PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 8, August 2016, Pages 3563–3568
http://dx.doi.org/10.1090/proc/13009
Article electronically published on February 3, 2016

VARIATIONAL PROBLEMS OF TOTAL MEAN CURVATURE OF SUBMANIFOLDS IN A SPHERE

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(Communicated by Guofang Wei)

Abstract. Let $\mathbf{H}$ be the mean curvature vector of an n-dimensional submanifold in a Riemannian manifold. The functional $\mathcal{H} = \int \|\mathbf{H}\|^n$ is called the total mean curvature functional. In this paper, we present the first variational formula of $\mathcal{H}$ and then, for a critical surface of $\mathcal{H}$ in the $(2+p)$-dimensional unit sphere $\mathbb{S}^{2+p}$, we establish the relationship between the integral of an extrinsic quantity of the surfaces and its Euler characteristic number.

1. Introduction

Let M^n $(n \geq 2)$ be a submanifold isometrically immersed in an $(n + p)$-dimensional space form with constant sectional curvature c. Let e_i $(1 \leq i \leq n)$ and e_α $(n+1 \leq \alpha \leq n+p)$ denote the local field of orthonormal tangent frames and the local field of orthonormal normal frames of M^n, respectively. The second fundamental form is denoted by h^α_{ij} in this paper. For each α, by setting $H^\alpha = 1/n\sum_i h^\alpha_{ii}$, we can write the mean curvature vector as follows: $\mathbf{H} = \sum_\alpha H^\alpha e_\alpha$. The square of the norm of the second fundamental form is denoted by S, and the square of the norm of the trace-free tensor $h^\alpha_{ij} - n\|\mathbf{H}\|^2\delta_{ij}$ by ρ^2. Then

$$\rho^2 = S - n\|\mathbf{H}\|^2,$$

which is a nonnegative function on M^n and vanishes at umbilical points. The functional

$$\mathcal{H}(M) = \int_M \|\mathbf{H}\|^n dv,$$

where dv denotes the volume element of M^n, is called the total mean curvature. When the ambient space is Euclidean space $(c = 0)$, considering the compact submanifolds without topology restriction, B. Y. Chen [3] proved that $\mathcal{H}$ has a positive lower bound c_n (the volume of the unit n-sphere) and this lower bound is only attained at the n-sphere. If the ambient space is a sphere, then we see that the lower bound zero is attached at all compact minimal submanifolds without topology restriction. This stimulates us to consider the variational problem of $\mathcal{H}$. By calculating the first variational, we have the following result.

Received by the editors July 30, 2015 and, in revised form, October 15, 2015.
2010 *Mathematics Subject Classification.* Primary 53C17, 53C40, 53C42.
Key words and phrases. Submanifolds, total mean curvature, variation, Euler characteristic.
The authors were supported by project numbers 11161056 and 11531012 of the National Natural Science Foundation of China.

Proposition 1.1. *The Euler-Lagrange equation of $\mathcal{H}$ is*

$$\Delta(\|\mathbf{H}\|^{n-2}H^{\alpha}) + \|\mathbf{H}\|^{n-2}\sum_{\beta}(\sum_{i,j} h_{ij}^{\alpha}h_{ij}^{\beta} + (nc - n\|\mathbf{H}\|^2)\delta_{\alpha\beta})H^{\beta} = 0, \tag{1.1}$$

where α is any integer between $n+1$ and $n+p$.

For convenience, we denote a solution of the equation by an $\mathcal{H}$-submanifold. In particular, for 2-dimensional submanifolds in $\mathbb{S}^{2+p}$, the equation is reduced to

$$\Delta H^{\alpha} + \sum_{\beta}(\sum_{i,j} h_{ij}^{\alpha}h_{ij}^{\beta} - 2\|\mathbf{H}\|^2\delta_{\alpha\beta})H^{\beta} + 2H^{\alpha} = 0, \tag{1.2}$$

where α is any integer between $n+1$ and $n+p$. In particular, the equation of an $\mathcal{H}$-surface in $\mathbb{S}^3$ is

$$\Delta H + (\rho^2 + 2)H = 0, \tag{1.3}$$

where H is the mean curvature function of M^2.

Remark 1.1. Another famous functional is the Chen-Willmore functional [2]: $\mathcal{W}(M) = \frac{1}{2}\int_M \rho^2 dv$, which is invariant under the conformal group of $\mathbb{S}^{2+p}$. It is well known that the Euler-Lagrange equation of the functional $\mathcal{W}$ is as follows (see [1, 9, 14] or [7]):

$$\Delta H^{\alpha} + \sum_{\beta}(\sum_{i,j} h_{ij}^{\alpha}h_{ij}^{\beta} - 2\|\mathbf{H}\|^2\delta_{\alpha\beta})H^{\beta} = 0.$$

A surface satisfying this equation is called a Willmore surface or a $\mathcal{W}$-surface. In particular, the equation of a $\mathcal{W}$-surface in $\mathbb{S}^3$ is

$$\Delta H + \rho^2 H = 0.$$

Marques and Neves proved the famous Willmore conjecture: suppose M^2 is a compact surface with $\chi(M) = 0$; then $\mathcal{W}(M) \geq 2\pi^2$ (see [5]). For a $\mathcal{W}$-surface with codimension p in $\mathbb{S}^{2+p}$, Guo and Li [6, 9] proved the inequality

$$\int_M \rho^2[2 - (2 - \frac{1}{p})\rho^2]dv \leq 0.$$

For an $\mathcal{H}$-surface in $\mathbb{S}^{2+p}$ we establish the relationship between the integral of a function of ρ^2 and the Euler characteristic $\chi(M)$ of M. We state the main theorem of this paper as follows:

Theorem 1.1. *For a compact $\mathcal{H}$-surface with codimension p in $\mathbb{S}^{2+p}$, we have*

$$\int_M \{\rho^2[1 - (2 - \frac{1}{p})\rho^2] + 2\}dv \leq 4\pi\chi(M), \tag{1.4}$$

and the equality holds if and only if M is a geodesic 2-sphere, a Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$ in $\mathbb{S}^3$, or a Veronese surface in $\mathbb{S}^4$.

In particular, for a compact $\mathcal{H}$-surface in a 3-sphere we have

$$\int_M (\rho^2 + 1)(2 - \rho^2)dv \leq 4\pi\chi(M). \tag{1.5}$$

As for any compact minimal surface in a 3-sphere, it holds that $\int_M (2 - \rho^2)dv = 4\pi\chi(M)$. Thus, from Theorem 1.1 we reobtain the Simons's inequality [12]: $\int_M S(2 - S)dv \leq 0$. So, Theorem 1.1 can be seen as a generalization of Simons's theorem on the minimal surfaces to $\mathcal{H}$-surfaces. In our proof of the theorem, we will employ techniques similar to those used by H. Li in [10,11], and by S. S. Chern et al. in [4].

2. Proof of Theorem 1.1

We continue to use the notation in Section 1. Locally, let ω_i be the dual basis of orthonormal tangent basis e_i. Let ω_{ij} and $\omega_{\alpha\beta}$ denote the connection on the tangent bundle and the connection on the normal bundle of the submanifold M, respectively. If ξ is a field of normal vectors on M, then the covariation derivatives $\xi^\alpha_{,i}$ and $\xi^\alpha_{,ij}$ of the component ξ^α of ξ are defined as follows:

$$\xi^\alpha_{,i}\omega_i = d\xi^\alpha + \xi^\beta\omega_{\beta\alpha},$$

$$\xi^\alpha_{,ij}\omega_j = d\xi^\alpha_{,i} + \xi^\alpha_{,j}\omega_{ji} + \xi^\beta_{,i}\omega_{\beta\alpha}.$$

The covariation derivatives of the second fundamental form h^α_{ij} are defined as follows:

$$h^\alpha_{ij,k}\omega_k = dh^\alpha_{ij} + h^\alpha_{lj}\omega_{li} + h^\alpha_{il}\omega_{lj} + h^\beta_{ij}\omega_{\beta\alpha},$$

$$h^\alpha_{ij,kl}\omega_l = dh^\alpha_{ij,k} + h^\alpha_{lj,k}\omega_{li} + h^\alpha_{il,k}\omega_{lj} + h^\alpha_{ij,l}\omega_{lk} + h^\beta_{ij,k}\omega_{\beta\alpha}.$$

Let R_{ijkl} and $R_{\alpha\beta ij}$ be the curvatures with respect to ω_{ij} and $\omega_{\alpha\beta}$, respectively. From the theory of submanifolds in a sphere, we have the following Codazzi equation, Gaussian equation, Ricci equation and Ricci identity:

$$h^\alpha_{ij,k} - h^\alpha_{ik,j} = 0, \tag{2.1}$$

$$R_{ijkl} = \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}) + \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}, \tag{2.2}$$

$$R_{\alpha\beta ij} = \sum_k (h^\alpha_{ik}h^\beta_{kj} - h^\beta_{ik}h^\alpha_{kj}), \tag{2.3}$$

$$h^\alpha_{ij,kl} - h^\alpha_{ij,lk} = h^\alpha_{mj}R_{mikl} + h^\alpha_{im}R_{mjkl} + h^\beta_{ij}R_{\beta\alpha kl}. \tag{2.4}$$

In the following, we suppose that the submanifold M is compact. We define the operator $L : \Gamma(T^\perp M) \to C^\infty(M)$ as

$$L(\xi) = \sum_{\alpha,i,j} (h^\alpha_{ij} - nH^\alpha\delta_{ij})\xi^\alpha_{,ij}.$$

It is easy to check that:

$$\int_M L(\xi)dv = 0. \tag{2.5}$$

We need the following two lemmas.

Lemma 2.1 (see [8])**.**

$$\|\nabla h\|^2 - \frac{3n^2}{n+2}\|\nabla \mathbf{H}\|^2 \geq 0, \tag{2.6}$$

where $\|\nabla h\|^2 = \sum_{\alpha,i,j}(h^\alpha_{ij,k})^2$, $\|\nabla \mathbf{H}\|^2 = \sum_{\alpha,i}(H^\alpha_{,i})^2$.

Lemma 2.2.

$$\int_M \frac{n^2(n-1)}{n+2}H^\alpha\Delta H^\alpha + h^\alpha_{ij}h^\alpha_{kl}R_{kijl} + h^\alpha_{ik}h^\alpha_{kj}R_{ij} + h^\alpha_{ik}h^\beta_{kj}R_{\beta\alpha ij} \leq 0. \tag{2.7}$$

Proof. Using the Codazzi equation and the Ricci identity, we have

$$nH^\alpha_{,ij} - h^\alpha_{ij,kk} = h^\alpha_{kk,ij} - h^\alpha_{ij,kk} = h^\alpha_{ik,kj} - h^\alpha_{ik,jk} = h^\alpha_{lk}R_{likj} + h^\alpha_{il}R_{lkkj} + h^\beta_{ik}R_{\beta\alpha kj}.$$

Noting that

$$\frac{1}{2}\Delta S = \frac{1}{2}\Delta \sum_{\alpha,i,j}(h^\alpha_{ij})^2 = \|\nabla h\|^2 + \sum_{\alpha,i,j,k} h^\alpha_{ij}h^\alpha_{ij,kk}$$

and

$$\frac{1}{2}\Delta\|\mathbf{H}\|^2 = \|\nabla\mathbf{H}\|^2 + H^\alpha\Delta H^\alpha,$$

we have

$$L(n\mathbf{H}) = nh^\alpha_{ij}H^\alpha_{,ij} - n^2H^\alpha\Delta H^\alpha$$

$$= \frac{1}{2}\Delta(S - n^2\|\mathbf{H}\|^2) - \|\nabla h\|^2 + n^2\|\nabla\mathbf{H}\|^2 + h^\alpha_{ij}h^\alpha_{lk}R_{likj} + h^\alpha_{ij}h^\alpha_{il}R_{lkkj} + h^\alpha_{ij}h^\beta_{ik}R_{\beta\alpha kj}$$

$$\leq \frac{1}{2}\Delta(S - n^2\|\mathbf{H}\|^2) + \frac{n^2(n-1)}{n+2}\|\nabla\mathbf{H}\|^2 - h^\alpha_{ij}h^\alpha_{kl}R_{iklj} - h^\alpha_{ik}h^\alpha_{kj}R_{ij} - h^\alpha_{ik}h^\beta_{kj}R_{\beta\alpha ij}.$$

We used Lemma 2.1 in the last step. From (2.5) we see that Lemma 2.2 holds. □

Proof of Theorem 1.1. Setting

$$Q = h^\alpha_{ij}h^\alpha_{kl}R_{iklj} + h^\alpha_{ik}h^\alpha_{kj}R_{ij} + h^\alpha_{ik}h^\beta_{kj}R_{\beta\alpha ij},$$

denoting $A_\alpha = (h^\alpha_{ij})$, and using the Gaussian equation, we have

$$Q = -\sum_{\alpha\neq\beta} N(A_\alpha A_\beta - A_\beta A_\alpha) - \sum_{\alpha,\beta}[tr(A_\alpha A_\beta)]^2 + n\rho^2 + \sum_{\alpha,\beta} nH^\beta tr(A^2_\alpha A_\beta),$$

where $N(A)$ denotes the normal of an $n\times n$ matrix A, which is defined as $N(A) = tr(AA^t)$. Let B be a trace-free tensor defined by

$$A_\alpha = B_\alpha + H^\alpha I.$$

Then $\rho^2 = \sum_\alpha tr(B^2_\alpha)$. We can rewrite Q as follows:

$$Q = -\sum_{\alpha\neq\beta} N(B_\alpha B_\beta - B_\beta B_\alpha) - \sum_{\alpha,\beta}[tr(B_\alpha B_\beta)]^2 + n\rho^2 + \sum_{\alpha,\beta} nH^\beta tr(B^2_\alpha B_\beta) + n\|\mathbf{H}\|^2\rho^2.$$

In the following we assume $n = 2$. From equation (1.2) of the $\mathcal{H}$-surface, we have

$$\Delta H^\alpha + \sum_\beta[tr(B_\alpha B_\beta) + 2H^\alpha H^\beta - 2\|\mathbf{H}\|^2\delta_{\alpha\beta}]H^\beta + 2H^\alpha = 0.$$

Substituting these into the inequality in Lemma 2.2 (i.e. (2.7)), we have

$$0 \geq -\sum_{\alpha\neq\beta} N(B_\alpha B_\beta - B_\beta B_\alpha) - \sum_{\alpha,\beta}[tr(B_\alpha B_\beta)]^2$$

$$+2\rho^2 + 2\|\mathbf{H}\|^2\rho^2 - 2\|\mathbf{H}\|^2 - H^\alpha H^\beta tr(B_\alpha B_\beta) + 2H^\beta tr(B^2_\alpha B_\beta). \tag{2.8}$$

We will estimate the rest of the terms. Set $\sigma_{\alpha\beta} = tr(B_\alpha B_\beta)$. Then the $p\times p$-matrix $(\sigma_{\alpha\beta})$ is symmetric and can be assumed to be diagonal for a suitable choice of $\{e_\alpha\}$. So we see that

$$\sum_{\alpha,\beta} H^\alpha H^\beta tr(B_\alpha B_\beta) = \sum_\alpha (H^\alpha)^2 tr(B^2_\alpha) \leq \sum_\alpha (H^\alpha)^2 \sum_\beta tr(B^2_\beta) = \|\mathbf{H}\|^2\rho^2.$$

As B_α is a symmetric 2×2-matrix and $tr(B_\alpha) = 0$, it is easy to check that

$$\sum_{\alpha,\beta} 2H^\beta tr(B_\alpha^2 B_\beta) = 0.$$

So, from (2.8) we have

$$0 \geq -\sum_{\alpha\neq\beta} N(B_\alpha B_\beta - B_\beta B_\alpha) - \sum_{\alpha,\beta}[tr(B_\alpha B_\beta)]^2$$

$$+2\rho^2 + \|\mathbf{H}\|^2\rho^2 - 2\|\mathbf{H}\|^2. \tag{2.9}$$

Making use of the famous inequality [4]

$$\sum_{\alpha\neq\beta} N(B_\alpha B_\beta - B_\beta B_\alpha) + \sum_{\alpha,\beta}[tr(B_\alpha B_\beta)]^2 \leq (2 - \frac{1}{p})\rho^4, \tag{2.10}$$

from (2.9) we have

$$0 \geq \int_M [2 - (2 - \frac{1}{p})\rho^2]\rho^2 - 2\|\mathbf{H}\|^2. \tag{2.11}$$

Noting that the Gaussian equation (2.2) and the Gauss-Bonnet theorem imply

$$\int_M 2\|\mathbf{H}\|^2 = 4\pi\chi(M) - \int_M (2 - \rho^2),$$

we see that the inequality (2.11) implies the inequality (1.4) in Theorem 1.1. Since the equality of (1.4) holds implies the equality of (2.11) holds, thus $\|\mathbf{H}\|^2\rho^2 = 0$ and the equality of (2.6) holds. This leads to $\|\mathbf{H}\| \equiv 0$ or $\rho^2 \equiv 0$. But the latter also implies $\|\mathbf{H}\| \equiv 0$, which can be seen from the equation of $\mathcal{H}$-surfaces. Now that M is minimal and the equality of (2.11) holds, we have $\int S[2 - (2 - \frac{1}{p})S] = 0$, which implies that M is a geodesic sphere, or a Clifford torus in $\mathbb{S}^3$ or a Veronese surface in $\mathbb{S}^4$[4]. This completes the proof of Theorem 1.1. $\square$

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