.e., if $\ell_\tau : C_k(\Delta) \rightarrow C_{k-1}(\tilde{\text{lk}}_\tau \Delta)$ induces a surjection in homology for any face $\tau \in \Delta$. It is easy to check that a simplicial homology theory is resolving if and only if it satisfies Theorem 1.2.

Example 5.5. The homology theory that takes Δ to $H_*(\mathcal{M}\Delta)$, where $\mathcal{M}\Delta$ is the moment-angle complex of Δ , cf. [DJ91], is resolving.

In particular, it is not necessary for the homology theory to satisfy invariance under homotopy equivalence in order to be resolving.

6. DIRAC'S THEOREM: MINIMAL HOMOLOGY CUTS AND REVERSE PROPAGATION

In this section, we examine one of the most fundamental characterizations of chordal graphs, the theorem of Dirac [Dir61], and observe that it is the basic case $k = 1$ of what we call a reverse propagation phenomenon: i.e., a result that asserts

that decomposition k -chordal complexes contain nontrivial decomposition $(k-1)$ -chordal complexes.

6.1. The classical case: Dirac's Theorems for graphs. Dirac's Theorems are perhaps the most influential contributions to the theory of chordal graphs; they find applications, for instance, to colorings and computational graph theory. Recall that a cut relative to vertices v, w of a graph $G = (V, E)$, is a set of vertices $V' \subset V \setminus \{v, w\}$ such that every path from v to w must contain a vertex of V' , i.e., V' separates v from w . The cut is minimal if no proper subset of V' separates v from w . Dirac's first theorem characterizes relative minimal cuts:

Theorem 6.1 (Dirac's Minimal Cut theorem). *In every chordal graph, the induced subgraph on a relative minimal cut is a complete graph.*

By iteratively applying the Minimal Cut theorem, until only one vertex is left in a connected component, one immediately derives Dirac's Theorem on elimination orders in a chordal graph.

Theorem 6.2 (Dirac's Reverse Propagation Theorem). *In every chordal graph, there is a vertex such that the induced subgraph on its link is complete.*

To justify the name we gave this theorem, notice that a complete graph is merely a resolution 0-chordal complex. This motivates us to search for higher-dimensional reverse propagation; more accurately, we are led to consider the following question, to be answered in the next subsection:

In every decomposition k -chordal complex, must there be a face whose extended link induces a subcomplex which is resolution $(k-1)$ -chordal?

6.2. Dirac's Theorem in higher dimensions. Before we address this problem, let us introduce another notion of higher chordality:

Dirac complexes. Let us call a simplicial complex Δ k -Dirac if one of the following conditions holds:

- (1) $\text{Sk}_k \Delta = \text{Sk}_k \tau$ for some simplex τ , or
- (2) there exists a face σ of Δ , $\dim \sigma \leq k-1$, such that $\Delta - \sigma$ is k -Dirac and the extended link $\tilde{\text{lk}}_{\sigma} \Delta$ is $(k-1)$ -Dirac and satisfies $\text{Cl}_{k-1} \tilde{\text{lk}}_{\sigma} \Delta = \tilde{\text{lk}}_{\sigma}(\text{Cl}_k \Delta)$.

Note that a graph is chordal if and only if it is 1-Dirac. Compared to resolution and decomposition chordality, the Dirac property is combinatorial rather than homological, and one can therefore expect it to be more restrictive than these homological notions:

Proposition 6.3. *Every k -Dirac complex is decomposition k -chordal, for every ring of coefficients R .*

Proof. Let Δ be k -Dirac. If Δ has a complete k -skeleton it is clearly δ_k -chordal; else let $\sigma \in \Delta^{(\leq k-1)}$ be such that $\Delta - \sigma$ is k -Dirac and $\tilde{\text{lk}}_{\sigma} \Delta$ is $(k-1)$ -Dirac with $\text{Cl}_{k-1} \tilde{\text{lk}}_{\sigma} \Delta = \tilde{\text{lk}}_{\sigma}(\text{Cl}_k \Delta)$. By Lemma 3.1(2), we need to show that $\text{Cl}_k \Delta$ is ρ_k -chordal. We proceed by double induction, on dimension and number of faces.

Consider a k -cycle z in $\text{Cl}_k \Delta$, $z = z_0 + z_1$ where z_0 is the restriction of z to faces containing σ , and $z_1 = z - z_0$. Thus, z_0 is a (relative) cycle in $(\text{st}_{\sigma}(\text{Cl}_k \Delta), \tilde{\text{lk}}_{\sigma}(\text{Cl}_k \Delta))$. As $\tilde{\text{lk}}_{\sigma} \Delta$ is $(k-1)$ -Dirac, and hence by induction $\text{Cl}_{k-1} \tilde{\text{lk}}_{\sigma} \Delta$ is ρ_{k-1} -chordal, by the Extended cone Lemma 4.2, and the requirement $\text{Cl}_{k-1} \tilde{\text{lk}}_{\sigma} \Delta = \tilde{\text{lk}}_{\sigma}(\text{Cl}_k \Delta)$, there is

a (relative) k -chain c_0 that resolves z_0 ; that is, $c_0^{(0)} = z_0^{(0)}$ and $\partial c_0 = z_0 + b$ where b is a k -chain in $\widetilde{\text{lk}}_\sigma \text{Cl}_k \Delta$, hence also in $\text{Cl}_k(\Delta - \sigma)$. As both $z_0 + b$ and $z_0 + z_1$ are k -cycles, we get that $z_1 - b$ is a k -cycle in $\text{Cl}_k(\Delta - \sigma)$. As $\Delta - \sigma$ is k -Dirac, by induction $z_1 - b$ has a resolution c_1 . Thus, $c = c_0 + c_1$ resolves z . $\square$

The converse to this proposition does not hold:

Example 6.4. Recall that the dunce hat is the quotient topological space obtained from identifying the edges of a triangle with an orientation that is not cyclic; see e.g. [Zee64] for more on this remarkable space. Let D denote any *flag* triangulation of the Dunce hat, e.g. the barycentric subdivision of some triangulation of it; minimal 8 vertex triangulations are known, e.g. [BL13] (or any other contractible 2-complex without free edges, i.e., where every edge is contained in at least two facets, will do). Since D is contractible it does not support a 2-cycle; hence, D is decomposition 2-chordal. However, the extended links of all edges and vertices contain induced circles of length at least 4, and hence are not 1-Dirac.

Remark 6.5. The notion of k -Dirac is more general than the combinatorial chordality notions of [Woo11] and [Emt10], and therefore more general than the one of [HVT08] as well. All these notions imply the existence of a linear resolution and are incomparable as explained by Woodroffe [Woo11, Example 4.8]. The approach of both is similar: they provide combinatorial criteria on the hypergraph/clutter of minimal nonfaces of Δ and use these criteria to show the existence of a linear resolution. Both criteria are inductive in nature. In fact, choosing σ as in the definition of k -Dirac to be a vertex is always possible for such complexes. For an example of Δ which is k -Dirac for any $k \geq d$, and its missing faces form a $(d + 1)$ -uniform clutter whose complement $(d + 1)$ -clutter is not chordal by any of [Woo11] and [Emt10], take Δ (for $d = 2$) to be the join of an edge with a square boundary, union with the two diagonals of the square. Indeed, one can verify that Δ is k -Dirac for any $k \geq 2$ (again, choosing σ in the definition to be a vertex is always possible), but no vertex of Δ is *simplicial* in the sense of [Woo11], and similarly for [Emt10].

In [CLLS09] another combinatorial definition of elimination order was introduced, and was shown to imply decomposition k -chordality [CLLS09, Thm.5.2]. Compared to our definition of k -Dirac, their face σ is always $(k - 1)$ -dimensional and $\widetilde{\text{lk}}_\sigma \Delta$ is always the join of $\partial\sigma$ with a nonempty simplex, and in particular satisfies the conditions in our definition. Further, their complex must be the k -clique complex of a pure k -dimensional complex, thus the complex Δ above does not satisfy the definition in [CLLS09] either.

Cuts and homology cuts. We need an analogue of cuts for higher-dimensional complexes; this is straightforward: denote by $\Gamma_k \Delta$ the graph whose vertices are the k -faces of Δ , and two are connected by an edge if their intersection is a $(k - 1)$ -face of Δ . A $(k - 1)$ -dimensional subcomplex $\mathcal{C}$ of a simplicial complex Δ is a k -cut relative to two faces σ and τ if every path in $\Gamma_k \Delta$ from a k -face containing σ to a k -face containing τ must pass through one of the $(k - 1)$ -faces of $\mathcal{C}$. When σ and τ are understood, we say that $\mathcal{C}$ is a relative k -cut, or just a k -cut, for short. The component $\Delta_\sigma^\mathcal{C}$ of σ in Δ w.r.t. $\mathcal{C}$ is the subcomplex of Δ induced by k -faces that are connected to $\text{st}_\sigma \Delta$ by a path in $\Gamma_k \Delta$ that does not intersect $\mathcal{C}$; the component $\Delta_\tau^\mathcal{C}$ is defined analogously. A k -cut is minimal if it is inclusion minimal.

We say that a relative k -cut $\mathcal{C}$ is a homology k -cut if for every $(k-1)$ -cycle z in $\mathcal{C}$, and for $L = \Delta_\sigma^{\mathcal{C}}$ or $\Delta_\tau^{\mathcal{C}}$ (or both, in which case we say the homology k -cut is two-sided), there is a k -chain c in L such that

$$c^{(0)} \cap \mathcal{C} = z^{(0)} \quad \text{and} \quad \partial c = z.$$

The next subsection clarifies the hierarchy between these definitions.

Existence of minimal cuts and homology cuts. First, we note that minimal k -cuts always exist:

Proposition 6.6. *Let Δ denote any simplicial complex that has at least two k faces. Then Δ has a minimal k -cut (relative to any two k -faces σ and τ of Δ).*

Proof. Clearly, Δ admits a cut, e.g. $\mathcal{C} = \partial\sigma$; passing to an inclusion minimal subcut yields the claim. $\square$

We can guarantee minimal cuts of a special form:

Theorem 6.7. *Let Δ denote any k -dimensional simplicial complex that has at least two k faces. Assume $\Gamma_k\Delta$ is connected. Then Δ has a face σ of dimension at most $k-1$ whose extended link is a minimal k -cut.*

Proof. By assumption, Δ is not the k -simplex, hence for some $(k-1)$ -face τ , $\tilde{\text{lk}}_\tau$ is a k -cut. Assume $\tilde{\text{lk}}_\tau$ is not minimal, otherwise we are done. Then there is a $(k-1)$ -face τ' of $\tilde{\text{lk}}_\tau$ such that $\text{st}_{\tau'} \subseteq \text{st}_\tau$. There is exactly one k -face σ' which contains both τ and τ' and thus $\text{st}_{\sigma'} = 2\sigma'$, and $\tilde{\text{lk}}_{\sigma'}$ is also a k -cut. Let σ be the intersection of all facets of σ' whose (closed) star equals σ' . (This intersection is nonempty as $\Gamma_k\Delta$ is connected.) Then $\tilde{\text{lk}}_\sigma$ is a minimal k -cut. $\square$

The cuts in the above theorem are also homology cuts:

Corollary 6.8. *Under the assumptions of Theorem 6.7, the minimal k -cut $\tilde{\text{lk}}_\sigma \Delta$ is a homology cut.*

Proof. The component $L = \text{st}_\sigma$ is acyclic, hence any $(k-1)$ -cycle z in $\tilde{\text{lk}}_\sigma$ is the boundary of some k -chain c in L . In fact, z must be of the form $z = \partial\sigma * z'$ for some cycle z' , hence c can be chosen to be $c = \sigma * z'$. Thus, for $k > 1$, c and z have the same vertex support. For $k = 1$, Dirac's Minimal Cut theorem implies the claim. $\square$

However, two-sided homology cuts are harder to guarantee, and not every homology cut is two-sided:

Example 6.9 (No two-sided homology cuts). Consider a triangulation of the dunce hat D ; see also Example 6.4. The extended link of each face σ is *not* a two-sided homology 2-cut, else, a 1-cycle z in $\tilde{\text{lk}}_\sigma$ (which exists!) would be the boundary of two 2-chains s, t , one in each component, and thus D would contain the 2-cycle $s - t$, contradicting that the dunce hat is acyclic.

Reverse propagation. Nevertheless, when two-sided homology k -cuts exist and a few additional mild conditions are satisfied, reverse propagation holds, which we discuss next.

Theorem 6.10. *Let Δ be a decomposition k -chordal simplicial complex, and let $\sigma \in \Delta$ be a k -face. Assume $\tilde{\text{lk}}_\sigma \Delta^{(k-1)}$ is a two-sided homology k -cut for $\Delta^{(k)}$ (relative to two k -faces), and that $\tilde{\text{lk}}_\sigma \Delta^{(k)}$ is a $(k+1)$ -cut for $\Delta^{(k+1)}$. Then, $\tilde{\text{lk}}_\sigma \Delta$ is decomposition $(k-1)$ -chordal.*

Proof. The proof follows the ideas of Dirac: Let z denote any $(k-1)$ -cycle in $\tilde{\text{lk}}_\sigma$. By the definition of two-sided homology cuts, there exist k -chains s and t , one in each component defined by the k -cut $L = \tilde{\text{lk}}_\sigma^{(k-1)}$, such that $\partial s = \partial t = z$ and $s^{(0)} \cap L^{(0)} = t^{(0)} \cap L^{(0)} = z^{(0)}$.

Then the chain $\hat{z} = s - t$ is a k -cycle, so by δ_k -chordality of Δ it has a resolution $\hat{c}$. Since $L' = \tilde{\text{lk}}_\sigma^{(k)}$ is a $(k+1)$ -cut, it separates $\hat{c}$ into chains $\hat{s}$ and $\hat{t}$, with $\hat{c} = \hat{s} - \hat{t}$. Then $\partial \hat{s}$ and s are identical on the restriction to k -faces not in L' ; similarly for $\partial \hat{t}$ and t . Thus, $s - \partial \hat{s} = -\partial \hat{t} + t$ forms the desired resolution of z . $\square$

We end with the following problem:

Problem 6.11. For any $k > 1$, find interesting families of complexes Δ for which there exists a face σ satisfying the conditions in Theorem 6.10.

ACKNOWLEDGMENTS

The authors thank Satoshi Murai and the referee for pointing them to the relevance of [HS13] and [CLLS09] respectively, and Isabella Novik and the referee for helpful remarks on the presentation. The third author also thanks Isabella Novik for the research assistant positions funded through NSF Grant DMS-1069298 and NSF Grant DMS-1361423.

REFERENCES

- [AS14] K. A. Adiprasito and R. Sanyal, *Relative Stanley–Reisner Theory and Upper Bound Theorems for Minkowski sums*, preprint, arXiv:1405.7368.
- [BL13] B. Benedetti and F. H. Lutz, *The dunce hat and a minimal non-extendably collapsible 3-ball*, Electronic Geometry Model No. 2013.10.001. Available online (<http://www.eg-models.de/2013.10.001>) (2013).
- [CF13a] E. Connon and S. Faridi, *Chorded complexes and a necessary condition for a monomial ideal to have a linear resolution*, J. Combin. Theory Ser. A **120** (2013), no. 7, 1714–1731, DOI 10.1016/j.jcta.2013.05.009. MR3092695
- [CF13b] E. Connon and S. Faridi, *A criterion for a monomial ideal to have a linear resolution in characteristic 2*, math arXiv 1306.2857 (2013).
- [CLLS09] Raul Cordovil, Manoel Lemos, and Cláudia Linhares Sales, *Dirac’s theorem on simplicial matroids*, Ann. Comb. **13** (2009), no. 1, 53–63, DOI 10.1007/s00026-009-0012-2. MR2529719 (2010j:05082)
- [DJ91] Michael W. Davis and Tadeusz Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), no. 2, 417–451, DOI 10.1215/S0012-7094-91-06217-4. MR1104531 (92i:52012)
- [Dir61] G. A. Dirac, *On rigid circuit graphs*, Abh. Math. Sem. Univ. Hamburg **25** (1961), 71–76. MR0130190 (24 #A57)
- [ER98] John A. Eagon and Victor Reiner, *Resolutions of Stanley–Reisner rings and Alexander duality*, J. Pure Appl. Algebra **130** (1998), no. 3, 265–275, DOI 10.1016/S0022-4049(97)00097-2. MR1633767 (99h:13017)

- [Emt10] Eric Emtander, *A class of hypergraphs that generalizes chordal graphs*, Math. Scand. **106** (2010), no. 1, 50–66. MR2603461 (2011b:05179)
- [Frö90] Ralf Fröberg, *On Stanley-Reisner rings*, Topics in algebra, Part 2 (Warsaw, 1988), Banach Center Publ., vol. 26, PWN, Warsaw, 1990, pp. 57–70. MR1171260 (93f:13009)
- [Gre98] Mark L. Green, *Generic initial ideals*, Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math., vol. 166, Birkhäuser, Basel, 1998, pp. 119–186. MR1648665 (99m:13040)
- [HVT08] Huy Tài Hà and Adam Van Tuyl, *Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers*, J. Algebraic Combin. **27** (2008), no. 2, 215–245, DOI 10.1007/s10801-007-0079-y. MR2375493 (2009a:05145)
- [Hac03] M. Hachimori, *Nonshellable but constructible 2-dimensional simplicial complex.*, Electronic Geometry Model (2003) (English).
- [HS13] J. Herzog and H. Srinivasan, *A note on the subadditivity problem for maximal shifts in free resolutions*, math arXiv 1303.6214 (2013).
- [Kal87] Gil Kalai, *Rigidity and the lower bound theorem. I*, Invent. Math. **88** (1987), no. 1, 125–151, DOI 10.1007/BF01405094. MR877009 (88b:52014)
- [KM06] Gil Kalai and Roy Meshulam, *Intersections of Leray complexes and regularity of monomial ideals*, J. Combin. Theory Ser. A **113** (2006), no. 7, 1586–1592, DOI 10.1016/j.jcta.2006.01.005. MR2259083 (2007e:13033)
- [Lee96] C. W. Lee, *P.L.-spheres, convex polytopes, and stress*, Discrete Comput. Geom. **15** (1996), no. 4, 389–421, DOI 10.1007/BF02711516. MR1384883 (97c:52024)
- [LB63] C. G. Lekkerkerker and J. Ch. Boland, *Representation of a finite graph by a set of intervals on the real line*, Fund. Math. **51** (1962/1963), 45–64. MR0139159 (25 #2596)
- [Olt09] Anda Olteanu, *Constructible ideals*, Comm. Algebra **37** (2009), no. 5, 1656–1669, DOI 10.1080/00927870802210001. MR2526328 (2010b:13030)
- [Sta96] Richard P. Stanley, *Combinatorics and commutative algebra*, 2nd ed., Progress in Mathematics, vol. 41, Birkhäuser Boston, Inc., Boston, MA, 1996. MR1453579 (98h:05001)
- [TWW95] Tiong-Seng Tay, Neil White, and Walter Whiteley, *Skeletal rigidity of simplicial complexes. I*, European J. Combin. **16** (1995), no. 4, 381–403, DOI 10.1016/0195-6698(95)90019-5. MR1337142 (96e:52052)
- [Woo11] Russ Woodroffe, *Chordal and sequentially Cohen-Macaulay clutters*, Electron. J. Combin. **18** (2011), no. 1, Paper 208, 20. MR2853065 (2012j:05472)
- [Zee64] E. C. Zeeman, *On the dunce hat*, Topology **2** (1964), 341–358. MR0156351 (27 #6275)

SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540
 – AND – EINSTEIN INSTITUTE OF MATHEMATICS, UNIVERSITY OF JERUSALEM, JERUSALEM, ISRAEL
E-mail address: `adiprasito@math.huji.ac.il`

EINSTEIN INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM, ISRAEL
E-mail address: `nevo@math.huji.ac.il`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON AT SEATTLE, SEATTLE, WASHINGTON 98105
E-mail address: `samper@math.washington.edu`