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SINGULAR INTEGRALS WITH ANGULAR INTEGRABILITY

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Abstract. In this note we prove a class of sharp inequalities for singular integral operators in weighted Lebesgue spaces with angular integrability.

1. Introduction

We consider singular integral operators

$$Tf(x) := \text{P.V.} \int_{\mathbb{R}^n} f(x-y)K(y)\, dy, \tag{1.1}$$

where the kernel K satisfies the following conditions:

$$|y|^n|K| \le C, \quad |y|^{n+1}|\nabla K| \le C, \quad |\widehat{K}| \le C.$$

Here $C > 0$ is a constant and $\widehat{\ }$ denotes the Fourier transform. The main example we have in mind is the directional Riesz transform, which corresponds to the choice $K(y) := |y|^{-(n+1)} y \cdot \theta$, $\theta \in \mathbb{S}^{n-1}$.

The study of the boundedness of these operators in weighted Lebesgue spaces $L^p(w(x)dx)$, for $1 < p < \infty$ and $0 < w \in L^1_{loc}(\mathbb{R}^n)$, is a classical problem in harmonic analysis: in particular, Stein [13] proved it for the (sharp range of) homogeneous weights $w(x) = |x|^{\alpha p}$, $-n/p < \alpha < n - n/p$. The result was later extended by Coifman and Fefferman [2] to any A_p weight.

While the weighted L^p-theory has been extensively studied, less is known in the case of Lebesgue norms with different integrability in the radial and angular directions, namely

$$\|f\|_{L^p_{|x|}L^{\widetilde{p}}_\theta} := \left(\int_0^{+\infty} \|f(\rho\,\cdot)\|^p_{L^{\widetilde{p}}(\mathbb{S}^{n-1})} \rho^{n-1} d\rho \right)^{\frac{1}{p}}. \tag{1.2}$$

These mixed radial-angular spaces have been successfully used in recent years to improve several results in the framework of partial differential equations; see e.g. [1,8,10–12,14,15]. Notice that when $p = \widetilde{p}$ the norms reduce to the usual L^p norms. Notice also that, neglecting the constants, they are increasing in $\widetilde{p}$, and that they behave as the L^p norms under homogeneous rescaling, namely $f(\cdot) \to f(\lambda\,\cdot)$, $\lambda > 0$.

In a recent paper A. Córdoba [3] proved, among other things, the $L^p_{|x|}L^2_\theta$ boundedness for operators of the form (1.1). Here we give an extension of this result to the weighted setting.

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Theorem 1.1. *Let $n \geq 2$, $1 < p < \infty$, $1 < \widetilde{p} < \infty$ and $-n/p < \alpha < n - n/p$. Then*

$$\||x|^\alpha T\phi\|_{L^p_{|x|}L^{\widetilde{p}}_\theta} \leq C\||x|^\alpha \phi\|_{L^p_{|x|}L^{\widetilde{p}}_\theta}, \tag{1.3}$$

where C is a constant depending only on $\alpha, p, \widetilde{p}, n$.

Let us point out that the case $\alpha = 0$, $1 < \widetilde{p} < \infty$ in inequality (1.3) may be deduced by the application of Córdoba's argument; see [6, Theorem 2.6]. Therefore the novelty of Theorem 1.1 is that it covers all the possible homogeneous weights of the kind $|x|^{\alpha p}$.

Remark 1.1. Condition $-n/p < \alpha$ turns out to be necessary by testing the inequality on functions ϕ such that $\phi = 0$ and $T\phi > 0$ in a neighborhood of the origin. On the other hand, condition $\alpha < n - n/p$ turns out to be necessary for the same reason by considering the dual inequality.

Remark 1.2. One of the estimates (1.3) has been used in [6, Theorem 1.5] to deduce information about the regularity of weak solutions of the $3d$ Navier–Stokes problem with initial velocities satisfying good angular integrability properties.

We write $A \lesssim B$ if $A \leq CB$ with a constant C depending only on $\alpha, p, \widetilde{p}, n$. We write $A \simeq B$ if both $A \lesssim B$ and $B \lesssim A$.

2. Proof

We know by [6, Theorem 2.6] that inequality (1.3) is true in the case $\alpha = 0$, that is,

$$\|Tg\|_{L^p_{|x|}L^{\widetilde{p}}_\theta} \lesssim \|g\|_{L^p_{|x|}L^{\widetilde{p}}_\theta}. \tag{2.1}$$

Following Stein [13], we now show that the weighted case (1.3) can be then deduced by the unweighted one. The next lemma represents the core of the proof.

Lemma 2.1. *Let $n \geq 2$, $1 < p < \infty$, $1 \leq \widetilde{p} \leq \infty$, $-n/p < \alpha < n - n/p$ and*

$$F(x, y) := \frac{|1 - (|x|/|y|)^\alpha|}{|x - y|^n}; \tag{2.2}$$

then

$$\left\| \int_{\mathbb{R}^n} F(x, y)\phi(y)dy \right\|_{L^p_{|x|}L^{\widetilde{p}}_\theta} \leq C\|\phi\|_{L^p_{|x|}L^{\widetilde{p}}_\theta}. \tag{2.3}$$

Assume indeed this has been proved and first apply inequality (2.1) with the choice $g := |\cdot|^\alpha f$ to have

$$\|T(|x|^\alpha f)\|_{L^p_{|x|}L^{\widetilde{p}}_\theta} \lesssim \||x|^\alpha f\|_{L^p_{|x|}L^{\widetilde{p}}_\theta}. \tag{2.4}$$

Then notice that

$$\begin{aligned} \big|T(|x|^\alpha f) - |x|^\alpha Tf\big| &\leq \int_{\mathbb{R}^n} \big|K(x - y)(|y|^\alpha - |x|^\alpha)f(y)\big|dy \\ &\lesssim \int_{\mathbb{R}^n} \frac{||y|^\alpha - |x|^\alpha|}{|x - y|^n}|f(y)|dy = \int_{\mathbb{R}^n} \frac{|1 - (|x|/|y|)^\alpha|}{|x - y|^n}|y|^\alpha|f(y)|dy, \end{aligned} \tag{2.5}$$

so that by using Lemma 2.1 with $\phi = |\cdot|^\alpha f$ we obtain

$$\|T(|x|^\alpha f) - |x|^\alpha Tf\|_{L^p_{|x|}L^{\widetilde{p}}_\theta} \lesssim \||x|^\alpha f\|_{L^p_{|x|}L^{\widetilde{p}}_\theta}. \tag{2.6}$$

Then, the desired estimate (1.3) follows by (2.4) and (2.5) and triangle inequality. Thus it only remains to prove Lemma 2.1.

The idea is to use a change of variables which resembles the standard polar coordinates. In this variant the integration over the sphere is replaced by integration over the special orthogonal group $SO(n)$ and the radial integration is replaced by integration over the multiplicative group of the positive real numbers. This method works efficiently when homogeneous power weights are involved; see e.g. [5, 7].

Proof of Lemma 2.1. By using the isomorphism

$$\mathbb{S}^{n-1} \simeq SO(n)/SO(n-1)$$

we can rewrite integrals on $\mathbb{S}^{n-1}$ as follows:

$$\int_{\mathbb{S}^{n-1}} g(y)dS(y) \simeq \int_{SO(n)} g(Ae)dA, \qquad n \geq 2,$$

where dA is the left Haar measure on $SO(n)$, and $e \in \mathbb{S}^{n-1}$ is a fixed unit vector. Thus, via polar coordinates, a generic integral can be rewritten as

$$\begin{aligned}\int_{\mathbb{R}^n} F(x,y)\phi(x)dy &= \int_0^\infty \int_{\mathbb{S}^{n-1}} F(x,\rho\omega)\phi(\rho\omega)dS_\omega \rho^{n-1}d\rho \\ &\simeq \int_0^\infty \int_{SO(n)} F(x,\rho Be)\phi(\rho Be)dB\rho^{n-1}d\rho.\end{aligned}$$

Hence the $L^{\widetilde{p}}_\theta$ norm can be written as

$$\begin{aligned}&\left\| \int_{\mathbb{R}^n} F(|x|\theta,y)\phi(y)dy \right\|_{L^{\widetilde{p}}_\theta(\mathbb{S}^{n-1})} \simeq \left\| \int_{\mathbb{R}^n} F(|x|Ae,y)\phi(y)dy \right\|_{L^{\widetilde{p}}_A(SO(n))} \\ &\leq \int_0^\infty \left\| \int_{SO(n)} F(|x|Ae,\rho Be)\phi(\rho Be)dB \right\|_{L^{\widetilde{p}}_A(SO(n))} \rho^{n-1}d\rho\end{aligned}$$

where e is any fixed unit vector. We choose F as in (2.2) and we change variables $B \to AB^{-1}$ in the inner integral. By the invariance of the measure this is equivalent to

$$\begin{aligned}&= \int_0^\infty \left\| \int_{SO(n)} \frac{|1-(|x|/\rho)^\beta|}{|AB^{-1}(|x|Be-\rho e)|^n}\phi(\rho AB^{-1}e)dB \right\|_{L^{\widetilde{p}}_A(SO(n))} \rho^{n-1}d\rho \\ &= \int_0^\infty \left\| \int_{SO(n)} \frac{|1-(|x|/\rho)^\beta|}{||x|Be-\rho e|^n}\phi(\rho AB^{-1}e)dB \right\|_{L^{\widetilde{p}}_A(SO(n))} \rho^{n-1}d\rho.\end{aligned}$$

Notice that the integral

$$\int_{SO(n)} \frac{|1-(|x|/\rho)^\beta|}{||x|Be-\rho e|^n}\ |\phi(\rho AB^{-1}e)|dB = G * \phi(A)$$

is a convolution on $SO(n)$ of the functions

$$G(A) = \frac{|1-(|x|/\rho)^\beta|}{||x|Ae-\rho e|^n}, \qquad H(A) = |\phi(\rho Ae)|.$$

We can thus apply Young's inequality on $SO(n)$ (see for instance [9, Theorem 1.2.12]) to obtain, for any $1 \le \widetilde{p} \le \infty$, the estimate

(2.7)
$$\left\| \int_{\mathbb{R}^n} F(|x|\theta, y)\phi(y)dy \right\|_{L^{\widetilde{p}}_\theta(\mathbb{S}^{n-1})} \lesssim \int_0^\infty \left\| \frac{|1-(|x|/\rho)^\beta|}{|(|x|e-\rho\theta)|^n} \right\|_{L^1_\theta(\mathbb{S}^{n-1})} \|\phi(\rho\theta)\|_{L^{\widetilde{p}}_\theta(\mathbb{S}^{n-1})} \rho^{n-1} d\rho$$
$$= \int_0^\infty \left\| \frac{|1-(|x|/\rho)^\beta|}{|(\frac{|x|}{\rho}-\theta)|^n} \right\|_{L^1_\theta(\mathbb{S}^{n-1})} \|\phi(\rho\theta)\|_{L^{\widetilde{p}}_\theta(\mathbb{S}^{n-1})} \frac{d\rho}{\rho}$$

where we switched back to the coordinates of $\mathbb{S}^{n-1}$. Then we notice

(2.8)
$$\left\| \int_{\mathbb{R}^n} F(x,y)\phi(y)dy \right\|_{L^p_{|x|}L^{\widetilde{p}}_\theta} \lesssim \left\| \left\| \int_{\mathbb{R}^n} F(|x|\theta, y)\phi(y)dy \right\|_{L^{\widetilde{p}}_\theta(\mathbb{S}^{n-1})} dS_\theta |x|^{\frac{n}{p}} \right\|_{L^p(\mathbb{R}^+(\cdot), d|x|/|x|)}$$

where $\mathbb{R}^+(\cdot)$ is the moltiplicative group of positive real numbers equipped with its Haar measure $d\rho/\rho$. Using (2.7) allows one to estimate (2.8) with

$$\lesssim \left\| \int_0^\infty \left(\frac{|x|}{\rho}\right)^{\frac{n}{p}} \left\| \frac{|1-(|x|/\rho)^\beta|}{\left|\left(\frac{|x|}{\rho}e-\theta\right)\right|^n} \right\|_{L^1_\theta(\mathbb{S}^{n-1})} \rho^{\frac{n}{p}} \|\phi(\rho\theta)\|_{L^{\widetilde{p}}_\theta(\mathbb{S}^{n-1})} \frac{d\rho}{\rho} \right\|_{L^p(\mathbb{R}^+(\cdot), d|x|/|x|)}.$$

Notice that the inner term is a convolution on $\mathbb{R}^+(\cdot)$ of the functions

$$g(\rho) = \rho^{\frac{n}{p}} \left\| \frac{|1-\rho^\beta|}{|(\rho e-\theta)|^n} \right\|_{L^1_\theta(\mathbb{S}^{n-1})}, \qquad h(\rho) = \rho^{\frac{n}{p}} \|\phi(\rho\theta)\|_{L^{\widetilde{p}}_\theta(\mathbb{S}^{n-1})}.$$

Thus we can apply again Young's inequality to estimate further (2.8) with

$$\lesssim \left\| \|g(\rho)\|_{L^1_\theta(\mathbb{S}^{n-1})} \right\|_{L^1(\mathbb{R}^+(\cdot), d\rho/\rho)} \left\| \|h(\rho)\|_{L^{\widetilde{p}}_\theta(\mathbb{S}^{n-1})} \right\|_{L^p(\mathbb{R}^+(\cdot), d\rho/\rho)}, \tag{2.9}$$

for all $1 \le p \le \infty$. Once we have noticed that

$$\left\| \|h(\rho)\|_{L^{\widetilde{p}}_\theta(\mathbb{S}^{n-1})} \right\|_{L^p(\mathbb{R}^+(\cdot), d\rho/\rho)} = \|\phi\|_{L^p_{|x|}L^{\widetilde{p}}_\theta},$$

the concluding step of the proof of the lemma is represented by showing that the first term of (2.9) is bounded. We split the integral

$$\int_0^{+\infty} \rho^{\frac{n}{p}} \int_{\mathbb{S}^{n-1}} \frac{|1-\rho^\beta|}{|\rho e-\theta|^n}\, dS_\theta\, \frac{d\rho}{\rho} = \int_0^{\frac{1}{2}} (\cdot) + \int_{\frac{1}{2}}^2 (\cdot) + \int_2^{+\infty} (\cdot) =: I + II + III$$

and we bound separately the three terms.

If $0 < \rho < 1/2$, then $|\rho e - \theta| \ge 1/2$. Thus, since $|1-\rho^\beta| < 1+\rho^\beta$,

$$I \lesssim \int_0^{\frac{1}{2}} (\rho^{\frac{n}{p}-1} + \rho^{\frac{n}{p}-1+\beta}) d\rho < \infty$$

provided that $p < \infty$ and $\beta > -n/p$.

If $1/2 \le \rho \le 2$, we notice that $|1-\rho^\beta| \lesssim |1-\rho|$ and we use that (see for instance Lemma 2.1 in [5])

$$\int_{\mathbb{S}^{n-1}} |\rho e - \theta|^{-n} dS_\theta \simeq \frac{1}{|1-\rho|}$$

to bound

$$II \simeq \int_{\frac{1}{2}}^{2} \rho^{\frac{n}{p}-1} \frac{|1-\rho^{\beta}|}{|1-\rho|} d\rho \lesssim \int_{\frac{1}{2}}^{2} \rho^{\frac{n}{p}-1} d\rho < \infty. \tag{2.10}$$

Finally, if $2 < \rho < +\infty$, then $|\rho e - \theta| \geq |\rho| - |\theta| \geq |\rho|/2$. Thus, since $|1-\rho^{\beta}| < 1+\rho^{\beta}$,

$$III \lesssim \int_{2}^{+\infty} (\rho^{\frac{n}{p}-1-n} + \rho^{\frac{n}{p}-1+\beta-n}) d\rho < \infty$$

provided that $p > 1$ and $\beta < n - n/p$ and that concludes the proof. □

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