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ON THE ASYMPTOTIC MEAN VALUE PROPERTY FOR PLANAR p-HARMONIC FUNCTIONS

ÁNGEL ARROYO AND JOSÉ G. LLORENTE

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ABSTRACT. We show that p-harmonic functions in the plane satisfy a nonlinear asymptotic mean value property for $p > 1$. This extends previous results of Manfredi and Lindqvist for a certain range of p's.

1. INTRODUCTION

It is well known that harmonic functions in euclidean domains are those continuous functions satisfying the usual mean value property. Actually, harmonic functions can be characterized by the so-called asymptotic mean value property

$$u(x) = \fint_{B(x,r)} u(y)dy + o(r^2)$$

as $r \to 0$.

It is a challenging problem to try to find similar characterizations for solutions of other nonlinear differential operators such as the p-laplacian. We recall that a function $u \in W^{1,p}_{loc}(\Omega)$ is p-harmonic in a domain $\Omega \subset \mathbb{R}^d$ if it is a weak solution of the p-laplace equation

$$div(|\nabla u|^{p-2}\nabla u) = 0.$$

If $u \in C^2$ and $\nabla u \neq 0$, then direct computation shows that

$$\triangle_p u \equiv div(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2}\Big[(p-2)\frac{\triangle_\infty u}{|\nabla u|^2} + \triangle u\Big], \tag{1.1}$$

where $\triangle_\infty$ is the so-called infinity laplacian which, for C^2 functions, is given by

$$\triangle_\infty u = \sum_{i,j=1}^{d} u_{x_i,x_j} u_{x_i} u_{x_j}.$$

On the other hand, it follows essentially from Taylor's formula that

$$\lim_{r\to 0} \frac{1}{r^2}\Big[\frac{1}{2}\big(\sup_{B(x,r)} u + \inf_{B(x,r)} u\big) - u(x)\Big] = \frac{\triangle_\infty u(x)}{2|\nabla u(x)|^2}, \tag{1.2}$$

$$\lim_{r\to 0} \frac{1}{r^2}\Big(\fint_{B(x,r)} u - u(x)\Big) = \frac{\triangle u(x)}{2(d+2)}, \tag{1.3}$$

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where $B(x,r)$ denotes the open ball centered at x of radius r. From (1.1), (1.2) and (1.3) it can be shown that if $u \in C^2$, $\triangle_p u = 0$ and $\nabla u(x) \neq 0$, then

$$u(x) = \frac{p-2}{p+d} \cdot \frac{1}{2}\Big(\sup_{B(x,r)} u + \inf_{B(x,r)} u \Big) + \frac{2+d}{p+d} \fint_{B(x,r)} u(y)dy + o(r^2) \tag{1.4}$$

as $r \to 0$. Since p-harmonic functions are $C^{1,\alpha}$ for some $0 < \alpha < 1$ but not C^2 in general ([U], [L]), it is not clear whether (1.4) should be true in the general case. However, in [PMR] the authors proved that if $u \in C(\Omega) \cap W^{1,p}_{loc}(\Omega)$, then u is p-harmonic in Ω if and only if (1.4) holds in a weak (viscosity) sense. From [JLM] and [PMR], it follows that if u is continuous and satisfies (1.4) in the classical sense, then u is p-harmonic.

More information is available when $d = 2$. If u is p-harmonic in a planar domain, then the complex gradient

$$\frac{\partial u}{\partial z} = \frac{1}{2}(u_x - iu_y)$$

is a quasiregular mapping (so the critical points are isolated, unless u is constant) and u is C^∞ in $\{\nabla u \neq 0\}$ (see [BI], [IM]). Lindqvist and Manfredi have recently proven that in the plane p-harmonicity is equivalent to the asymptotic mean value property (in the classical sense) for a certain range of p's. Hereafter we denote by $D(x,r)$ the open disc of center x and radius r.

Theorem ([LM])**.** *Let $\Omega \subset \mathbb{R}^2$ be a domain and let $1 < p < p_0 = 9.52\ldots$. Then $u \in C(\Omega) \cap W^{1,p}_{loc}(\Omega)$ is p-harmonic in Ω if and only if the asymptotic expansion*

$$u(x) = \frac{p-2}{p+2} \cdot \frac{1}{2}\Big(\sup_{D(x,r)} u + \inf_{D(x,r)} u \Big) + \frac{4}{p+2} \fint_{D(x,r)} u(y)dy + o(r^2) \tag{1.5}$$

holds at each $x \in \Omega$, as $r \to 0$.

Our main result is an extension of Lindqvist and Manfredi's theorem to the whole range of p's.

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be a domain and let $1 < p < \infty$. Then $u \in C(\Omega) \cap W^{1,p}_{loc}(\Omega)$ is p-harmonic in Ω if and only if the asymptotic expansion* (1.5) *holds at each $x \in \Omega$, as $r \to 0$.*

It is enough to prove that a p-harmonic function satisfies (1.5) at a critical point. Indeed, by the previous comments, continuous functions satisfying (1.5) are p-harmonic and, on the other hand, p-harmonic functions satisfy (1.5) at noncritical points. Therefore, we will focus on the local behavior of a p-harmonic function around a critical point of multiplicity n. As in [LM], the method exploits the power series expansion of the complex gradient in the hodographic plane that was obtained in [IM].

2. The hodographic representation of u

Let u be a p-harmonic function with a critical point of multiplicity n at the origin and let

$$\frac{\partial u}{\partial z} = \frac{1}{2}(u_x - iu_y)$$

be the complex gradient of u. We can represent

$$\frac{\partial u}{\partial z}(z) = (\chi(z))^n;$$

here χ is quasiconformal in a neighborhood of the origin and $\chi(0)=0$ (see [BI]). In the hodographic plane, $\xi=\chi(z)$, and, according to [IM], the inverse of χ is given by

$$z=H(\xi)=\sum_{k=n+1}^{\infty}\left[A_k\left(\frac{\xi}{|\xi|}\right)^k+\varepsilon_k\overline{A_k}\left(\frac{\overline{\xi}}{|\xi|}\right)^k\right]\left(\frac{\xi}{|\xi|}\right)^{-n}|\xi|^{\lambda_k}, \tag{2.1}$$

where $A_k\in\mathbb{C}$, $A_{n+1}\neq 0$ and

$$\sum_{k=n+1}^{\infty}k\,|A_k|^2<\infty. \tag{2.2}$$

Furthermore,

$$\varepsilon_k=\frac{\lambda_k+n-k}{\lambda_k+n+k}\,,\quad \lambda_k=\frac{1}{2}\left(\sqrt{4k^2(p-1)+n^2(p-2)^2}-np\right). \tag{2.3}$$

From (2.3) it is easy to check that

$$0<\lambda_k<\frac{k^2-n^2}{n}\,,\quad |\varepsilon_k|<\frac{k-n}{k+n}. \tag{2.4}$$

Equation (2.1) can be interpreted as the "hodographic representation" of the point $z=x+iy$ near the origin. Note that, also from [IM], it follows that if H is given by (2.1) for certain coefficients A_k, ε_k and λ_k satisfying (2.2) and (2.3), then there are a quasiconformal change of variables $\xi=\chi(z)$ and a p-harmonic function u such that

$$\frac{\partial u}{\partial z}(z)=(\chi(z))^n$$

locally around the origin.

Therefore, if $\xi=re^{i\theta}$, we can rewrite (2.1) as

$$H(re^{i\theta})=e^{-in\theta}\sum_{k=n+1}^{\infty}r^{\lambda_k}\varphi_k(\theta), \tag{2.5}$$

where

$$\varphi_k(\theta)=A_ke^{ik\theta}+\varepsilon_k\overline{A_k}e^{-ik\theta}$$

for each $k=n+1,n+2,\ldots$. We can split $H(\xi)$ into its real and imaginary parts, i.e., $H(\xi)=\tilde{z}(\xi)=\tilde{x}(\xi)+i\tilde{y}(\xi)$.

Let us denote by $\tilde{u}$ the hodographic representation of u, i.e.,

$$\tilde{u}(\xi)=(u\circ H)(\xi). \tag{2.6}$$

Moreover, we can easily write $\frac{\partial u}{\partial z}$ in terms of ξ as

$$\frac{\partial u}{\partial z}(z)=\frac{\partial u}{\partial z}(H(\xi))=\xi^n$$

or, equivalently, if $\xi=re^{i\theta}$,

$$\begin{cases} u_x=2r^n\cos(n\theta),\\ u_y=-2r^n\sin(n\theta).\end{cases} \tag{2.7}$$

Proposition 2.1. *u is a p-harmonic function with a critical point of order n at $z = 0$ if and only if its hodographic representation $\tilde{u}$ has the following power series expansion in a neighborhood of $\xi = 0$:*

$$\tilde{u}(\xi) = u(0) + 4 \sum_{k=n+1}^{\infty} \mu_k \, |\xi|^{n+\lambda_k} \, \mathfrak{Re} \left\{ A_k \left(\frac{\xi}{|\xi|} \right)^k \right\},$$

where $\mu_k = \dfrac{\lambda_k}{\lambda_k + n + k}$. Moreover, $0 \le \mu_k < 1 - \frac{n}{k}$.

Proof. Let u be p-harmonic with a critical point of order n at the origin. Using (2.7), we compute $\tilde{u}_r$:

$$\tilde{u}_r = (u \circ H)_r = u_x \tilde{x}_r + u_y \tilde{y}_r = 2r^n \left[\tilde{x}_r \cos(n\theta) - \tilde{y}_r \sin(n\theta) \right],$$

the expression in brackets being equal to $\mathfrak{Re} \left\{ e^{in\theta} H_r \right\}$. Replacing (2.5) in the previous equation we get

$$\tilde{u}_r = 2r^n \mathfrak{Re} \left\{ e^{in\theta} H_r \right\} = 4 \sum_{k=n+1}^{\infty} \lambda_k (1 + \varepsilon_k) r^{n+\lambda_k - 1} \mathfrak{Re} \left\{ A_k e^{ik\theta} \right\}.$$

Integrate with respect to r to complete the proof. The bound on μ_k follows from (2.4). The converse follows from the comments at the beginning of the section. $\square$

Remark. From now on, we can assume without loss of generality that $u(0) = 0$.

3. Quantitative injectivity estimates for the first term of H

We define the mapping $\mathcal{A}(\xi)$ as the first term in the power series expansion of H:

$$\mathcal{A}(\xi) = \left[A_{n+1} \left(\frac{\xi}{|\xi|} \right)^{n+1} + \varepsilon_{n+1} \overline{A_{n+1}} \left(\frac{\overline{\xi}}{|\xi|} \right)^{n+1} \right] \left(\frac{\xi}{|\xi|} \right)^{-n} |\xi|^{\lambda_{n+1}}. \tag{3.1}$$

We define $\widetilde{\mathfrak{U}}(\xi)$ to be the first term in the power series expansion of $\tilde{u}$,

$$\widetilde{\mathfrak{U}}(\xi) = 4\mu_{n+1} \, |\xi|^{n+\lambda_{n+1}} \, \mathfrak{Re} \left\{ A_{n+1} \left(\frac{\xi}{|\xi|} \right)^{n+1} \right\}. \tag{3.2}$$

For simplicity, we will use hereafter the notation $a \lesssim b$ (resp. $a \approx b$) to indicate that $a \le Cb$ (resp. $C^{-1}a \le b \le Ca$) for some positive constant C independent of a and b.

Lemma 3.1. *The following estimates hold in a neighborhood of $\xi = 0$:*

$$\left| \tilde{u}(\xi) - \widetilde{\mathfrak{U}}(\xi) \right| \lesssim |\xi|^{n+\lambda_{n+2}}, \tag{3.3}$$

$$|H(\xi) - \mathcal{A}(\xi)| \lesssim |\xi|^{\lambda_{n+2}}, \tag{3.4}$$

$$|\mathcal{A}(\xi)| \approx |H(\xi)| \approx |\xi|^{\lambda_{n+1}}. \tag{3.5}$$

Proof. From (2.2), (2.4) and the fact that $0 \le \mu_k < 1$ we get in particular that the sequence (A_k) is bounded and that $|\varepsilon_k| < 1$ for all k. Since (λ_k) is increasing, (3.3), (3.4) and (3.5) follow from the estimate

$$\sum_{k=n+2}^{\infty} |\xi|^{\lambda_k} = O(|\xi|^{\lambda_{n+2}}). \tag{3.6}$$

Now an elementary computation shows that there is $C = C(p) > 0$ such that $\lambda_k - \lambda_{n+2} \geq C(k - (n+2))$ for all $k \geq n+2$. This implies (3.6) and proves the lemma. □

Now, we study the behavior of $\mathcal{A}$ and we give an injectivity estimate. For this purpose, we will need the help of the following elementary lemma whose proof is omitted.

Lemma 3.2. *Let $\rho > 0$, $\lambda > 0$ and $t \in \mathbb{R}$. Then for all $k \in \mathbb{N}$,*

$$\left|\rho e^{ikt} - 1\right| \leq k\left|\rho e^{it} - 1\right|. \tag{3.7}$$

Furthermore, if $\Lambda > 1$ and if $\Lambda^{-1} \leq \rho \leq \Lambda$, then there is a constant $C = C(\lambda, \Lambda) > 0$ such that

$$\left|\rho^{\lambda} e^{it} - 1\right| \geq C\rho^{\lambda-1}\left|\rho e^{it} - 1\right|. \tag{3.8}$$

Lemma 3.3. *The mapping $\mathcal{A} : \mathbb{C} \to \mathbb{C}$ is bijective and satisfies*

$$|\mathcal{A}(\xi) - \mathcal{A}(\zeta)| \geq C\left||\xi|^{\lambda_{n+1}-1}\xi - |\zeta|^{\lambda_{n+1}-1}\zeta\right|, \tag{3.9}$$

where $C = (1 - (2n+1)|\varepsilon_{n+1}|)|A_{n+1}|$.

Proof. First, we observe that from (2.4) for $k = n+1$ we obtain that $0 < \lambda_{n+1} < 2 + \dfrac{1}{n}$ and that

$$|\varepsilon_{n+1}| < \frac{1}{2n+1}.$$

We show first that $\mathcal{A}$ is surjective. We write $\lambda \equiv \lambda_{n+1}$, $\varepsilon \equiv \varepsilon_{n+1}$ and $A \equiv A_{n+1}$. Then

$$\mathcal{A}(re^{i\theta}) = r^{\lambda}e^{i\theta}\big(A + \varepsilon\overline{A}e^{-i2(n+1)\theta}\big).$$

Assume, for simplicity, that $A = 1$. Then we can write

$$\mathcal{A}(re^{i\theta}) = r^{\lambda}|1 + \varepsilon\, e^{-i2(n+1)\theta}|e^{if(\theta)},$$

where $f(\theta) = \theta + \arg(1 + \varepsilon\, e^{-i2(n+1)\theta})$ and

$$m(\theta) = |1 + \varepsilon\, e^{-i2(n+1)\theta}| = \sqrt{1 + \varepsilon^2 + 2\varepsilon\cos(2(n+1)\theta)}. \tag{3.10}$$

To prove that $\mathcal{A}$ is surjective, let $w = se^{it} \in \mathbb{C}$ such that $w \neq 0$ (if $w = 0$ it is obvious that $\mathcal{A}(0) = 0$). Since $f(0) = 0$ and $f(2\pi) = 2\pi$, by continuity we can pick $k \in \mathbb{Z}$ and $\theta \in [0, 2\pi]$ such that $t + 2k\pi \in [0, 2\pi]$ and $f(\theta) = t + 2k\pi$. Then $e^{if(\theta)} = e^{it}$. For that θ, choose $r > 0$ so that

$$r^{\lambda}m(\theta) = s.$$

Then we have shown that $\mathcal{A}(re^{i\theta}) = w$, so the surjectiveness of $\mathcal{A}$ follows.

To finish the proof of the lemma, it is enough to prove (3.9), which is a quantitative form of injectiveness. By (3.1),

$$\begin{aligned} |\mathcal{A}(\xi) - \mathcal{A}(\zeta)| \ &\geq\ |A_{n+1}|\left||\xi|^{\lambda}\frac{\xi}{|\xi|} - |\zeta|^{\lambda}\frac{\zeta}{|\zeta|}\right| \\ &-\ |A_{n+1}|\,|\varepsilon|\left||\xi|^{\lambda}\left(\frac{\overline{\xi}}{|\xi|}\right)^{2n+1} - |\zeta|^{\lambda}\left(\frac{\overline{\zeta}}{|\zeta|}\right)^{2n+1}\right|. \end{aligned} \tag{3.11}$$

Now apply (3.7) with $\rho = \left|\frac{\xi}{\zeta}\right|^{\lambda}$, $e^{it} = \frac{\xi/\zeta}{|\xi/\zeta|}$ and $k = 2n+1$, and multiply both sides of the inequality by $|\zeta|^{\lambda}$. Then

$$\left| |\xi|^{\lambda} \left(\frac{\xi}{|\xi|}\right)^{2n+1} - |\zeta|^{\lambda} \left(\frac{\zeta}{|\zeta|}\right)^{2n+1} \right| \leq (2n+1) \left| |\xi|^{\lambda} \frac{\xi}{|\xi|} - |\zeta|^{\lambda} \frac{\zeta}{|\zeta|} \right| .$$

Replacing this expression in (3.11) we obtain

$$|\mathcal{A}(\xi) - \mathcal{A}(\zeta)| \geq |A_{n+1}| \left(1 - (2n+1)\, |\varepsilon|\right) \left| |\xi|^{\lambda} \frac{\xi}{|\xi|} - |\zeta|^{\lambda} \frac{\zeta}{|\zeta|} \right| ,$$

so the proof is finished. □

Lemma 3.4. *Let $\Lambda > 1$. Then there is a constant $C = C(n, p, \Lambda, |A_{n+1}|) > 0$ such that for any ξ, $\zeta \in \mathbb{C}$ with $\Lambda^{-1} |\zeta| \leq |\xi| \leq \Lambda |\zeta|$ we have*

$$|\mathcal{A}(\xi) - \mathcal{A}(\zeta)| \geq C\, |\xi|^{\lambda_{n+1}-1}\, |\xi - \zeta| . \tag{3.12}$$

Proof. Apply (3.8) with $\rho = \left|\frac{\xi}{\zeta}\right|$ and $e^{it} = \frac{\xi/\zeta}{|\xi/\zeta|}$ and multiply by $|\zeta|^{\lambda}$ to obtain

$$\left| |\xi|^{\lambda} \frac{\xi}{|\xi|} - |\zeta|^{\lambda} \frac{\zeta}{|\zeta|} \right| \geq C\, |\xi|^{\lambda-1}\, |\xi - \zeta| . \tag{3.13}$$

Then the lemma follows from (3.9) together with (3.13). □

4. The perturbation method

Given ξ in the hodographic plane, set $z = H(\xi)$, $\zeta = \mathcal{A}^{-1}(z)$ and $w = H(\zeta)$. Then

$$\xi = \chi(z)\ , \quad \zeta = \chi(w) = \mathcal{A}^{-1}(H(\xi)).$$

Since $|z| = |\mathcal{A}(\zeta)| \approx |H(\zeta)| = |w|$, by (3.5) it follows from quasiconformality ([A]) that $|\xi| = |\chi(z)| \approx |\chi(w)| = |\zeta|$. We recall that $u(z) = (\tilde{u} \circ \chi)(z)$ and that $\widetilde{\mathfrak{U}}$ is given by (3.2). Following [LM], define

$$\mathfrak{U}(z) = (\widetilde{\mathfrak{U}} \circ \mathcal{A}^{-1})(z) \tag{4.1}$$

and note that $\mathfrak{U}$ is p-harmonic from Proposition 2.1.

Lemma 4.1. *Let $\Lambda > 1$. There is a constant $C = C(n, p, \Lambda, |A_{n+1}|) > 0$ such that for any ξ, $\zeta \in \mathbb{C}$ with $\Lambda^{-1}|\zeta| \leq |\xi| \leq \Lambda|\zeta|$ then*

$$\left|\widetilde{\mathfrak{U}}(\xi) - \widetilde{\mathfrak{U}}(\zeta)\right| \leq C|\xi|^{n} |\mathcal{A}(\xi) - \mathcal{A}(\zeta)|.$$

Proof. From (3.2), the fact that $0 \leq \mu_k < 1$ if $k \geq n$ and direct computation it follows that

$$\left|\widetilde{\mathfrak{U}}(\xi) - \widetilde{\mathfrak{U}}(\zeta)\right| \leq C|A_{n+1}|\, |\xi|^{n+\lambda_{n+1}-1}\, |\xi - \zeta| ,$$

where $C = C(n, \Lambda) > 0$. Then the conclusion follows from Lemma 3.4. □

Corollary 4.1. *Let ξ and ζ be as in the beginning of the section. Then the following estimate holds in a neighborhood of $\xi = 0$:*

$$\left|\widetilde{\mathfrak{U}}(\xi) - \widetilde{\mathfrak{U}}(\zeta)\right| \lesssim |\xi|^{n+\lambda_{n+2}}.$$

Proof. Use the fact that $|\xi| \approx |\zeta|$, Lemma 4.1 and estimate (3.4). □

Now we are ready to prove the following singular expansion of a p-harmonic function.

Proposition 4.1. *Let u be a p-harmonic function with a critical point of order n at $z = 0$ and $u(0) = 0$. Then u can be written as*

$$u(z) = \mathfrak{U}(z) + O(|z|^{\frac{n+\lambda_{n+2}}{\lambda_{n+1}}}) \tag{4.2}$$

in a neighborhood of $z = 0$.

Proof. By (2.6) and (4.1) we can write

$$u(z) - \mathfrak{U}(z) = \tilde{u}(\xi) - \widetilde{\mathfrak{U}}(\zeta) = \widetilde{\mathfrak{U}}(\xi) - \widetilde{\mathfrak{U}}(\zeta) + \tilde{u}(\xi) - \widetilde{\mathfrak{U}}(\xi).$$

By (3.3), (3.5) and Corollary 4.1 we get

$$|u(z) - \mathfrak{U}(z)| \lesssim |\xi|^{n+\lambda_{n+2}} \approx |H(\xi)|^{\frac{n+\lambda_{n+2}}{\lambda_{n+1}}} = |z|^{\frac{n+\lambda_{n+2}}{\lambda_{n+1}}},$$

so the proof is finished. □

5. Proof of Theorem 1

As before, we will write λ and ε instead of λ_{n+1} and ε_{n+1}, respectively. We can assume without loss of generality that $A_{n+1} = 1$. Then

$$\mathcal{A}(re^{i\theta}) = r^{\lambda} e^{-in\theta}(e^{i(n+1)\theta} + \varepsilon e^{-i(n+1)\theta})$$

and $\left|\mathcal{A}(re^{i\theta})\right| = r^{\lambda} m(\theta)$, where $m(\theta)$ is given by (3.10). Furthermore,

$$\widetilde{\mathfrak{U}}(re^{i\theta}) = 4\mu r^{n+\lambda} \cos((n+1)\theta),$$

where $\mu = \mu_{n+1}$.

Denote by $D_R = D(0, R)$ the open disc centered at 0 with radius $R > 0$ and define the "hodographic disc" $\widetilde{D_R}$ as $\mathcal{A}^{-1}(D_R)$. Then, a point $re^{i\theta}$ of the hodographic plane belongs to $\widetilde{D_R}$ if and only if $\left|\mathcal{A}(re^{i\theta})\right| < R$. Then, $\widetilde{D_R}$ can be described, in polar coordinates, as

$$\widetilde{D_R} = \left\{ re^{i\theta} \ : \ r < \left(\frac{R}{m(\theta)}\right)^{1/\lambda} \right\}.$$

Now we define the function $J(\zeta)$ as the absolute value of the jacobian of $\mathcal{A}(\zeta)$. Computing $J(\zeta)$ in polar coordinates we get

$$J(re^{i\theta}) = \lambda r^{2(\lambda-1)}\big(1 - (2n+1)\varepsilon^2 - 2n\varepsilon\cos(2(n+1)\theta)\big). \tag{5.1}$$

(Observe that, since $|\varepsilon| < (2n+1)^{-1}$, the expression in the right hand side of (5.1) is positive.)

Lemma 5.1. *The p-harmonic function $\mathfrak{U}(z)$ given by (4.1) satisfies the following properties, for small enough $R > 0$:*

$$\sup_{D_R} \mathfrak{U} + \inf_{D_R} \mathfrak{U} = 0, \tag{5.2}$$

$$\int_{D_R} \mathfrak{U} = 0. \tag{5.3}$$

Proof. By (4.1), we need to study the behavior of $\widetilde{\mathfrak{U}}(\xi)$ in $\widetilde{D_R}$. Then (5.2) is a direct consequence of the symmetries of $\widetilde{D_R}$. To show (5.3), observe that, by a change of variables

$$\int_{D_R} \mathfrak{U}(z)dz = \int_{\widetilde{D_R}} \widetilde{\mathfrak{U}}(\zeta)J(\zeta)d\zeta \tag{5.4}$$

and using polar coordinates in (5.4), we get

$$\int_{D_R} \mathfrak{U}(z)dz = 4\mu\lambda \int_0^{2\pi} \int_0^{r(\theta)} r^{n+3\lambda-1} \cos((n+1)\theta) j(\theta) dr d\theta, \tag{5.5}$$

where

$$r(\theta) = \Big(\frac{R}{m(\theta)}\Big)^{1/\lambda}, \quad j(\theta) = 1 - (2n+1)\varepsilon^2 - 2n\varepsilon\cos(2(n+1)\theta)$$

and $m(\theta)$ is given by (3.10). Now (5.3) follows directly from (5.5) and the symmetry properties of $m(\theta)$ and $j(\theta)$. □

Lemma 5.2. *The inequality*

$$\frac{n+\lambda_{n+2}}{\lambda_{n+1}} > 2 \tag{5.6}$$

holds for each $1 < p < \infty$ *and each* $n \geq 1$.

Proof. From (2.3) and some computation it follows that inequality (5.6) is equivalent to

$$n(p+2)\sqrt{n^2p^2 + 16(n+1)(p-1)} > n^2p^2 + (-2n^2+8n)p - (2n^2+8n). \tag{5.7}$$

Now we distinguish two cases. If $n = 1$, then (5.7) becomes

$$(p+2)\sqrt{p^2 + 32(p-1)} > p^2 + 6p - 10.$$

If the right hand side is negative, then the inequality follows. Otherwise, squaring the previous inequality we get

$$2p^3 + 7p^2 + 10p - 19 > 0,$$

which holds for each $p > 1$ since the left hand side is increasing in p and vanishes for $p = 1$. This proves (5.7) when $n = 1$.

Now assume $n \geq 2$ and observe that $\sqrt{n^2p^2 + 16(n+1)(p-1)} \geq np$ for each $p > 1$. Then (5.7) would follow if

$$n^2p(p+2) > n^2p^2 + (-2n^2+8n)p - (2n^2+8n),$$

which is equivalent to

$$(2n-4)p + n + 4 > 0,$$

and holds trivially if $n \geq 2$. This finishes the proof of the lemma. □

Proof of Theorem 1. As stated in the introduction, we only need to prove that planar p-harmonic functions satisfy (1.5) since the converse is well known. We also discussed there that (1.5) need only be checked at a critical point. Therefore, we can assume that $x = 0$, $u(0) = 0$ and that 0 is a critical point of u.

Let $r > 0$ be small enough. By Proposition 4.1 and Lemma 5.1,

$$\int_{D_r} u = O\big(r^{\frac{n+\lambda_{n+2}}{\lambda_{n+1}}}\big) \tag{5.8}$$

and

$$\frac{1}{2}\Big(\sup_{D_r} u + \inf_{D_r} u\Big) = O\big(r^{\frac{n+\lambda_{n+2}}{\lambda_{n+1}}}\big). \tag{5.9}$$

Finally, combine (5.8), (5.9) and divide by r^2 to obtain that for any $\alpha \in \mathbb{R}$

$$\frac{1}{r^2}\Big[\alpha\big(\frac{1}{2}\sup_{D_r} u + \frac{1}{2}\inf_{D_r} u\big) + (1-\alpha)\fint_{D_r} u\Big] = O\big(r^{\frac{n+\lambda_{n+2}}{\lambda_{n+1}}-2}\big). \tag{5.10}$$

By Lemma 5.2 the exponent of r in the right hand side is strictly positive. Therefore, taking limits as $r \to 0$, we show that (1.5) holds at the origin and we conclude the proof. □

Remarks.

(1) The reason why our method produces a better exponent in (4.2) and therefore makes possible the extension of the result in [LM] to the full range $1 < p < \infty$ is that we use estimate (3.12), which is stronger than the Hölder estimate for $\mathcal{A}^{-1}$ used in [LM] (Section 2.2).

(2) Even though our arguments work equally for all $p > 1$, it is pointed out in [LM] (Section 2.1) that the result is meaningful for $p > 2$, since in the range $1 < p < 2$ it always holds thanks to the Hölder regularity of the second derivatives. On the other hand, it can be checked that

$$\frac{n+\lambda_{n+2}}{\lambda_{n+1}} \geq 1 + \frac{\lambda_{n+2}}{\lambda_{n+1}^2}$$

for all $p \geq 2$ and each $n \in \mathbb{N}$.

(3) The proof actually shows that if x is a critical point of the p-harmonic function u, then (1.5) still holds at x if the coefficients $(p-2)/(p+2)$ and $4/(p+2)$ are replaced by α and $1-\alpha$ for arbitrary α.

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain
E-mail address: `arroyo@mat.uab.cat`

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain
E-mail address: `jgllorente@mat.uab.cat`