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A THEOREM OF LOHWATER AND PIRANIAN

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Abstract. By a well-known theorem of Lohwater and Piranian, for any set E on $|z| = 1$ of type F_σ and of measure zero there exists a bounded analytic function in $|z| < 1$ which fails to have radial limits exactly at the points of E. We show that this theorem is an immediate corollary of Fatou's interpolation theorem of 1906.

1. Introduction

Denote by Δ and T the open unit disk and the unit circle in the complex plane, respectively. Recall that the disk algebra A is the algebra of all continuous functions on the closed unit disk $\overline{\Delta}$ that are analytic on Δ. Let m be the Lebesgue measure on T. The following important classical result is due to P. Fatou.

Theorem A (Fatou, 1906)**.** *Let K be a closed subset of T such that $m(K) = 0$. Then there exists a function $f \in A$ which vanishes precisely on K.*

Theorem A and its elementary proof (which also shows that the real part of f is positive on Δ) can be found in such standard references as [2, pp. 80-81] and [4, pp. 29-30].

Recall that a set is of type F_σ if it is a countable union of closed sets. We formulate a well-known theorem of A. J. Lohwater and G. Piranian [5, Theorem 6], [1, Theorem 2.7].

Theorem B (Lohwater and Piranian, 1957)**.** *Let E on T be of type F_σ and of measure zero. Then there exists a bounded analytic function in Δ which fails to have radial limits exactly at the points of E.*

The proof of Theorem B is long and complicated. Our purpose here is to show how Theorem B follows as an elementary and almost instant corollary of Theorem A.

2. Proof of Theorem B

Let $E = \bigcup_{n=1}^{\infty} E_n$, where E_n is closed and $m(E_n) = 0$. By Theorem A we have $f_n \in A$ such that $f_n = 0$ precisely on E_n. As mentioned above, also $\Re f_n(z) > 0$ for $z \in \Delta$. This provides a single valued analytic function $\log f_n(z) = \log |f_n(z)| + i \arg f_n(z)$ in Δ with $|\arg f_n(z)| < \pi/2$. Obviously $\log |f_n(z)| \to -\infty$ as $z \in \Delta$

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approaches any point of E_n, while $\log f_n(z)$ has finite limits at each point of the open set $T \setminus E_n$. The radial limit of

$$\varphi_n(z) = e^{i \log f_n(z)} = e^{-\arg f_n(z)}[\cos(\log |f_n(z)|) + i \sin(\log |f_n(z)|)]$$

exists for each $\zeta \in T \setminus E_n$. On each radius ending on E_n the oscillation of φ_n exceeds $e^{-\pi/2}$.

The bounded analytic function $f(z) = \sum_{n=1}^{\infty} 1000^{-n} \varphi_n(z)$ has radial limits at $\zeta \in T \setminus E$ since each φ_n does, and the series converges uniformly. If $\zeta_0 \in E$, let q be the smallest index such that $\zeta_0 \in E_q$. Then $\sum_{n=1}^{q-1} 1000^{-n} \varphi_n(z)$ has a finite radial limit at ζ_0. Since at ζ_0 the oscillation of the radial limit of $1000^{-q} \varphi_q(z)$ exceeds $1000^{-q} e^{-\pi/2}$ and the remainder $\sum_{n=q+1}^{\infty} 1000^{-n} \varphi_n(z)$ does not exceed $1000^{-q} \frac{e^{\pi/2}}{999}$, the oscillation of the radial limit of f at ζ_0 is larger than some positive number (say, $0.5e^{-\pi/2} 1000^{-q}$). The proof is over.

Remark. The Lohwater-Piranian theorem was an important step toward the solution of finding necessary and sufficient conditions on a set $E \subset T$ for it to be the (exact) set of points at which a bounded analytic function in Δ fails to have radial limits. That such a set must have measure zero and be of type $G_{\delta\sigma}$ had been known for a long time [1, Theorems 2.1 and 2.6]. The complete solution to this problem was obtained by S. V. Kolesnikov [3], with the following generalization of Theorem B.

Theorem C (Kolesnikov, 1994)**.** *Let $E \subset T$ be of type $G_{\delta\sigma}$ and of measure* 0*. Then there exists a bounded analytic function in Δ which fails to have radial limits exactly at the points of E.*

Kolesnikov's construction is extraordinarily elaborate; see Piranian's review [6] for a detailed summary.

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