

## A NOTE ON “REGULARITY LEMMA FOR DISTAL STRUCTURES”

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ABSTRACT. In a recent paper, Chernikov and Starchenko prove that graphs defined in distal theories have strong regularity properties, generalizing previous results about graphs defined by semi-algebraic relations. We give a shorter, purely model-theoretic proof of this fact, though with no explicit bounds.

In their recent paper [CS], Chernikov and Starchenko prove that graphs defined in distal theories have the *strong Erdős-Hajnal property*: if  $R \subseteq M^m \times M^n$  is a definable relation in some distal structure  $M$ , there is  $\alpha > 0$  such that if  $A \subseteq M^m$ ,  $B \subseteq M^n$  are finite sets, then there are  $A_0 \subseteq A$ ,  $B_0 \subseteq B$  with  $|A_0| \geq \alpha|A|$ ,  $|B_0| \geq \alpha|B|$  and  $R \cap A_0 \times B_0$  is either empty or equal to  $A_0 \times B_0$ . In fact, they prove a more general statement for hypergraphs and generically stable measures instead of counting measure; see Theorem 2.2 below. This generalizes a theorem of Alon et al. [APP<sup>+</sup>05] who proved it if  $R$  is semi-algebraic.

Chernikov and Starchenko’s proof uses both the theory of Keisler measures in NIP and some combinatorial arguments. The purpose of this note is to remove the latter and give a purely model-theoretic (and in fact quite short) proof of the strong Erdős-Hajnal property in distal theories. We also answer a small question raised in [CS] about uniformly cutting finite sets and show that this property is equivalent to having no generically stable types. We keep this note short and refer to the excellent introduction of [CS] for background.

### 1. GENERICALLY STABLE AND SMOOTH MEASURES

We record here some facts about generically stable and smooth measures in NIP theories. Everything appeared in previous works, possibly in different terms. See e.g. [Sim15, Chapter 7] for details.

Throughout this note, we assume that our ambient theory  $T$  is NIP. Recall that a (Keisler) measure  $\mu(x)$  over some set  $A$  is a finitely additive probability measure on the boolean algebra of  $A$ -definable sets (in the free variable  $x$ ). Such a measure extends uniquely to a regular Borel measure on  $S_x(A)$ .

Of special importance are measures  $\mu(x)$  which can be approximated by average measures on finite sets. Such measures are called *generically stable*. Here is one way to define them, where  $\text{Av}(a_1, \dots, a_n; X)$  means  $\frac{1}{n}|\{i : a_i \in X\}|$ .

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**Definition 1.1.** A measure  $\mu(x)$  over a model  $M$  is generically stable if for every formula  $\phi(x; y)$  and  $\epsilon$ , there are  $a_1, \dots, a_n \in M$  such that for any  $b \in M^{|y|}$ ,  $|\mu(\phi(x; b)) - \text{Av}(a_1, \dots, a_n; \phi(x; b))| \leq \epsilon$ .

In particular, the normalized counting measure on a finite set  $\{a_1, \dots, a_n\}$  is generically stable. The VC-theorem implies that the number  $n$  above depends only on  $\phi(x; y)$  and  $\epsilon$ :

*Fact 1.2.* Let  $\phi(x; y)$  be a formula and  $\epsilon > 0$ , then there is an integer  $n$  such that for any generically stable measure  $\mu(x)$  on a model  $M$ , there are  $a_1, \dots, a_n \in M$  such that for any  $b \in M^{|y|}$ ,  $|\mu(\phi(x; b)) - \text{Av}(a_1, \dots, a_n; \phi(x; b))| \leq \epsilon$ .

In fact, the number  $n$  depends only on the VC-dimension of  $\phi$  (and  $\epsilon$ ).

**Corollary 1.3.** Let  $I$  be an index set and for  $i \in I$ ,  $\mu_i(x)$  a generically stable measure on a model  $M_i$ . For  $\mathcal{U}$  an ultrafilter on  $I$ , the ultraproduct  $\prod_{\mathcal{U}} \mu_i$  is a generically stable measure on  $\prod_{\mathcal{U}} M_i$ .

*Proof.* Let  $\tilde{\mu} = \prod_{\mathcal{U}} \mu_i$  and fix a formula  $\phi(x; y)$  and  $\epsilon > 0$ . For each  $i$ , take  $a_1(i), \dots, a_n(i)$  in  $M_i$  given by Fact 1.2 for  $\mu_i$ . Define  $\bar{a}_k := [a_k(i) : i \in I] \in \prod_{\mathcal{U}} M_i$ . Then for any parameter  $b$ , we have  $|\tilde{\mu}(\phi(x; b)) - \text{Av}(\bar{a}_1, \dots, \bar{a}_k; \phi(x; b))| \leq \epsilon$ . Hence  $\tilde{\mu}$  is generically stable.  $\square$

A measure  $\mu(x)$  over  $M$  is called *smooth* if it has a unique extension to any bigger model  $N$ . We have the following equivalent condition (which follows easily by compactness; see [Sim15, Lemma 7.8]).

*Fact 1.4.* A measure  $\mu(x)$  over  $M$  is smooth if and only if for every formula  $\phi(x; y)$  and  $\epsilon > 0$ , there are finitely many formulas  $\psi_i(y)$  and  $\theta_i^1(x), \theta_i^2(x)$  all over  $M$  such that:

- (1) the formulas  $\psi_i(y)$  partition  $y$ -space;
- (2) for all  $i$  and  $b \in M^{|y|}$ , if  $M \models \psi_i(b)$ , then

$$M \models (\theta_i^1(x) \rightarrow \phi(x; b) \rightarrow \theta_i^2(x));$$

- (3) for all  $i$ ,  $\mu(\theta_i^2(x)) - \mu(\theta_i^1(x)) \leq \epsilon$ .

Let  $\mu(x)$  and  $\nu(y)$  be two measures over the same base  $M$ . We say that a measure  $\omega(x, y)$  over  $M$  is a *product measure* of  $\mu(x)$  and  $\lambda(y)$  if  $\omega(\phi(x) \wedge \psi(y)) = \mu(\phi(x)) \cdot \lambda(\psi(y))$  for any two definable sets  $\phi(x)$  and  $\psi(y)$ . We let  $(\mu \times \lambda)(x, y)$  denote the partial measure defined on the boolean algebra generated by rectangles  $\phi(x) \wedge \psi(y)$  and giving such a set the product measure  $\mu(\phi(x)) \cdot \lambda(\psi(y))$  (this determines the measure on the boolean algebra since a boolean combination of rectangles can be written as a disjoint union of rectangles). Note that if  $\lambda(y)$  is a type, then any extension of  $\mu(x) \cup \lambda(y)$  is a product measure.

We say that two measures  $\mu(x)$  and  $\lambda(y)$  over  $M$  are *weakly orthogonal* if there is a unique product measure of  $\mu(x)$  and  $\lambda(y)$ .

**Lemma 1.5.** Two measures  $\mu(x)$  and  $\lambda(y)$  over  $M$  are weakly orthogonal if and only if the following property holds:

For any  $M$ -definable set  $D(x, y)$  and  $\epsilon > 0$ , there are two definable sets  $D_\epsilon^-(x, y)$  and  $D_\epsilon^+(x, y)$  such that:

- (i)  $D_\epsilon^-$  and  $D_\epsilon^+$  are a union of rectangles  $A_i(x) \times B_i(y)$ ;
- (ii)  $D_\epsilon^- \subseteq D \subseteq D_\epsilon^+$ ;
- (iii)  $(\mu \times \lambda)(D_\epsilon^+) - (\mu \times \lambda)(D_\epsilon^-) \leq \epsilon$ .

*Proof.* Right to left: the conditions imply that any product measure must give  $D$  measure  $\inf_{\epsilon>0}(\mu \times \lambda)(D_\epsilon^+) = \sup_{\epsilon<0}(\mu \times \lambda)(D_\epsilon^-)$ . Hence there is a unique such measure.

For the converse, we use Lemma 7.3 in [Sim15]. It says that given a boolean algebra  $\Omega$  of definable sets and a partial measure  $\mu_0$  on  $\Omega$ , we can extend  $\mu_0$  to a Keisler measure  $\mu$  and additionally impose  $\mu(D) = r$  for some definable set  $D$  and real  $r \in [0, 1]$  as long as there is no obvious obstruction, namely any  $B \in \Omega$ ,  $B \subseteq D$  has measure  $\leq r$  and any  $B \in \Omega$ ,  $D \subseteq B$  has measure  $\geq r$ . We apply this here with  $\Omega$  being the algebra generated by rectangles. □

**Lemma 1.6.** *Let  $\mu(x)$  be a measure over  $M$ . Then  $\mu(x)$  is smooth if and only if it is weakly orthogonal to all measures over  $M$ .*

*Proof.* By definition, a measure is smooth if and only if it is weakly orthogonal to all types over  $M$ . Hence right to left is fine.

Left to right is [HPS13, Corollary 2.5] and follows easily from Fact 1.4: a formula  $\phi(x; y)$  is well approximated from below by the union of rectangles  $P(x, y) := \bigvee_i \theta_i^0(x) \wedge \psi_i(y)$  and from above by  $Q(x, y) := \bigvee_i \theta_i^1(x) \wedge \psi_i(y)$ . If  $\omega(x, y)$  is a product measure of  $\mu(x)$  and some  $\lambda(y)$ , then  $\omega(Q) - \omega(P)$  cannot be more than  $\epsilon$ . □

If  $R(x_1, \dots, x_n)$  is a relation and sets  $A_i \subseteq M^{|x_i|}$  are given, we say that the tuple  $(A_1, \dots, A_n)$  is  $R$ -homogeneous if either for all  $(x_1, \dots, x_n) \in A_1 \times \dots \times A_n$ ,  $R(x_1, \dots, x_n)$  holds or for all  $(x_1, \dots, x_n) \in A_1 \times \dots \times A_n$ ,  $\neg R(x_1, \dots, x_n)$  holds.

**Corollary 1.7.** *Let  $\mu(x)$  be a smooth measure and  $\lambda(y)$  any Keisler measure. Let  $R(x; y)$  be a formula over  $M$  and  $\epsilon > 0$ . Then there is an  $M$ -definable partition  $P_1, \dots, P_n$  of  $x$ -space and  $Q_1, \dots, Q_m$  of  $y$ -space such that*

$$\sum \mu(P_i)\lambda(Q_j) < \epsilon,$$

where the sum runs over all pairs  $(i, j)$  such that  $(P_i, Q_j)$  is not  $R$ -homogeneous.

*Proof.* By the previous lemmas, there are two definable sets  $P, Q \subseteq M^{|x|+|y|}$  which are a union of rectangles  $A_i(x) \times B_i(y)$  such that:

1.  $P \subseteq R \subseteq Q$ ;
2.  $\omega(Q) - \omega(P) < \epsilon$ , for any product measure  $\omega(x, y)$  of  $\mu(x)$  and  $\lambda(y)$ .

We then obtain the partition  $(P_i)_i$  by taking all the atoms in the boolean algebra generated by the  $A_i$ 's and the same for  $(Q_j)_j$  and the  $B_i$ 's. □

This generalizes immediately to a product of  $n$ -many smooth measures (or indeed  $n - 1$  smooth measures and an arbitrary one).

**Corollary 1.8.** *Let  $\mu_1(x_1), \dots, \mu_n(x_n)$  be smooth measures over  $M$ . Fix a formula  $R(x_1, \dots, x_n)$  and  $\epsilon > 0$ . Then for each  $i$ , there is an  $M$ -definable partition  $P(i, 1), \dots, P(i, m(i))$  of  $x_i$ -space such that*

$$\sum \mu_1(P(1, i_1)) \cdots \mu_n(P(n, i_n)) < \epsilon,$$

where the sum runs over all tuples  $(i_1, \dots, i_n)$  for which the family  $(P(1, i_1), \dots, P(n, i_n))$  is not  $R$ -homogeneous.

2. DISTALITY AND THE MAIN RESULT

The class of distal theories was introduced in [Sim13] to capture the notion of an order-like, or purely unstable, NIP theory. There are various equivalent definitions of distality in terms of indiscernible sequences, invariant types, measures, arbitrary types, etc. We only need to know here that a theory is distal if and only if all generically stable measures are smooth. Thus we have:

**Lemma 2.1.** *Let  $T$  be distal; then an ultraproduct of smooth measures is smooth.*

*Proof.* This follows immediately from Corollary 1.3 and the fact that generically stable measures and smooth measures coincide. □

We now have all we need to prove the main theorem. The following is Corollary 4.6 in [CS], which is the final statement in Sections 3 and 4 of that paper. We prove the version with parameters directly, although as observed in the proof of Corollary 4.6, it would follow at once from the simpler parameter-free version.

**Theorem 2.2.** *Let  $T$  be distal,  $R_y(x_1, \dots, x_d) = R(x_1, \dots, x_d; y)$  a definable relation. Given  $\alpha > 0$ , there is  $\delta > 0$  such that: for any parameter  $b$ , for any generically stable (equiv. smooth) product measure  $\omega$  on  $M^{|x_1|} \times \dots \times M^{|x_d|}$  with  $\omega(R_b) \geq \alpha$ , there are definable sets  $A_i \subseteq M^{|x_i|}$  with  $\omega|_{x_i}(A_i) > \delta$  for all  $i$  such that  $\prod_i A_i \subseteq R_b$ .*

*Moreover, each  $A_i$  is defined by an instance of a formula that depends only on  $R$  and  $\alpha$ .*

*Proof.* This is a direct consequence of Corollary 1.8 and Lemma 2.1. We now give the details. First, by Corollary 1.8, given  $\omega$  and  $R_b$ , we can find sets  $A_i \subseteq M^{|x_i|}$  such that  $\prod A_i \subseteq R_b$  and  $\omega|_{x_i}(A_i) > \delta$  for some  $\delta > 0$  depending on all the data.

Now assume that the conclusion is false. For simplicity of notation, assume  $L$  is countable and list all tuples of the form  $(\theta_1(x_1; y_1), \dots, \theta_d(x_d; y_d))$  as  $(\bar{\theta}_j : j < \omega)$ , where  $\bar{\theta}_j = (\theta_1^j(x_1; y_{1,j}), \dots, \theta_d^j(x_d; y_{d,j}))$ . Given an integer  $n$ , there is a smooth measure  $\omega = \omega(n)$  over a model  $M = M(n)$  and a parameter  $b = b(n) \in M(n)$  such that  $\omega(R_b) \geq \alpha$  and for any  $j \leq n$  and any choice of parameters  $b_{i,j}$  from  $M$ , setting  $A_i = \theta_i^j(M; b_{i,j})$ , either for some  $i$ ,  $\omega|_{x_i}(A_i) \leq 1/n$ , or it is not the case that  $\prod_i A_i \subseteq R_b$ .

Take a non-principal ultraproduct of the  $\omega(n)$  to obtain a measure  $\tilde{\omega}$  over some model  $\tilde{M}$  and some  $\tilde{b} \in \tilde{M}$  such that  $\tilde{\omega}(R_{\tilde{b}}) \geq \alpha$ . By Lemma 2.1,  $\tilde{\omega}$  is smooth. Hence by the first paragraph, we can find  $A_i$ 's and  $\delta$  for  $\tilde{\omega}$  and  $R_{\tilde{b}}$  as in the statement. The same  $\delta$  and formulas defining the  $A_i$ 's will work for almost all factors, contradicting the construction. □

We can also prove the asymmetric version which is Theorem 3.6 in [CS].

**Theorem 2.3.** *Let  $T$  be distal and  $R(x, y)$  a definable relation. Let  $\beta \in (0, \frac{1}{2})$ . There are  $\alpha \in (0, 1)$  and formulas  $\phi_1(x; z_1), \phi_2(y; z_2)$  such that:*

*For any Keisler measure  $\lambda(x)$  and generically stable measure  $\mu(y)$  both on  $M$ , there are parameters  $c_1 \in M^{|z_1|}$  and  $c_2 \in M^{|z_2|}$  with  $\lambda(\phi_1(x; c_1)) \geq \alpha$ ,  $\mu(\phi_2(y; c_2)) \geq \beta$  and the pair  $\phi_1(M; c_1), \phi_2(M; c_2)$  is  $R$ -homogeneous.*

*Proof.* Choose  $R$  and  $\beta \in (0, \frac{1}{2} - \epsilon)$  and fix a pair of measures  $\lambda(x), \mu(y)$  as in the statement. As  $\mu$  is smooth, we can apply Fact 1.4 to it, with parameter  $\epsilon$ . It gives us formulas  $\psi_i(x)$  and  $\theta_i^k(y)$ . Take  $i$  such that  $\lambda(\psi_i(x)) =: \alpha > 0$  and set  $\phi_1(x) = \psi_i(x)$ . At least one of  $\theta_i^1(y)$  or  $\neg\theta_i^2(y)$  has measure  $\geq \beta$ . Let  $\phi_2(y)$  be

equal to it. Then the pair  $(\phi_1(x), \phi_2(y))$  (with hidden parameters from  $M$ ) is as required for the given pair of measures. We conclude by a compactness argument as in Theorem 2.2.  $\square$

### 3. EQUIPARTITIONS

In Remark 5.11 of [CS], it is asked whether in distal theories, one can cut finite sets uniformly. We answer this question positively.

**Definition 3.1.** Let  $\mathcal{S}$  be a sort. We say that  $T$  uniformly cuts finite sets in  $\mathcal{S}$  if for every  $\epsilon > 0$ , there is a formula  $\chi(x; y)$  such that for any sufficiently large finite set  $A$  in  $\mathcal{S}$ , and  $r \in [0, 1]$ , there is a parameter  $b$  for which

$$\left| \frac{|\chi(A; b)|}{|A|} - r \right| \leq \epsilon.$$

We will say that  $T$  uniformly cuts generically stable measures on  $\mathcal{S}$  if the conclusion of Proposition 5.12 in [CS] holds, namely: For every formula  $\phi(x; y)$  and  $\epsilon > 0$ , there is some  $\chi(x; z)$  such that for any generically stable measure  $\mu$  on  $M$  with  $\mu(\{c\}) = 0$  for any singleton  $c \in M^{|x|}$ , if  $0 \leq r \leq \mu(\phi(x; a))$ , then we can find  $b \in M$  with  $|\mu(\phi(x; a) \cap \chi(x; b)) - r| \leq \epsilon$ .

In the following proof, we write  $a \approx_\epsilon b$  for  $|a - b| \leq \epsilon$ .

**Lemma 3.2.** Let  $\mu(x)$  be generically stable over  $M$  and  $p \in S_x(M)$  such that  $\mu(\{p\}) > 0$ . Then  $p$  is generically stable.

*Proof.* This can be seen in various ways. Here is an argument which does not use any additional fact about generically stable measures. Fix a formula  $\phi(x; y)$  and  $\epsilon > 0$ . Set  $\alpha = \mu(\{p\})$ . By regularity of  $\mu$  seen as a measure on  $S_x(M)$ , there is a formula  $\theta(x) \in L(M)$  such that  $p \vdash \theta(x)$  and  $\mu(\theta(x)) \leq (1 + \epsilon)\alpha$ . Set  $\phi'(x; y) = \phi(x; y) \wedge \theta(x)$  and without loss, assume that for some  $b_0$ ,  $\phi'(x; b_0) = \theta(x)$ . Take  $a_1, \dots, a_n$  given by Definition 1.1 for  $\mu$ ,  $\phi'$  and  $\epsilon' := \epsilon\alpha$ . Reordering, there is  $k \leq n$  such that  $a_i \models \theta(x)$  if and only if  $i \leq k$ . We have  $|k/n| = (1 + \delta)\alpha$  for  $\delta \leq 2\epsilon$ . Then for any  $b$ ,  $p(\phi(x; b)) \approx_\epsilon \frac{1}{\alpha}\mu(\phi'(x; b)) \approx_\epsilon \frac{1}{\alpha} \text{Av}(a_1, \dots, a_n; \phi'(x; b)) = \text{Av}(a_1, \dots, a_k; \phi(x; b)) \frac{k}{n} \frac{1}{\alpha} \approx_\delta \text{Av}(a_1, \dots, a_k; \phi(x; b))$ . Hence  $p$  is generically stable.  $\square$

**Proposition 3.3.** Let  $T$  be NIP and  $\mathcal{S}$  a sort. The following are equivalent:

- (1)  $T$  uniformly cuts finite sets in  $\mathcal{S}$ ;
- (2)  $T$  uniformly cuts generically stable measures on  $\mathcal{S}$ ;
- (3) any generically stable type concentrating on  $\mathcal{S}$  is realized.

*Proof.* (3)  $\Rightarrow$  (2): First, let  $\mu(x)$  be a fixed generically stable measure over  $M$  with  $\mu(\{c\}) = 0$  for all  $c$ . Then if  $\mu(\{p\}) > 0$  for some  $p \in S_x(M)$ ,  $p$  is generically stable. By (3) and the assumption on  $\mu$ , this does not happen. But then by regularity of  $\mu$ , for any  $p$  in the support of  $\mu$ , we can find a clopen set  $U_p \subseteq S_x(M)$  containing  $p$  of measure  $\mu(U_p) < \epsilon$ . By compactness of the support, we can extract a finite cover which we can refine to be composed of disjoint sets. Hence we obtain a finite partition of  $x$ -space into definable sets of measures  $\leq \epsilon$ . By Corollary 1.3, the size and the formulas involved in this partition can in fact be chosen uniformly in  $\mu$ . By standard coding techniques, we construct a formula  $\chi(x; z)$  such that any finite union of members of this partition is equal to some  $\chi(x; b)$ . Then  $\chi(x; z)$  has the required properties (and does not depend on  $\phi(x; y)$ ).

(2)  $\Rightarrow$  (1): This is a simple compactness argument. If (1) does not hold, then for a given  $\epsilon$ , there is a sequence  $A_n$  of finite sets,  $|A_n| \geq n$  such that for any formula  $\chi(x; y)$ , for  $n$  large enough,  $\chi(x; y)$  cannot be used to cut  $A_n$  with precision  $\epsilon$  as in Definition 3.1. Let  $\mu_n$  be the normalized counting measure on  $A_n$  and let  $\bar{\mu}$  be a non-principal ultraproduct of the  $\mu_n$ 's. Then  $\bar{\mu}$  is generically stable and  $\bar{\mu}(\{c\}) = 0$  for all  $c$ . Assumption (2) applied to  $\phi(x; a) = \top$  gives us a formula  $\chi(x; y)$  which can be used to partition the space in sets of arbitrary  $\bar{\mu}$ -measures up to  $\epsilon$ . But then the same  $\chi(x; y)$  works for almost all the measures  $\mu_n$  contradicting the assumption.

(1)  $\Rightarrow$  (3): Assume that (3) fails. Then there is a non-constant totally indiscernible sequence  $I = (a_i : i < \omega)$  of elements of  $\mathcal{S}$  (take a Morley sequence of a generically stable type). Then by NIP, for any formula  $\chi(x; y)$  there is  $N$  such that for any  $b$ , either  $\chi(I; b)$  or  $I \setminus \chi(I; b)$  has size  $\leq N$ . Thus taking large finite sets of  $I$ , we see that (1) cannot hold.  $\square$

Since distal theories satisfy (3), this gives a proof of [CS, Proposition 5.12] for all distal theories (that proposition asserts (2) above assuming distality and (1)).

We end with a question. It is proved in [CS] that there is no infinite distal field of characteristic  $p$ . In general, it would be very interesting to understand what are the possible obstructions to an NIP theory having a distal expansion. One can also ask about expansions that uniformly cut finite sets.

**Question 3.4.** Does  $\text{ACF}_p$  have an NIP expansion which uniformly cuts finite sets?

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