SINGULAR INTEGRALS WITH ANGULAR INTEGRABILITY

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Abstract. In this note we prove a class of sharp inequalities for singular integral operators in weighted Lebesgue spaces with angular integrability.

1. Introduction

We consider singular integral operators
\[ Tf(x) := \text{P.V.} \int_{\mathbb{R}^n} f(x - y)K(y) \, dy, \]
where the kernel \( K \) satisfies the following conditions:
\[ |y|^n |K| \leq C, \quad |y|^{n+1} |\nabla K| \leq C, \quad |\hat{K}| \leq C. \]
Here \( C > 0 \) is a constant and \( \hat{\cdot} \) denotes the Fourier transform. The main example we have in mind is the directional Riesz transform, which corresponds to the choice \( K(y) := |y|^{-(n+1)}y \cdot \theta, \ \theta \in \mathbb{S}^{n-1}. \)

The study of the boundedness of these operators in weighted Lebesgue spaces \( L^p(w(x)dx), \) for \( 1 < p < \infty \) and \( 0 < w \in L^1_{\text{loc}}(\mathbb{R}^n), \) is a classical problem in harmonic analysis: in particular, Stein [13] proved it for the (sharp range of) homogeneous weights \( w(x) = |x|^{\alpha p}, \ -n/p < \alpha < n - n/p. \) The result was later extended by Coifman and Fefferman [2] to any \( A_p \) weight.

While the weighted \( L^p \)-theory has been extensively studied, less is known in the case of Lebesgue norms with different integrability in the radial and angular directions, namely
\[ \|f\|_{L^p_{|x|^{\frac{n}{p}}}L^\tilde{p}_\theta} := \left( \int_0^{+\infty} \|f(\rho \cdot)\|^p_{L^\tilde{p}(\mathbb{S}^{n-1})} \rho^{n-1} \, d\rho \right)^{\frac{1}{p}}. \]

These mixed radial-angular spaces have been successfully used in recent years to improve several results in the framework of partial differential equations; see e.g. [1,8,10–12,14,15]. Notice that when \( p = \tilde{p} \) the norms reduce to the usual \( L^p \) norms. Notice also that, neglecting the constants, they are increasing in \( \tilde{p}, \) and that they behave as the \( L^p \) norms under homogeneous rescaling, namely \( f(\cdot) \to f(\lambda \cdot), \ \lambda > 0. \)

In a recent paper A. Córdoba [3] proved, among other things, the \( L^p_{|x|^{\frac{n}{p}}}L^\tilde{p}_\theta \) boundedness for operators of the form (1.1). Here we give an extension of this result to the weighted setting.
Theorem 1.1. Let $n \geq 2$, $1 < p < \infty$, $1 < \tilde{p} < \infty$ and $-n/p < \alpha < n - n/p$. Then
\begin{equation}
\| |x|^\alpha T\phi\|_{L^p_{|x|} L^{\tilde{p}}_{\theta}} \leq C\| |x|^\alpha \phi\|_{L^p_{|x|} L^{\tilde{p}}_{\theta}},
\end{equation}
where $C$ is a constant depending only on $\alpha, p, \tilde{p}, n$.

Let us point out that the case $\alpha = 0$, $1 < \tilde{p} < \infty$ in inequality (1.3) may be deduced by the application of Córdoba’s argument; see [6, Theorem 2.6]. Therefore the novelty of Theorem 1.1 is that it covers all the possible homogeneous weights of the kind $|x|^\alpha$.

Remark 1.1. Condition $-n/p < \alpha$ turns out to be necessary by testing the inequality on functions $\phi$ such that $\phi = 0$ and $T\phi > 0$ in a neighborhood of the origin. On the other hand, condition $\alpha < n - n/p$ turns out to be necessary for the same reason by considering the dual inequality.

Remark 1.2. One of the estimates (1.3) has been used in [6, Theorem 1.5] to deduce information about the regularity of weak solutions of the 3d Navier–Stokes problem with initial velocities satisfying good angular integrability properties.

We write $A \lesssim B$ if $A \leq CB$ with a constant $C$ depending only on $\alpha, p, \tilde{p}, n$. We write $A \simeq B$ if both $A \lesssim B$ and $B \lesssim A$.

2. Proof

We know by [6, Theorem 2.6] that inequality (1.3) is true in the case $\alpha = 0$, that is,
\begin{equation}
\| Tg\|_{L^p_{|x|} L^{\tilde{p}}_{\theta}} \lesssim \| g\|_{L^p_{|x|} L^{\tilde{p}}_{\theta}}.
\end{equation}

Following Stein [13], we now show that the weighted case (1.3) can be then deduced by the unweighted one. The next lemma represents the core of the proof.

Lemma 2.1. Let $n \geq 2$, $1 < p < \infty$, $1 \leq \tilde{p} \leq \infty$, $-n/p < \alpha < n - n/p$ and
\begin{equation}
F(x, y) := \frac{1 - (|x|/|y|)^\alpha}{|x - y|^n};
\end{equation}
then
\begin{equation}
\left\| \int_{\mathbb{R}^n} F(x, y) \phi(y) dy \right\|_{L^p_{|x|} L^{\tilde{p}}_{\theta}} \leq C\| \phi\|_{L^p_{|x|} L^{\tilde{p}}_{\theta}}.
\end{equation}

Assume indeed this has been proved and first apply inequality (2.1) with the choice $g := | \cdot |^\alpha f$ to have
\begin{equation}
\| T(|x|^\alpha f)\|_{L^p_{|x|} L^{\tilde{p}}_{\theta}} \lesssim \| |x|^\alpha f\|_{L^p_{|x|} L^{\tilde{p}}_{\theta}}.
\end{equation}
Then notice that
\begin{equation}
| T(|x|^\alpha f) - |x|^\alpha Tf | \leq \int_{\mathbb{R}^n} |K(x - y)(|y|^\alpha - |x|^\alpha) f(y)| dy \lesssim \int_{\mathbb{R}^n} \frac{|y|^\alpha - |x|^\alpha}{|x - y|^n} |f(y)| |y|^\alpha f(y) dy = \int_{\mathbb{R}^n} \frac{1 - (|x|/|y|)^\alpha}{|x - y|^n} |y|^\alpha f(y) dy,
\end{equation}
so that by using Lemma 2.1 with $\phi = | \cdot |^\alpha f$ we obtain
\begin{equation}
\| T(|x|^\alpha f) - |x|^\alpha Tf\|_{L^p_{|x|} L^{\tilde{p}}_{\theta}} \lesssim \| |x|^\alpha f\|_{L^p_{|x|} L^{\tilde{p}}_{\theta}}.
\end{equation}
Then, the desired estimate (1.3) follows by (2.4) and (2.5) and triangle inequality. Thus it only remains to prove Lemma 2.1.

The idea is to use a change of variables which resembles the standard polar coordinates. In this variant the integration over the sphere is replaced by integration over the special orthogonal group $SO(n)$ and the radial integration is replaced by integration over the multiplicative group of the positive real numbers. This method works efficiently when homogeneous power weights are involved; see e.g. [5,7].

**Proof of Lemma 2.1** By using the isomorphism

$$S^{n-1} \simeq SO(n)/SO(n-1)$$

we can rewrite integrals on $S^{n-1}$ as follows:

$$\int_{S^{n-1}} g(y) dS(y) \simeq \int_{SO(n)} g(Ae) dA, \quad n \geq 2,$$

where $dA$ is the left Haar measure on $SO(n)$, and $e \in S^{n-1}$ is a fixed unit vector. Thus, via polar coordinates, a generic integral can be rewritten as

$$\int_{R^n} F(x,y) \phi(x) dy = \int_{0}^{\infty} \int_{S^{n-1}} F(x,\rho \omega) \phi(\rho \omega) dS_\omega \rho^{n-1} d\rho \simeq \int_{0}^{\infty} \int_{SO(n)} F(x,\rho Be) \phi(\rho Be) dB \rho^{n-1} d\rho.$$

Hence the $L_{\theta}^\gamma$ norm can be written as

$$\left\| \int_{R^n} F(|x|\theta, y) \phi(y) dy \right\|_{L_{\theta}^\gamma(S^{n-1})} \simeq \left\| \int_{R^n} F(|x|Ae, y) \phi(y) dy \right\|_{L_{\theta}^\gamma(SO(n))}$$

$$\leq \int_{0}^{\infty} \left\| \int_{SO(n)} F(|x|Ae, \rho Be) \phi(\rho Be) dB \right\|_{L_A^\gamma(SO(n))} \rho^{n-1} d\rho$$

where $e$ is any fixed unit vector. We choose $F$ as in (2.2) and we change variables $B \to AB^{-1}$ in the inner integral. By the invariance of the measure this is equivalent to

$$= \int_{0}^{\infty} \left\| \int_{SO(n)} \frac{|1 - (|x|/\rho)^\beta|}{|AB^{-1}(|x|Be - pe)|^n} \phi(\rho AB^{-1}e) dB \right\|_{L_A^\gamma(SO(n))} \rho^{n-1} d\rho$$

$$= \int_{0}^{\infty} \left\| \int_{SO(n)} \frac{|1 - (|x|/\rho)^\beta|}{|x|Be - pe|^n} \phi(\rho AB^{-1}e) dB \right\|_{L_A^\gamma(SO(n))} \rho^{n-1} d\rho.$$

Notice that the integral

$$\int_{SO(n)} \frac{|1 - (|x|/\rho)^\beta|}{|x|Be - pe|^n} \phi(\rho AB^{-1}e) dB = G \ast \phi(A)$$

is a convolution on $SO(n)$ of the functions

$$G(A) = \frac{|1 - (|x|/\rho)^\beta|}{|x|Ae - pe|^n}, \quad H(A) = |\phi(\rho Ae)|.$$
We can thus apply Young’s inequality on $SO(n)$ (see for instance [9, Theorem 1.2.12]) to obtain, for any $1 \leq \bar{p} \leq \infty$, the estimate
\[
(2.7) \quad \left\| \int_{\mathbb{R}^n} F(|x|, y) \phi(y) dy \right\|_{L_{\bar{p}}^p(S^{n-1})} \lesssim \int_0^\infty \left\| \frac{1 - (|x|/\rho)^\beta}{|x|/\rho - \theta} \right\|_{L_{\bar{p}}^p(S^{n-1})} \|\phi(\rho\theta)\|_{L_{\bar{p}}^p(S^{n-1})} \rho^{n-1} d\rho
\]
\[
= \int_0^\infty \left\| \frac{1 - (|x|/\rho)^\beta}{|x|/\rho - \theta} \right\|_{L_{\bar{p}}^p(S^{n-1})} \|\phi(\rho\theta)\|_{L_{\bar{p}}^p(S^{n-1})} \frac{d\rho}{\rho}
\]
where we switched back to the coordinates of $S^{n-1}$. Then we notice
\[
(2.8) \quad \left\| \int_{\mathbb{R}^n} F(|x|, y) \phi(y) dy \right\|_{L_{\bar{p}}^p} \leq \left\| \int_{\mathbb{R}^n} F(|x|, y) \phi(y) dy \right\|_{L_{\bar{p}}^p(S^{n-1})} dS_\theta |x|^{\frac{\bar{n}}{p}} \left\| L_{\bar{p}}^p(S^{n-1}) \right\|_{L^p(\mathbb{R}^+(\cdot), d|x|/|x|)}
\]
where $\mathbb{R}^+(\cdot)$ is the moltiplicative group of positive real numbers equipped with its Haar measure $d\rho/\rho$. Using (2.7) allows one to estimate (2.8) with
\[
\lesssim \left\| \int_0^\infty \left| \frac{1 - (|x|/\rho)^\beta}{|x|/\rho - \theta} \right| \rho^{\frac{n}{p}} \|\phi(\rho\theta)\|_{L_{\bar{p}}^p(S^{n-1})} \frac{d\rho}{\rho} \right\|_{L^p(\mathbb{R}^+(\cdot), d|x|/|x|)}
\]
Notice that the inner term is a convolution on $\mathbb{R}^+(\cdot)$ of the functions
\[
g(\rho) = \rho^{\frac{n}{p}} \left\| \frac{1 - (|x|/\rho)^\beta}{|x|/\rho - \theta} \right\|_{L_{\bar{p}}^p(S^{n-1})}, \quad h(\rho) = \rho^{\frac{n}{p}} \|\phi(\rho\theta)\|_{L_{\bar{p}}^p(S^{n-1})}.
\]
Thus we can apply again Young’s inequality to estimate further (2.8) with
\[
(2.9) \quad \lesssim \left\| g(\rho) \right\|_{L_{\bar{p}}^p(S^{n-1})} \left\| L^1(\mathbb{R}^+(\cdot), d\rho/\rho) \right\| \left\| h(\rho) \right\|_{L_{\bar{p}}^p(S^{n-1})} \left\| L^p(\mathbb{R}^+(\cdot), d\rho/\rho) \right\|,
\]
for all $1 \leq p \leq \infty$. Once we have noticed that
\[
\left\| h(\rho) \right\|_{L_{\bar{p}}^p(S^{n-1})} \left\| L^p(\mathbb{R}^+(\cdot), d\rho/\rho) \right\| = \left\| \phi \right\|_{L_{\bar{p}}^p(S^{n-1})},
\]
the concluding step of the proof of the lemma is represented by showing that the first term of (2.9) is bounded. We split the integral
\[
\int_0^{+\infty} \rho^{\frac{n}{p}} \int_{S^{n-1}} \frac{1 - (|x|/\rho)^\beta}{|x|/\rho - \theta} dS_\theta \frac{d\rho}{\rho} = \int_0^{\frac{1}{2}} (\cdot) + \int_{\frac{1}{2}}^{1} (\cdot) + \int_1^{+\infty} (\cdot) =: I + II + III
\]
and we bound separately the three terms.

If $0 < \rho < 1/2$, then $|\rho e - \theta| \geq 1/2$. Thus, since $|1 - \rho^\beta| < 1 + \rho^\beta$,
\[
I \lesssim \int_0^{\frac{1}{2}} (\rho^{\frac{n}{p} - 1} + \rho^{\frac{n}{p} - 1 + \beta}) d\rho < \infty
\]
provided that $p < \infty$ and $\beta > -n/p$.

If $1/2 \leq \rho \leq 2$, we notice that $|1 - \rho^\beta| \lesssim |1 - \rho|$ and we use that (see for instance Lemma 2.1 in [3])
\[
\int_{S^{n-1}} |\rho e - \theta|^{-n} dS_\theta \simeq \frac{1}{|1 - \rho|}
\]
to bound
\[(2.10) \quad II \sim \int_{\frac{1}{2}}^{2} \rho^{n-1} \frac{|1 - \rho^\beta|}{|1 - \rho|} d\rho \lesssim \int_{\frac{1}{2}}^{2} \rho^{n-1} d\rho < \infty.
\]

Finally, if \(2 < \rho < +\infty\), then \(|\rho e - \theta| \geq |\rho| - |\theta| \geq |\rho|/2\). Thus, since \(|1 - \rho^\beta| < 1 + \rho^\beta\),
\[III \lesssim \int_{2}^{+\infty} (\rho^{\frac{n}{p}-1-n} + \rho^{\frac{n}{p}-1+\beta-n}) d\rho < \infty\]
provided that \(p > 1\) and \(\beta < n - n/p\) and that concludes the proof. \(\square\)

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REFERENCES


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