

ON THE BREIMAN CONJECTURE

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(Communicated by Mark M. Meerschaert)

ABSTRACT. Let Y_1, Y_2, \dots be positive, nondegenerate, i.i.d. G random variables, and independently let X_1, X_2, \dots be i.i.d. F random variables. In this note we show that for $F \in \mathcal{F}$ in a specified class of distributions \mathcal{F} , whenever $\sum X_i Y_i / \sum Y_i$ converges in distribution to a nondegenerate limit then G necessarily belongs to the domain of attraction of a stable law with index less than 1. The class \mathcal{F} contains those nondegenerate X with a finite second moment and those X in the domain of attraction of a stable law with index $1 < \alpha < 2$.

1. INTRODUCTION AND RESULTS

Let Y, Y_1, \dots be positive, nondegenerate, i.i.d. random variables with distribution function [df] G , and independently let X, X_1, \dots be i.i.d. nondegenerate random variables with df F . Let ϕ_X denote the characteristic function [cf] of X . We shall use the notation $Y \in D(\beta)$ to mean that Y is in the domain of attraction of a stable law of index $0 < \beta < 1$, and $Y \in D(0)$ will denote that $1 - G$ is slowly varying at infinity. Furthermore, $\mathcal{RV}_\infty(\rho)$ will signify the class of positive measurable functions regularly varying at infinity with index ρ , and $\mathcal{RV}_0(\rho)$ the class of positive measurable functions regularly varying at zero with index ρ . In particular, using this notation $Y \in D(\beta)$, with $0 \leq \beta < 1$, if and only if $\overline{G} := 1 - G \in \mathcal{RV}_\infty(-\beta)$.

For each integer $n \geq 1$ set

$$(1) \quad T_n = \sum_{i=1}^n X_i Y_i / \sum_{i=1}^n Y_i.$$

Notice that $\mathbb{E}|X| < \infty$ implies that T_n is stochastically bounded. Theorem 4 of Breiman [2] says that T_n converges in distribution along the full sequence $\{n\}$ for every X with finite expectation, and with at least one limit law being nondegenerate, if and only if

$$(2) \quad Y \in D(\beta), \text{ with } 0 \leq \beta < 1.$$

Let \mathcal{X} denote the class of nondegenerate random variables X with $\mathbb{E}|X| < \infty$ and let \mathcal{X}_0 denote those $X \in \mathcal{X}$ such that $\mathbb{E}X = 0$. At the end of his paper Breiman conjectured that if for *some* $X \in \mathcal{X}$, T_n converges in distribution to some nondegenerate random variable T , written

$$(3) \quad T_n \rightarrow_d T, \text{ as } n \rightarrow \infty, \text{ with } T \text{ nondegenerate, then (2) holds.}$$

Received by the editors August 6, 2015 and, in revised form, October 28, 2015.

2010 *Mathematics Subject Classification*. Primary 60F05.

The research of the first author was funded by a postdoctoral fellowship of the Alexander von Humboldt Foundation.

By Proposition 2 (in the case $\beta = 0$) and Theorem 3 (in the case $0 < \beta < 1$) of [2], for any $X \in \mathcal{X}$, (2) implies (3), in which case T , in the case $0 < \beta < 1$, has a distribution related to the arcsine law. Using this fact, we see that his conjecture can be restated to be: for any $X \in \mathcal{X}$, (2) is equivalent to (3).

It has proved to be surprisingly challenging to resolve. Mason and Zinn [8] partially verified Breiman’s conjecture. They established that whenever X is non-degenerate and satisfies $\mathbb{E}|X|^p < \infty$ for some $p > 2$, then (2) is equivalent to (3). In this note we further extend this result.

Theorem 1. *Assume that for some $X \in \mathcal{X}_0$, $1 < \alpha \leq 2$, positive slowly varying function L at zero and $c > 0$,*

$$(4) \quad \frac{-\log(\Re \phi_X(t))}{|t|^\alpha L(|t|)} \rightarrow c, \text{ as } t \rightarrow 0.$$

Whenever (3) holds, then $Y \in D(\beta)$ for some $\beta \in [0, 1)$.

Let \mathcal{F} denote the class of random variables that satisfy the conditions of the theorem. Applying our theorem in combination with Proposition 2 and Theorem 3 of [2] we get the following corollary.

Corollary 1. *Whenever $X - \mathbb{E}X \in \mathcal{F}$, (2) is equivalent to (3).*

Remark 1. It can be inferred from Theorem 8.1.10 of Bingham et al. [1] (see also Theorems 1 and 5 of Pitman [9]) that for $X \in \mathcal{X}_0$, (4) holds for some $1 < \alpha < 2$, positive slowly varying function L at zero and $c > 0$ if and only if X satisfies $\mathbb{P}\{|X| > x\} \sim L(1/x)x^{-\alpha}c\Gamma(\alpha)\frac{2}{\pi} \sin(\frac{\pi\alpha}{2})$. Note that a random variable $X \in \mathcal{X}_0$ in the domain of attraction of a stable law of index $1 < \alpha < 2$ satisfies (4). For $\alpha = 2$ there is no simple condition equivalent to (4). By Theorem 5 of Pitman [9], for $\alpha = 2$ condition (4) implies that

$$(5) \quad \frac{1 - \Re \phi_X(t)}{t^2} \sim \int_0^{t^{-1}} u\mathbb{P}\{|X| > u\}du, \text{ as } t \downarrow 0.$$

Also a random variable $X \in \mathcal{X}_0$ with variance $0 < \sigma^2 < \infty$ fulfills (4) with $\alpha = 2$, $L = 1$ and $c = \sigma^2/2$. Theorem 3 in [9] states that $\mathbb{P}\{|X| > x\} \in \mathcal{RV}_\infty(-2)$ implies (5), from which, combined with Proposition 1.5.9a of [1], condition (4) follows.

Remark 2. Consult Kevei and Mason [7] for a fairly exhaustive study of the asymptotic distributions of T_n along subsequences, along with revelations of their unexpected properties.

The theorem follows from the two propositions below. First we need more notation. For any $\alpha \in (1, 2]$, define for $n \geq 1$

$$(6) \quad S_n(\alpha) = \frac{\sum_{i=1}^n Y_i^\alpha}{(\sum_{i=1}^n Y_i)^\alpha}.$$

Proposition 1. *Assume that the assumptions of the theorem hold. Then for some $0 < \gamma \leq 1$,*

$$(7) \quad \mathbb{E}S_n(\alpha) \rightarrow \gamma, \text{ as } n \rightarrow \infty.$$

The next proposition is interesting in its own right. It is an extension of Theorem 5.3 by Fuchs et al. [4], where $\alpha = 2$ (see also Proposition 3 of [8]).

Proposition 2. *If (7) holds with some $\gamma \in (0, 1]$, then $Y \in D(\beta)$, for some $\beta \in [0, 1)$, where $-\beta \in (-1, 0]$ is the unique solution of*

$$\text{Beta}(\alpha - 1, 1 - \beta) = \frac{\Gamma(\alpha - 1)\Gamma(1 - \beta)}{\Gamma(\alpha - \beta)} = \frac{1}{\gamma(\alpha - 1)}.$$

In particular, $Y \in D(0)$ for $\gamma = 1$.

Conversely, if $Y \in D(\beta)$, $0 \leq \beta < 1$, then (7) holds with

$$\gamma = \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(1 - \beta)} = \frac{1}{(\alpha - 1)\text{Beta}(\alpha - 1, 1 - \beta)}.$$

2. PROOFS

Set for each $n \geq 1$, $R_i = Y_i / \sum_{l=1}^n Y_l$, for $i = 1, \dots, n$. For notational ease we drop the dependence of R_i on $n \geq 1$. Consider the sequence of strictly decreasing continuous functions $\{\varphi_n\}_{n \geq 1}$ on $[1, \infty)$ defined by $\varphi_n(y) = \mathbb{E}(\sum_{i=1}^n R_i^y)$, $y \in [1, \infty)$. Note that each function φ_n satisfies $\varphi_n(1) = 1$. By a diagonal selection procedure for each subsequence of $\{n\}_{n \geq 1}$ there is a further subsequence $\{n_k\}_{k \geq 1}$ and a right continuous nonincreasing function ψ such that φ_{n_k} converges to ψ at each continuity point of ψ .

Lemma 1. *Each such function ψ is continuous on $(1, \infty)$.*

Proof. Choose any subsequence $\{n_k\}_{k \geq 1}$ and a right continuous nonincreasing function ψ such that φ_{n_k} converges to ψ at each continuity point of ψ in $(1, \infty)$. Select any $x > 1$ and continuity points $x_1, x_2 \in (1, \infty)$ of ψ such that $1 < x_1 < x < x_2 < \infty$. Set $\rho_1 = x_1 - 1$ and $\rho_2 = x_2 - 1$. Since $\rho_2/\rho_1 > 1$ we get by Hölder’s inequality

$$\sum_{i=1}^{n_k} R_i^{x_1} = \sum_{i=1}^{n_k} R_i^{\rho_1} R_i \leq \left(\sum_{i=1}^{n_k} R_i^{\rho_2} R_i \right)^{\rho_1/\rho_2} = \left(\sum_{i=1}^{n_k} R_i^{x_2} \right)^{\rho_1/\rho_2}.$$

Taking expectations and using Jensen’s inequality we get $\varphi_{n_k}(x_1) \leq (\varphi_{n_k}(x_2))^{\rho_1/\rho_2}$. Letting $n_k \rightarrow \infty$, we have $\psi(x_1) \leq (\psi(x_2))^{\rho_1/\rho_2}$. Since $x_1 < x$ and $x_2 > x$ can be chosen arbitrarily close to x we conclude by right continuity of ψ at x that $\psi(x-) = \psi(x+) = \psi(x)$. □

Proof of Proposition 1. For a complex z , we use the notation for the principal branch of the logarithm, $\text{Log}(z) = \log|z| + i \arg z$, where $-\pi < \arg z \leq \pi$, i.e. $z = |z| \exp(i \arg z)$. We see that for all t

$$\begin{aligned} \mathbb{E} \exp(itT_n) &= \mathbb{E} \left(\prod_{j=1}^n \phi_X(tR_j) \right) \\ &= \mathbb{E} \left(\prod_{j=1}^n \exp(\text{Log} \phi_X(tR_j)) \right). \end{aligned}$$

Since $\mathbb{E}X = 0$ we have $\Re \phi_X(u) = 1 - o_+(u)$, where $o_+(u) \geq 0$ and $o_+(u)/u \rightarrow 0$ as $u \rightarrow 0$, and $\Im \phi_X(u) = o(u)$. This when combined with

$$(\arctan \theta)' = \frac{1}{1 + \theta^2}$$

gives as $u \rightarrow 0$,

$$\arg \phi_X(u) = \arctan \left(\frac{\Im \phi_X(u)}{\Re \phi_X(u)} \right) = o(u).$$

Note that for all $|u| > 0$ sufficiently small so that $\Re \phi_X(u) > 0$,

$$\text{Log} \phi_X(u) = \text{Log}(\Re \phi_X(u) + i \Im \phi_X(u)) = \log \Re \phi_X(u) + \text{Log} \left(1 + i \frac{\Im \phi_X(u)}{\Re \phi_X(u)} \right),$$

where for the second term

$$\Re \text{Log} \left(1 + i \frac{\Im \phi_X(u)}{\Re \phi_X(u)} \right) = \frac{1}{2} \left(\frac{\Im \phi_X(u)}{\Re \phi_X(u)} \right)^2 (1 + o(u)), \text{ as } u \rightarrow 0.$$

Thus for every $\varepsilon > 0$ for all $|t| > 0$ sufficiently small and independent of $n \geq 1$ and R_1, \dots, R_n ,

$$1 - \varepsilon^2 t^2 \leq \cos(\varepsilon t) \leq \Re \left(\exp \left\{ \sum_{j=1}^n \text{Log} \left(1 + i \frac{\Im \phi_X(tR_j)}{\Re \phi_X(tR_j)} \right) \right\} \right) \leq e^{2^{-1} \varepsilon t^2} \leq 1 + \varepsilon t^2.$$

Thus we obtain

$$\begin{aligned} \mathbb{E} \exp \left\{ \sum_{j=1}^n \log \Re \phi_X(tR_j) \right\} (1 - \varepsilon^2 t^2) &\leq \mathbb{E} (\Re \exp(itT_n)) \\ &= \Re \mathbb{E} \exp(itT_n) \\ &\leq \mathbb{E} \exp \left\{ \sum_{j=1}^n \log \Re \phi_X(tR_j) \right\} (1 + \varepsilon t^2). \end{aligned}$$

We shall show (4) implies that (7) holds for some $0 < \gamma \leq 1$. Now using (4) we get for any $0 < \delta < c$ and all $|t|$ small enough independent of $n \geq 1$,

$$\begin{aligned} -\varepsilon t^2 + \log \mathbb{E} \exp \left(-(c + \delta) |t|^\alpha \left(\sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \right) &\leq \log (\Re \mathbb{E} \exp(itT_n)) \\ &\leq \varepsilon t^2 + \log \mathbb{E} \exp \left(-(c - \delta) |t|^\alpha \left(\sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \right). \end{aligned}$$

Next since $\log s / (1 - s) \rightarrow -1$ as $s \nearrow 1$, for all $|t|$ small enough independent of $n \geq 1$ and R_1, \dots, R_n (keeping in mind that $\sum_{i=1}^n R_i = 1$ and $1 < \alpha \leq 2$),

$$\begin{aligned} &\log \mathbb{E} \exp \left(-(c + \delta) |t|^\alpha \left(\sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \right) \\ &\geq - \left(1 + \frac{\delta}{2} \right) \mathbb{E} \left(1 - \exp \left(-(c + \delta) |t|^\alpha \left(\sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \right) \right) \end{aligned}$$

and

$$\begin{aligned} &\log \mathbb{E} \exp \left(-(c - \delta) |t|^\alpha \left(\sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \right) \\ &\leq - \left(1 - \frac{\delta}{2} \right) \mathbb{E} \left(1 - \exp \left(-(c - \delta) |t|^\alpha \left(\sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \right) \right). \end{aligned}$$

Further, since $(1 - \exp(-y)) / y \rightarrow 1$ as $y \searrow 0$, for all $|t|$ small enough independent of $n \geq 1$,

$$\begin{aligned} & - \left(1 + \frac{\delta}{2}\right) \mathbb{E} \left(1 - \exp \left(- (c + \delta) |t|^\alpha \left(\sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \right) \right) \\ & \geq - (1 + \delta) (c + \delta) |t|^\alpha \mathbb{E} \left(\sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \end{aligned}$$

and

$$\begin{aligned} & - \left(1 - \frac{\delta}{2}\right) \mathbb{E} \left(1 - \exp \left(- (c - \delta) |t|^\alpha \left(\sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \right) \right) \\ & \leq - (1 - \delta) (c - \delta) |t|^\alpha \mathbb{E} \left(\sum_{i=1}^n R_i^\alpha L(|t| R_i) \right). \end{aligned}$$

Therefore, for all $|t|$ small enough independent of n ,

$$\begin{aligned} & - \varepsilon t^2 - (1 + \delta) (c + \delta) |t|^\alpha \mathbb{E} \left(\sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \\ & \leq \log (\mathfrak{R} \mathfrak{e} \mathbb{E} \exp (itT_n)) \\ & \leq \varepsilon t^2 - (1 - \delta) (c - \delta) |t|^\alpha \mathbb{E} \left(\sum_{i=1}^n R_i^\alpha L(|t| R_i) \right). \end{aligned}$$

By the Potter’s bound, Theorem 1.5.6 (i) in [1], for all $A > 1$ and $1 < \alpha_1 < \alpha < \alpha_2$, for all $t > 0$ small enough independent of $n \geq 1$,

$$(8) \quad A^{-1} \sum_{i=1}^n R_i^{\alpha_2} \leq \sum_{i=1}^n R_i^\alpha L(|t| R_i) / L(|t|) \leq A \sum_{i=1}^n R_i^{\alpha_1}.$$

We see now that for all $n \geq 1$ and $0 < 4\varepsilon < c$, appropriate $1 < \alpha_1 < \alpha < \alpha_2$ and all $|t|$ small enough independent of n ,

$$\begin{aligned} & - \varepsilon t^2 - (1 + \varepsilon) (c + 2\varepsilon) |t|^\alpha L(|t|) \mathbb{E} S_n (\alpha_2) \\ & = - \varepsilon t^2 - (1 + \varepsilon) (c + 2\varepsilon) |t|^\alpha L(|t|) \mathbb{E} \left(\sum_{i=1}^n R_i^{\alpha_2} \right) \\ & \leq \log (\mathfrak{R} \mathfrak{e} \mathbb{E} \exp (itT_n)) \\ & \leq \varepsilon t^2 - (1 - \varepsilon) (c - 2\varepsilon) |t|^\alpha L(|t|) \mathbb{E} \left(\sum_{i=1}^n R_i^{\alpha_1} \right) \\ & = \varepsilon t^2 - (1 - \varepsilon) (c - 2\varepsilon) |t|^\alpha L(|t|) \mathbb{E} S_n (\alpha_1). \end{aligned}$$

Choose any subsequence $\{n_k\}_{k \geq 1}$ and a right continuous nonincreasing function ψ such that φ_{n_k} converges to ψ at each continuity point of ψ , which by Lemma 1 above is all $(1, \infty)$. We see that $\mathbb{E} S_{n_k} (\alpha) \rightarrow \psi(\alpha)$, $\mathbb{E} S_{n_k} (\alpha_1) \rightarrow \psi(\alpha_1)$ and $\mathbb{E} S_{n_k} (\alpha_2) \rightarrow \psi(\alpha_2)$, where necessarily $0 < \psi(\alpha_2) \leq \psi(\alpha) \leq \psi(\alpha_1) \leq 1$. We see that for all $|t|$ sufficiently small independent of the subsequence $n_k \geq 1$,

$$(9) \quad \begin{aligned} & - \varepsilon t^2 - (1 + \varepsilon) (c + 3\varepsilon) |t|^\alpha L(|t|) \psi(\alpha_2) \leq \log (\mathfrak{R} \mathfrak{e} \mathbb{E} \exp (itT)) \\ & \leq \varepsilon t^2 - (1 - \varepsilon) (c - 3\varepsilon) |t|^\alpha L(|t|) \psi(\alpha_1), \end{aligned}$$

where T is the nondegenerate limit in (3). Note that if $\psi(\alpha_1) = 0$, then because of monotonicity $\psi(\alpha_2) = 0$, we would have $\lim_{t \rightarrow 0} t^{-2} \mathbb{E}[1 - \cos(tT)] = 0$, which by an easy argument based on a classical probability inequality (see Lemma 1, p. 268 of Chow and Teicher [3]), implies that $\mathbb{P}\{T = 0\} = 1$, contrary to our assumptions. Therefore, $\psi(\alpha_1) > 0$.

From (9) we obtain $|t|$ sufficiently small independent of the subsequence $n_k \geq 1$,

$$-\varepsilon - (1 + \varepsilon)(c + 3\varepsilon)\psi(\alpha_2) \leq \log(\Re \mathbb{E} \exp(itT_{n_k})) / (|t|^\alpha L(|t|)) \leq \varepsilon - (1 - \varepsilon)(c - 3\varepsilon)\psi(\alpha_1),$$

where for $\alpha = 2$ we use that $\liminf_{t \searrow 0} L(t) > 0$; see Remark 1. Since $0 < 4\varepsilon < c$ can be made arbitrarily small and $0 \leq \psi(\alpha_1) - \psi(\alpha_2)$ can be made as close to zero as desired, by letting $n_k \rightarrow \infty$, we get that for all $|t|$ sufficiently small,

$$-\varepsilon - (1 + \varepsilon)(c + 4\varepsilon)\psi(\alpha) \leq \log(\Re \mathbb{E} \exp(itT)) / (|t|^\alpha L(|t|)) \leq \varepsilon - (1 - \varepsilon)(c - 4\varepsilon)\psi(\alpha),$$

which can happen only if $\psi(\alpha)$ does not depend on $\{n_k\}$. Thus (7) holds for some $0 < \gamma \leq 1$, namely $\gamma = \psi(\alpha)$. □

Proof of Proposition 2. To begin with, we note that whenever (7) holds, necessarily $\mathbb{E}Y = \infty$. To see this, write $D_n^{(1)} = \max_{1 \leq i \leq n} Y_i / (\sum_{i=1}^n Y_i)$ and observe that

$$\begin{aligned} \left(D_n^{(1)}\right)^\alpha &= \max_{1 \leq i \leq n} \frac{Y_i^\alpha}{\left(\sum_{i=1}^n Y_i\right)^\alpha} \leq S_n(\alpha) \\ &\leq \max_{1 \leq i \leq n} \frac{Y_i^{\alpha-1}}{\left(\sum_{i=1}^n Y_i\right)^{\alpha-1}} = \left(D_n^{(1)}\right)^{\alpha-1}. \end{aligned}$$

From these inequalities it is easy to prove that $\mathbb{E}S_n(\alpha) \rightarrow 0, n \rightarrow \infty$, if and only if

$$(10) \quad D_n^{(1)} \rightarrow_P 0, n \rightarrow \infty.$$

Proposition 1 of Breiman [2] says that (10) holds if and only if there exists a sequence of positive constants B_n converging to infinity such that

$$(11) \quad \sum_{i=1}^n Y_i/B_n \rightarrow_P 1, n \rightarrow \infty.$$

Since $\mathbb{E}Y < \infty$ obviously implies (11), it readily follows that $\mathbb{E}S_n(\alpha) \rightarrow 0, n \rightarrow \infty$, and thus (7) cannot hold.

We shall first prove the first part of Proposition 2. Following similar steps as in [8] we have that

$$\begin{aligned} \mathbb{E} \frac{\sum_{i=1}^n Y_i^\alpha}{\left(\sum_{i=1}^n Y_i\right)^\alpha} &= n \mathbb{E} \frac{Y_1^\alpha}{\left(\sum_{i=1}^n Y_i\right)^\alpha} \\ &= \frac{n}{\Gamma(\alpha)} \mathbb{E} \int_0^\infty Y_1^\alpha e^{-t \sum_{i=1}^n Y_i} t^{\alpha-1} dt \\ &= \frac{n}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \mathbb{E} \left(e^{-tY_1} Y_1^\alpha \right) \left(\mathbb{E} e^{-tY_1} \right)^{n-1} dt \\ &=: \frac{n}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \phi_\alpha(t) \phi_0(t)^{n-1} dt. \end{aligned}$$

Next, assuming (7) and arguing as in the proof of Theorem 3 in [2] we get

$$(12) \quad s \int_0^\infty t^{\alpha-1} \phi_\alpha(t) e^{s \log \phi_0(t)} dt \rightarrow \gamma \Gamma(\alpha), \quad s \rightarrow \infty,$$

where $0 < \gamma \leq 1$. For $y \geq 0$, let $q(y)$ denote the inverse of $-\log \phi_0(t)$. Changing the variables to $y = -\log \phi_0(t)$ and $t = q(y)$, we get from (12) that

$$s \int_0^\infty (q(y))^{\alpha-1} \phi_\alpha(q(y)) \exp(-sy) dq(y) \rightarrow \gamma \Gamma(\alpha), \text{ as } s \rightarrow \infty.$$

By Karamata's Tauberian theorem (see Theorem 1.7.1' on p. 38 of [1]), we conclude that

$$v^{-1} \int_0^v (q(x))^{\alpha-1} \phi_\alpha(q(x)) dx \rightarrow \gamma \Gamma(\alpha), \text{ as } v \searrow 0,$$

which, in turn, by the change of variable $y = q(x)$ gives

$$\frac{\int_0^t y^{\alpha-1} \phi_\alpha(y) dy}{-\log \phi_0(t)} \rightarrow \gamma \Gamma(\alpha), \text{ as } t \searrow 0.$$

Now using that $-\log \phi_0(t) \sim 1 - \phi_0(t)$ as $t \rightarrow 0$, we end up with

$$(13) \quad \lim_{t \rightarrow 0} \frac{\int_0^t y^{\alpha-1} \phi_\alpha(y) dy}{1 - \phi_0(t)} = \gamma \Gamma(\alpha).$$

Since $\phi_\alpha(y) = \int_0^\infty e^{-uy} u^\alpha G(du)$, by Fubini's theorem

$$\begin{aligned} \int_0^t y^{\alpha-1} \phi_\alpha(y) dy &= \int_0^\infty u^\alpha G(du) \int_0^t y^{\alpha-1} e^{-uy} dy \\ &= \int_0^\infty G(du) \int_0^{ut} z^{\alpha-1} e^{-z} dz \\ &= \int_0^\infty \bar{G}(z/t) z^{\alpha-1} e^{-z} dz \\ &= t^\alpha \int_0^\infty \bar{G}(u) u^{\alpha-1} e^{-ut} du. \end{aligned}$$

A partial integration gives

$$1 - \phi_0(t) = t \int_0^\infty \bar{G}(u) e^{-ut} du.$$

So (13) reads

$$(14) \quad t^{\alpha-1} \frac{\int_0^\infty \bar{G}(u) u^{\alpha-1} e^{-ut} du}{\int_0^\infty \bar{G}(u) e^{-ut} du} \rightarrow \gamma \Gamma(\alpha), \text{ as } t \searrow 0,$$

with $0 < \gamma \leq 1$. Let us define the function for $t > 0$

$$(15) \quad f(t) = \int_0^\infty \bar{G}(u) u^{\alpha-1} e^{-ut} du.$$

Clearly, f is monotone decreasing and since $\mathbb{E}Y = \infty$, $\lim_{t \rightarrow 0} f(t) = \infty$. We shall show that f is regularly varying at 0, which by Lemma 3 of Pitman [9], implies that \bar{G} is regularly varying at infinity. We use the identity

$$u^{1-\alpha} e^{-ut} = \frac{1}{\Gamma(\alpha-1)} \int_0^\infty y^{\alpha-2} e^{-(y+t)u} dy,$$

which holds for $u > 0$ and $\alpha \in (1, 2]$. (This is the *Weyl-transform*, or *Weyl-fractional integral* of the function e^{-ut} .) This identity combined with Fubini's theorem (everything is nonnegative) gives

$$\begin{aligned} \frac{1}{\Gamma(\alpha - 1)} \int_0^\infty y^{\alpha-2} f(y + t) dy &= \int_0^\infty \bar{G}(u) u^{\alpha-1} du \frac{1}{\Gamma(\alpha - 1)} \int_0^\infty y^{\alpha-2} e^{-(y+t)u} dy \\ &= \int_0^\infty \bar{G}(u) e^{-ut} du. \end{aligned}$$

So we can rewrite (14) as

$$(16) \quad \lim_{t \searrow 0} \frac{t^{\alpha-1} f(t)}{\int_0^\infty y^{\alpha-2} f(t + y) dy} = \frac{\gamma \Gamma(\alpha)}{\Gamma(\alpha - 1)} = \gamma(\alpha - 1).$$

A change of variable gives

$$\int_0^\infty y^{\alpha-2} f(t + y) dy = t^{\alpha-1} \int_1^\infty (u - 1)^{\alpha-2} f(ut) du,$$

and so we have

$$(17) \quad \lim_{t \searrow 0} \int_1^\infty (u - 1)^{\alpha-2} \frac{f(ut)}{f(t)} du = [\gamma(\alpha - 1)]^{-1}.$$

We can rewrite f as

$$f(t) = \int_0^\infty \bar{G}(u) u^{\alpha-1} e^{-ut} du = t^{-\alpha} \int_0^\infty \bar{G}(u/t) u^{\alpha-1} e^{-u} du,$$

from which we see that the function

$$g(t) = \int_0^\infty \bar{G}(u/t) u^{\alpha-1} e^{-u} du = t^\alpha f(t)$$

is bounded and nondecreasing. Substituting g into (17) we obtain

$$(18) \quad \lim_{t \rightarrow 0^+} \int_1^\infty (u - 1)^{\alpha-2} u^{-\alpha} \frac{g(ut)}{g(t)} du = [\gamma(\alpha - 1)]^{-1}.$$

Write $g_\infty(x) = g(x^{-1})$, $x > 0$. Then (18) has the form

$$(19) \quad \int_1^\infty (u - 1)^{\alpha-2} u^{-\alpha} \frac{g_\infty(x/u)}{g_\infty(x)} du = \frac{k * g_\infty(x)}{g_\infty(x)} \rightarrow [\gamma(\alpha - 1)]^{-1}, \quad \text{as } x \rightarrow \infty,$$

where

$$k(u) = \begin{cases} (u - 1)^{\alpha-2} u^{-\alpha+1}, & u > 1, \\ 0, & 0 < u \leq 1, \end{cases}$$

and

$$k * h(x) = \int_0^\infty h(x/u) k(u) / u du$$

is the *Mellin-convolution* of h and k . Note that the *Mellin-transform* of k ,

$$\begin{aligned} \tilde{k}(z) &= \int_1^\infty (u - 1)^{\alpha-2} u^{-\alpha-z} du = \int_0^1 (1 - v)^{\alpha-2} v^z dv \\ &= \frac{\Gamma(\alpha - 1) \Gamma(1 + z)}{\Gamma(\alpha + z)} = \text{Beta}(\alpha - 1, 1 + z), \end{aligned}$$

is convergent for $z > -1$. We apply a version of the Drasin-Shea theorem (Theorem 5.2.3 on p. 273 of [1]). To do this we must verify the following conditions.

1. \tilde{k} has a maximal convergent strip $a < \Re z < b$ such that $a < 0$ and $b > 0$, $\tilde{k}(a+) = \infty$ and $\tilde{k}(b-) = \infty$ if $b < \infty$. Our \tilde{k} satisfies this condition with $a = -1$ and $b = \infty$.

2. Our function of interest

$$g_\infty(x) = g(x^{-1}) = \int_0^\infty \bar{G}(ux)u^{\alpha-1}e^{-u}du, \quad x > 0,$$

is certainly positive and locally bounded.

3. Also our function g_∞ is of bounded decrease, since for $\lambda > 1$,

$$\frac{g_\infty(\lambda x)}{g_\infty(x)} = \lambda^{-\alpha} \frac{(\lambda x)^\alpha g(1/(\lambda x))}{x^\alpha g(1/x)} = \lambda^{-\alpha} \frac{f(1/(\lambda x))}{f(1/x)} \geq \lambda^{-\alpha},$$

so its lower Matuszewska index is at least $-\alpha$.

Therefore, by Theorem 5.2.3 of [1], whenever

$$(20) \quad \frac{k^M * g_\infty(x)}{g_\infty(x)} \rightarrow c, \quad \text{as } x \rightarrow \infty,$$

then $\tilde{k}(\rho) = c$ for some $\rho \in (-1, \infty)$. (In our case by (19), $c = [\gamma(\alpha - 1)]^{-1}$.) Moreover, since $\tilde{k}(z)$ is strictly decreasing on $(-1, \infty)$ and $\tilde{k}(0) = \frac{1}{\alpha-1}$, for any $0 < \gamma \leq 1$, the solution ρ to $\tilde{k}(\rho) = [\gamma(\alpha - 1)]^{-1}$ must lie in $(-1, 0]$. Theorem 5.2.3 of [1] also says that $g_\infty(x)$ is regularly varying at infinity with index $0 \geq \rho > -1$.

Next, since $g_\infty(x) = g(x^{-1}) = x^{-\alpha}f(x^{-1}) \in \mathcal{RV}_\infty(\rho)$, where $\tilde{k}(\rho) = c$, $g \in \mathcal{RV}_0(-\rho)$, which implies that $f \in \mathcal{RV}_0(-\rho - \alpha)$. Recalling that

$$f(t) = \int_0^\infty \bar{G}(u)u^{\alpha-1}e^{-ut}du,$$

the Karamata Tauberian theorem now gives that

$$\int_0^x \bar{G}(u)u^{\alpha-1}du \in \mathcal{RV}_\infty(\alpha + \rho).$$

Thus by Lemma 3 of Pitman [9], $\bar{G}(u) \in \mathcal{RV}_\infty(\rho)$.

This says that $Y \in D(\beta)$, where $\rho = -\beta \in (-1, 0]$ and β is the unique solution of

$$\text{Beta}(\alpha - 1, 1 - \beta) = \frac{\Gamma(\alpha - 1)\Gamma(1 - \beta)}{\Gamma(\alpha - \beta)} = \frac{1}{\gamma(\alpha - 1)}.$$

We now turn to the proof of the second part of Proposition 2. First consider the case $\beta = 0$. Let $0 \leq D_n^{(n)} \leq \dots \leq D_n^{(1)}$ denote the order statistics of $Y_1 / (\sum_{i=1}^n Y_i), \dots, Y_n / (\sum_{i=1}^n Y_i)$. We see that

$$\mathbb{E} \left(D_n^{(1)} \right)^\alpha \leq \mathbb{E} S_n(\alpha) = \sum_{i=1}^n \mathbb{E} \left(D_n^{(i)} \right)^\alpha \leq \mathbb{E} \left(D_n^{(1)} \right)^{\alpha-1} \leq 1.$$

Now $D_n^{(1)} \rightarrow_P 1$ if and only if $Y \in D(0)$. (See Theorem 1 of Haeusler and Mason [5] and their references.) Thus if $Y \in D(0)$, then (7) holds with $\gamma = 1$.

Now assume that $Y \in D(\beta)$, $0 < \beta < 1$. In this case, there exists a sequence of positive constants $\{a_n\}_{n \geq 1}$, such that $a_n^{-1} \sum_{i=1}^n Y_i \rightarrow_d U$, where U is a β -stable random variable, with characteristic function

$$\mathbb{E} e^{itU} = \exp \left\{ \beta \int_0^\infty (e^{itu} - 1)u^{-\beta-1}du \right\}.$$

Moreover, $Y^\alpha \in D(\beta/\alpha)$, and it is easy to check that $a_n^{-\alpha} \sum_{i=1}^n Y_i^\alpha \rightarrow_d V$, where V is a β/α -stable random variable, with cf

$$\mathbb{E}e^{tV} = \exp \left\{ \frac{\beta}{\alpha} \int_0^\infty (e^{tu} - 1) u^{-\beta/\alpha-1} du \right\}.$$

Since

$$\begin{aligned} \lim_{n \rightarrow \infty} n\mathbb{P}\{Y > a_n u, Y^\alpha > a_n^\alpha v\} &= \lim_{n \rightarrow \infty} n\bar{G}(a_n(u \vee v^{1/\alpha})) \\ &= u^{-\beta} \wedge v^{-\beta/\alpha} =: \Pi((u, \infty) \times (v, \infty)), \end{aligned}$$

for $u, v \geq 0$, $u+v > 0$, using Corollary 15.16 of Kallenberg [6] one can show that the joint convergence also holds, and the limiting bivariate Lévy measure is Π . That is,

$$\left(a_n^{-1} \sum_{i=1}^n Y_i, a_n^{-\alpha} \sum_{i=1}^n Y_i^\alpha \right) \rightarrow_d (U, V),$$

where the limiting bivariate random vector has cf

$$\begin{aligned} \mathbb{E}e^{i(sU+tV)} &= \exp \left\{ \int_{[0, \infty)^2} (e^{i(su+tv)} - 1) \Pi(du, dv) \right\} \\ &= \exp \left\{ \beta \int_0^\infty (e^{i(su+tu^\alpha)} - 1) u^{-\beta-1} du \right\}. \end{aligned}$$

Since $\mathbb{P}\{U > 0\} = \mathbb{P}\{V > 0\} = 1$, we obtain

$$S_n(\alpha) \rightarrow_d \frac{V}{U^\alpha}.$$

Thus since $S_n(\alpha) \leq 1$ for all $n \geq 1$,

$$\mathbb{E}S_n(\alpha) \rightarrow \mathbb{E} \left(\frac{V}{U^\alpha} \right) =: \gamma \leq 1.$$

Clearly $\mathbb{P}\{U < \infty\} = 1$, which implies that $0 < \mathbb{E} \left(\frac{V}{U^\alpha} \right) \leq 1$, and thus by the first part of Proposition 2,

$$0 < \gamma = \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(1 - \beta)} < 1.$$

□

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