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THREE-SPHERES THEOREMS FOR SUBELLIPTIC QUASILINEAR EQUATIONS IN CARNOT GROUPS OF HEISENBERG-TYPE

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Abstract. We study the arithmetic three-spheres theorems for subsolutions of subelliptic PDEs of p-harmonic type in Carnot groups of Heisenberg type for $1 < p < \infty$. In the presentation we exhibit the special cases of sub-Laplace equations ($p = 2$) and the case p is equal to the homogeneous dimension of a Carnot group. Corollaries include asymptotic behavior of subsolutions for small and large radii and the Liouville-type theorems.

1. Introduction

The Hadamard three-circles theorem [16] asserts that given three concentric circles with radii $0 < r_1 < r < r_2$ and a subharmonic function u in the plane, the maximum of u over a circle with radius r is a convex function of $\log r$, with coefficients depending on the ratios of r_1, r and r_2; see Chapter 12 in Protter–Weinberger [25].

The three-circles theorem can be generalized in various directions, encompassing subharmonic functions in $\mathbb{R}^n$ for $n > 2$, higher-dimensional concentric surfaces (e.g., three-spheres theorems), more general linear and quasilinear elliptic equations (see Brummelhuis [6], Granlund–Marola [15] and Miklyukov–Rasila–Vuorinen [22]), the heat equation (three-parabolas theorem) and coupled elliptic systems of equations (see Adamowicz [1] and further references therein). Moreover, one studies multiplicative variants of the three-circles theorems which play a role in proving the Riesz–Thorin convexity and interpolation theorems, and also in showing the unique continuation properties for PDEs; see Alessandrini–Rondi–Rosset–Vessella [2] and Garofalo–Lin [13].

The main goal of our studies is to generalize the classical Hadamard theorem to the setting of subelliptic quasilinear equations in Carnot groups of Heisenberg type (H-type for short). Perhaps the most important example of such an equation is the p-harmonic equation; see Heinonen–Holopainen [17] for basic properties of p-harmonic functions and relations to quasiregular maps, Domokos [11], Manfredi–Mingione [21], Mingione–Zatorska-Goldstein–Zhong [23] for regularity results, Capogna–Danielli–Garofalo [7], Garofalo–Tyson [14] for results in potential theory, and also see Bieske [4] and Nyström [24] for further topics on subelliptic

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equations of p-harmonic type. For $p = 2$ we retrieve the important sub-Laplace equation, while for p equal to the homogeneous dimension of the given Carnot group we obtain the counterpart of the n-harmonic equation in $\mathbb{R}^n$; see [17].

The structure of the paper is as follows. In Preliminaries we recall basic notions and definitions of the theory of Carnot groups and quasilinear PDEs. In particular, we discuss elementary properties of Heisenberg groups and H-type groups, define weak solutions and formulate some of the fundamental results in the potential theory on Carnot groups needed in the presentation. In Section 3 we state and prove various variants of the three-spheres theorems: for $p \neq Q$, $p = Q$ and $p = 2$, where Q denotes the homogeneous dimension of the Carnot group. Corollaries of such results include the behavior of subsolutions for small radii, large radii and Liouville-type theorems. Similar observations are known for subharmonic functions in $\mathbb{R}^2$, but they often fail to hold beyond the plane. Whereas we observe, perhaps surprisingly, a new phenomena that counterparts of such results in the Carnot setting remain valid regardless of the value of Q.

According to our best knowledge, the three-spheres theorems on Carnot groups have not yet been studied in the literature. In accordance with classical theory, it is natural that three-spheres theorems should be studied since they play a role in extending a set of tools available in the study of quasilinear equations to the setting of subsolutions to PDEs in sub-Riemannian geometry. We also hope that our discussion can be extended to several other elliptic equations.

2. Preliminaries

Carnot groups are simply connected nilpotent Lie groups with a stratified Lie algebra equipped with a left invariant sub-Riemannian metric and form the ideal models of sub-Riemannian geometry. Historically, sub-Riemannian geometry originates from the paper by Carathéodory [9] on thermodynamics and arises in many areas of mathematics and physics, such as control theory and neurobiology; see Chapter 3.2 in Capogna–Danielli–Pauls–Tyson [8]. From the PDEs and physics perspective such geometries arise in connection with elliptic-parabolic equations, particularly via the Hörmander theorem; see below or Bonfiglioli–Lanconelli–Uguzzoni [5].

A nilpotent Lie algebra $\mathfrak{g}$ is said to admit an s-step stratification if it decomposes as the direct sum of subspaces $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$, which satisfy the bracket generating property $\mathfrak{g}_{j+1} = [\mathfrak{g}_1, \mathfrak{g}_j]$, where $j = 1, \ldots, s-1$, and $\mathfrak{g}_s$ is contained in the center of $\mathfrak{g}$. If $\mathfrak{g}$ is equipped with an inner product $\langle \, , \, \rangle$ such that $\mathfrak{g}_i \perp \mathfrak{g}_j$ for every $i \neq j$, then the connected, simply connected, nilpotent Lie group G, with stratified Lie algebra $\mathfrak{g}$, is called a *Carnot group*.

A consequence of nilpotency is that the exponential map $\exp_G : \mathfrak{g} \to G$ is a bijective diffeomorphism. Furthermore, if $\log_G = \exp_G^{-1}$, then the Baker–Campbell–Hausdorff formula (see Helgason [19]) is defined for all $X, Y \in \mathfrak{g}$ by

(1)
$$\log_G(\exp_G(X)\exp_G(Y)) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\left([X,[X,Y]] + [Y,[Y,X]]\right) + \ldots,$$

and so the formula $X \star Y = \log_G(\exp_G(X)\exp_G(Y))$ defines a product on the Lie algebra $\mathfrak{g}$ so that $\exp_G$ becomes a group isomorphism. We call $(\mathfrak{g}, \star)$ the *normal model* and note that if $G = (\mathfrak{g}, \star)$, then $\exp_G$ and $\log_G$ are both the identity map of $\mathfrak{g}$. An orthonormal basis $\{E_i\}$ for $\mathfrak{g}$ then imparts coordinates on $(\mathfrak{g}, \star)$ by setting $x_i(X) = \langle X, E_i \rangle$ for each $X \in G$, which in turn provide coordinates on G by composing with $\log_G$. With these observations in mind we see that we only need consider the normal models and so from now on we assume $G = (\mathfrak{g}, \star)$.

A dilation by $t > 0$ of an element $X \in \mathfrak{g}$ is defined by $\delta_t(X) = \sum_{k=1}^{s} t^k \pi_k(X)$, where π_k is the projection onto $\mathfrak{g}_k$. In the normal model the Lie algebra automorphism and group isomorphism coincide, in particular δ_t is simultaneously a Lie algebra automorphism and a group isomorphism.

The subbundle $\mathcal{H} \subseteq TG$ obtained by left translating $\mathfrak{g}_1$ is called the *horizontal bundle*. The inner product on $\mathfrak{g}$ also defines an inner product on the tangent space $T_Y G$ at every point $Y \in G$ by left translation. In particular, for $X_1, X_2 \in T_Y G$ we have

$$\langle X_1, X_2 \rangle_Y = \langle (\tau_{Y^{-1}})_*(X_1), (\tau_{Y^{-1}})_*(X_2) \rangle,$$

where $\tau_{Y^{-1}}$ denotes left translation by $Y^{-1} = -Y \in G$ and $(\tau_{Y^{-1}})_*$ stands for the differential of the map $\tau_{Y^{-1}} : G \to G$. Here $T_0 G$ is identified with $\mathfrak{g}$.

The left invariant Haar measure on G is simply the Lebesgue measure on $\mathfrak{g}$. We denote by $|E|$ the measure of a set $|E|$ and note that $|\delta_t(E)| = t^Q |E|$ where $Q = \sum_i^s i \dim \mathfrak{g}_i$ is the *homogeneous dimension* of G.

A final observation is that the left invariant vector fields corresponding to a basis of $\mathfrak{g}_1$ automatically satisfy Hörmander's condition and so an elliptic-parabolic operator constructed from these fields is always hypoelliptic.

Example 1. The n-dimensional Heisenberg group $G = \mathbb{H}_n$ is the Carnot group with a 2-step Lie algebra and orthonormal basis $\{X_1, \ldots, X_{2n}, Z\}$ such that

$$\mathfrak{g}_1 = \operatorname{Span}\{X_1, \ldots, X_{2n}\}, \quad \mathfrak{g}_2 = \operatorname{Span}\{Z\}$$

and the nontrivial brackets are $[X_i, X_{n+i}] = Z$ for $i = 1, \ldots, n$. If $X = \sum_i x_i X_i + zZ$ and $Y = \sum_i y_i X_i + sZ$, then the Baker–Campbell–Hausdorff formula defines the group product

$$\begin{aligned} X \star Y &= X + Y + \frac{1}{2}[X, Y] \\ &= \sum_i (x_i + y_i) X_i + \Big(z + s + \frac{1}{2}(x_i y_{n+i} - x_{n+i} y_i)\Big) Z. \end{aligned} \tag{2}$$

In the given coordinates, X is identified with the point $(x_1, \ldots, x_{2n}, z) \in \mathbb{R}^{2n+1}$ and the left invariant vector fields corresponding to the given basis of $\mathfrak{g}$ take the form

$$\begin{aligned} \tilde{X}_i &= \frac{\partial}{\partial x_i} - \frac{x_{n+i}}{2} \frac{\partial}{\partial z}, \\ \tilde{X}_{n+i} &= \frac{\partial}{\partial x_{n+i}} + \frac{x_i}{2} \frac{\partial}{\partial z} \quad \text{where} \quad i = 1, \ldots, n, \quad \text{and} \quad \tilde{Z} = \frac{\partial}{\partial z}. \end{aligned} \tag{3}$$

The corresponding dual forms are, respectively,

$$dx_i \quad \text{and} \quad \theta = dz - \frac{1}{2} \sum_i (x_i dx_{n+i} - x_{n+i} dx_i).$$

The Heisenberg group is a particular case of our next example, the more general class of Carnot groups known as H-type groups. Such groups were introduced by Kaplan in [20] and arise as the nilpotent part of the Iwasawa decomposition of semisimple Lie groups of real rank one; see Cowling–Dooley–Korányi–Ricci [10].

Example 2. An H-type group is a connected, simply connected 2-step Carnot group whose Lie algebra satisfies the following additional property: For each $Z \in \mathfrak{g}_2$ the homomorphism $J_Z : \mathfrak{g}_1 \to \mathfrak{g}_1$ defined by

$$\langle J_Z X, Y\rangle = \langle Z, [X,Y]\rangle, \quad \text{for all} \quad X, Y \in \mathfrak{g}_1,$$

satisfies

$$J_Z^2 = -\langle Z, Z\rangle I.$$

Let $\{X_1, \dots, X_m, Z_1, \dots, Z_n\}$ be an orthonormal basis of $\mathfrak{g}$ such that $\mathfrak{g}_1 = \text{Span}\,\{X_1, \dots, X_m\}$ and $\mathfrak{g}_2 = \text{Span}\,\{Z_1, \dots, Z_n\}$. Then $[X_i, X_j] = \sum_k C_{ij}^k Z_k$ and

$$\langle J_{Z_k} X_i, X_j\rangle = \langle Z_k, [X_i, X_j]\rangle = C_{ij}^k.$$

If $X = \sum_i x_i X_i + \sum_k z_k Z_k$ and $Y = \sum_j y_j X_j + \sum_l s_l Z_l$, then

$$X \star Y = \sum_j (x_j + y_j) X_j + \sum_k \Big(z_k + s_k + \frac{1}{2} \sum_{ij} x_i y_j C_{ij}^k \Big) Z_k.$$

In the given coordinates, X is identified with the point $(x_1, \dots, x_m, z_1, \dots, z_n) \in \mathbb{R}^{m+n}$ and the left invariant vector fields corresponding to the given basis of $\mathfrak{g}$ take the form

$$\tilde{X}_j = \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{ik} x_i C_{ij}^k \frac{\partial}{\partial z_k} \quad \text{and} \quad \tilde{Z}_k = \frac{\partial}{\partial z_k}.$$

The corresponding dual basis is given, respectively, by the forms

$$dx_j \quad \text{and} \quad \theta_k = dz_k - \frac{1}{2} \sum_{ij} x_i C_{ij}^k dx_j.$$

If G is an H-type group and $\pi_i : \mathfrak{g} \to \mathfrak{g}_i$, $i = 1, 2$, are the natural projections, then we define the *Folland–Kaplan* gauge on $\mathfrak{g}$ by the following formula:

$$|X| := a(X)^{\frac{1}{4}} := \big(\langle \pi_1(X), \pi_1(X)\rangle^2 + 16 \langle \pi_2(X), \pi_2(X)\rangle \big)^{\frac{1}{4}}. \tag{4}$$

In coordinates we have

$$|X| = \big((x_1^2 + \cdots + x_m^2)^2 + 16 (z_1^2 + \cdots + z_n^2) \big)^{\frac{1}{4}};$$

moreover, it is easy to check that $|-X| = |X|$ and $|\delta_t(X)| = t|X|$.

Definition 1. Let G be an H-type group and let $U \subset G$ be an open subset of G. For $1 < p < \infty$ we say that a function $u : U \to \mathbb{R}$ belongs to the *horizontal Sobolev space* $HW^{1,p}(U)$ if $u \in L^p(U)$, and for all $i = 1, \dots, m$ the horizontal derivatives $\tilde{X}_i u$ exist in the distributional sense and are represented by elements of $L^p(U)$. The space $HW^{1,p}(U)$ is a Banach space with respect to the norm

$$\|u\|_{HW^{1,p}(U)} = \|u\|_{L^p(U)} + \|(\tilde{X}_1 u, \dots, \tilde{X}_m u)\|_{L^p(U)}.$$

In a similar way, we define the local spaces $HW^{1,p}_{loc}(U)$. We define space $HW^{1,p}_0(U)$ as a closure of $C_0^\infty(U)$ in $HW^{1,p}(U)$.

We remark that the definition holds as well in all Carnot groups; see, e.g., Section 2 in [17].

The horizontal gradient $\nabla_0 u$ of $u \in HW^{1,p}_{loc}(U)$ is given by the equation

$$\nabla_0 u = \sum_{i=1}^m (\tilde{X}_i u) \tilde{X}_i,$$

and for $1 < p < \infty$ u is said to be *weakly p-harmonic* if

$$\int_U \langle |\nabla_0 u|^{p-2}\nabla_0 u, \nabla_0 \phi\rangle dx = 0 \tag{5}$$

for all $\phi \in C_0^\infty(U)$.

We say that a weak p-harmonic solution of (5) is *p-harmonic* if it is additionally continuous.

In an analogous way we define *p-subsolutions* by replacing $=$ with $\leq$ in (5), and *p-supersolutions* by replacing $=$ with $\geq$ in (5). In these cases we require the test functions $\phi \in C_0^\infty(U)$ to be nonnegative. If $u \in C^2(U)$, then (5) is equivalent to

$$\sum_{i=1}^m \tilde{X}_i(|\nabla_0 u|^{p-2}\tilde{X}_i u) = 0.$$

In particular, for $p = 2$ and $p = Q$, respectively, we have two important special cases of the p-harmonic equations:

$$\sum_{i=1}^m \tilde{X}_i\tilde{X}_i u = 0 \qquad \text{(harmonic functions)}, \tag{6}$$

$$\sum_{i=1}^m \tilde{X}_i(|\nabla_0 u|^{Q-2}\tilde{X}_i u) = 0 \qquad \text{(Q-harmonic functions)}. \tag{7}$$

The first example of a fundamental solution for a p-harmonic function seems to have been Folland's explicit solution to the 2-harmonic equation on the Heisenberg group in [12]. Later, Kaplan generalized Folland's result to obtain explicit fundamental solutions to the 2-harmonic equation on H-type groups; see [20]. Still within the class of H-type groups, an explicit fundamental solution to the Q-harmonic equation was found by Heinonen–Holopainen (see [17]), and finally solutions for all values of p were found by Capogna–Danielli–Garofalo; see [7]. We encapsulate these results in the following lemma; cf. Proposition 3.1 in Garofalo–Tyson [14] and Theorem 1.3 in Balogh–Holopainen–Tyson [3] for the case $p = Q$ in general Carnot groups.

Lemma 1. *Let G be an H-type group and $1 < p < \infty$. If $p \neq Q$, then there exists a constant $c(p, Q)$ such that $u(X) = c(p, Q)|X|^{\frac{p-Q}{p-1}}$ is a solution of the p-harmonic equation* (5) *in $G \setminus \{0\}$. If $p = Q$, then there exists a constant $c_1(Q)$ such that $u(X) = c_1(Q)\log|X|$ is a solution of the Q-harmonic equation* (7) *in $G \setminus \{0\}$.*

More generally, if G is any Carnot group, then for $p = Q$ there exists a homogeneous norm N on G such that $u(X) = -\frac{1}{\gamma}\log N(X)$ for $\gamma = \gamma(G) > 0$ is a solution to the Q-harmonic equation (7) *in $G \setminus \{0\}$.*

Remark 1. The shortfall in the general case is that N is not explicitly known. However, Proposition 4.16 in [17] shows that estimates are available.

One of the fundamental tools used in this paper is the comparison principle for p-harmonic equations. Its proof is the same as for the p-harmonic operators in $\mathbb{R}^n$, relying only on the monotonicity properties of $\mathcal{A}$-harmonic operators (here $\mathcal{A}(\xi) = |\xi|^{p-2}\xi$). Therefore, we will omit the proof; cf. Lemma 3.18 in Heinonen–Kilpeläinen–Martio [18], see also Vodopyanov [27].

Lemma 2. *Let $u \in HW_{loc}^{1,p}(U)$ be a p-supersolution and $v \in HW_{loc}^{1,p}(U)$ be a p-subsolution of* (5) *in U. If $\eta = \min\{u - v, 0\} \in HW_0^{1,p}$, then $u \geq v$ a.e. in U.*

Another tool needed in our discussion is the *strong maximum principle*. It follows immediately from the Harnack inequality for subelliptic $\mathcal{A}$-harmonic equations; see Theorem 4.1 in Heinonen–Holopainen [17].

Lemma 3. *Let G be a Carnot group. A nonconstant p-harmonic function in a domain $\Omega \subset G$ cannot attain its supremum and infimum inside Ω.*

3. The three-spheres theorems for subelliptic equations

In this section we prove the main results of the paper, Theorems 2 and 5. Since the fundamental solutions of p-harmonic equations have different natures and forms for $p = Q$ and $p \neq Q$, where Q denotes the homogeneous dimension of an underlying H-type group, we split the discussion into two cases. However, in both cases the proof is based on the same approach. Upon establishing three-spheres theorems, we provide a number of corollaries regarding the asymptotic behavior of subsolutions near 0 and for large radii. Furthermore, for $p = Q$ we show the Liouville-type results; see Corollaries 3 and 4. Due to the importance of the harmonic case ($p = 2$) we also state the main results of this section specifically for the harmonic case as a separate result; see Theorem 4.

For the sake of simplicity, in what follows we will state and formulate our results for gauge-norm spheres centered at the identity element of an H-type group G. The general results follow immediately by left invariance and the definition that the sphere centered at $Y \in G$ with radius r is defined as the left translate by Y of the sphere of radius r centered at the identity. In terms of the normal model this means

$$B_r(Y) = \{X \in \mathfrak{g} : |(-Y) \star X| < r\}.$$

Let G be an H-type group and let $U \subset G$. For a function $u : U \to \mathbb{R}$ we define

$$M(Y, r) = \sup\{u(X) \,:\, X \in U,\ |(-Y) \star X| = r\} \tag{8}$$

and for simplicity we write $M(0, r) = M(r)$.

3.1. The p-harmonic equations ($p \neq Q$).

Theorem 2. *Let G be an H-type group and let $U \subset G$ be a domain containing the identity element of G. Assume $p \neq Q$ and let $u \in HW^{1,p}_{loc}(U)$ be a p-subsolution. Moreover, let us consider three concentric gauge-norm spheres with radii $r_1 < r < r_2$ contained in U and centered at the identity. Then*

$$M(r) \leq M(r_1) \frac{r^{\frac{p-Q}{p-1}} - r_2^{\frac{p-Q}{p-1}}}{r_1^{\frac{p-Q}{p-1}} - r_2^{\frac{p-Q}{p-1}}} + M(r_2) \frac{r_1^{\frac{p-Q}{p-1}} - r^{\frac{p-Q}{p-1}}}{r_1^{\frac{p-Q}{p-1}} - r_2^{\frac{p-Q}{p-1}}}. \tag{9}$$

Equality holds if and only if $u(X) \equiv \phi(r)$, where $r = |X|$ and ϕ is a function on the right-hand side of (9).

Proof. Since $p \neq Q$, Lemma 1 shows that the function $c(p, Q)|X|^{\frac{p-Q}{p-1}}$ is a solution to the p-harmonic equation (5) for $X \neq 0$. So is $\phi(|X|) = a + bc(p, Q)|X|^{\frac{p-Q}{p-1}}$ with $a, b \in \mathbb{R}$. We choose a and b so that

$$\begin{cases} \phi(r_1) = M(r_1), \\ \phi(r_2) = M(r_2). \end{cases}$$

In particular, by direct computations, we have that

$$\phi(|X|) = M(r_1)\frac{|X|^{\frac{p-Q}{p-1}} - r_2^{\frac{p-Q}{p-1}}}{r_1^{\frac{p-Q}{p-1}} - r_2^{\frac{p-Q}{p-1}}} + M(r_2)\frac{r_1^{\frac{p-Q}{p-1}} - |X|^{\frac{p-Q}{p-1}}}{r_1^{\frac{p-Q}{p-1}} - r_2^{\frac{p-Q}{p-1}}}.$$

Since $u \le \phi(r_2)$ on the sphere $|X| = r_2$ and $u \le \phi(r_1)$ on the sphere $|X| = r_1$, we apply the comparison principle Lemma 2 and get that $u(X) \le \phi(|X|)$ for all X such that $r_1 < |X| < r_2$. Therefore, $M(r) \le \phi(|X|)$ in the annuli $r_1 < |X| < r_2$ and the proof is completed. ☐

The proof for spheres not centered at the identity proceeds identically except one has to adjust the fundamental solution with a left translation (see Proposition 8 in [14]). More precisely, the fundamental solution to the p-harmonic equation with singularity at $Y \in G$ is given by $v(X) := u((-Y) \star X)$ where u is the fundamental solution to the p-harmonic equation in $G \setminus \{0\}$ with singularity at 0 as in Lemma 1.

Theorem 3. *Let G be an H-type group and let $U \subset G$ be a domain. Assume $p \ne Q$ and let $u \in HW^{1,p}_{loc}(U)$ be a p-subsolution. Moreover, let us consider three concentric gauge-norm spheres with radii $r_1 < r < r_2$ contained in U and centered at Y. Then*

$$M(Y,r) \le M(Y,r_1)\frac{r^{\frac{p-Q}{p-1}} - r_2^{\frac{p-Q}{p-1}}}{r_1^{\frac{p-Q}{p-1}} - r_2^{\frac{p-Q}{p-1}}} + M(Y,r_2)\frac{r_1^{\frac{p-Q}{p-1}} - r^{\frac{p-Q}{p-1}}}{r_1^{\frac{p-Q}{p-1}} - r_2^{\frac{p-Q}{p-1}}}. \tag{10}$$

Equality holds if and only if $u(X) \equiv \phi(r)$, where $r = |(-Y)\star X|$ and ϕ is a function on the right-hand side of (10).

As a corollary to Theorem 2 we obtain the following growth estimate of subsolutions near 0 and for large r; cf. Remarks (ii) and (iii) following Theorem 30 in Protter–Weinberger [25].

Corollary 1. *Under the assumptions of Theorem* 2*, let us suppose additionally that*

$$\liminf_{r_1 \to 0^+} r_1^{\frac{Q-p}{p-1}} M(r_1) \le 0. \tag{11}$$

Then $M(r)$ is a nondecreasing function of r and hence u is bounded above in some neighborhood of 0.

The corollary says that, if u is a subsolution unbounded in the neighborhood of 0, then there exists a sequence (X_n) such that $X_n \to 0$ and $u(X_n)/|X_n|^{\frac{p-Q}{p-1}} \to \infty$.

Proof. We follow the discussion in [25, Section 12] and let $r_1 \to 0^+$. In order to simplify the notation, let us define an auxiliary exponent $\alpha = \frac{p-Q}{p-1}$. The discussion splits into two cases.

Case 1 ($1 < p < Q$). Under our assumptions $\alpha < 0$ and $r_1 < r < r_2$. With this notation we have

$$\lim_{r_1 \to 0^+} \frac{r_1^\alpha - r^\alpha}{r_1^\alpha - r_2^\alpha} = 1.$$

Moreover,

$$\frac{r^\alpha - r_2^\alpha}{r_1^\alpha - r_2^\alpha} = r_1^{-\alpha}\frac{r^\alpha - r_2^\alpha}{1 - \left(\frac{r_1}{r_2}\right)^{-\alpha}} \ge r_1^{-\alpha}(r^\alpha - r_2^\alpha) \ge 0.$$

Hence for $r_1 \to 0^+$ we have by (9) and by assumption (11) that

$$\begin{aligned}(12)\quad M(r) &\leq M(r_1)\frac{r^\alpha - r_2^\alpha}{r_1^\alpha - r_2^\alpha} + M(r_2)\frac{r_1^\alpha - r^\alpha}{r_1^\alpha - r_2^\alpha} \\ &= M(r_1)r_1^{-\alpha}\frac{r^\alpha - r_2^\alpha}{1-\left(\frac{r_1}{r_2}\right)^{-\alpha}} + M(r_2)\frac{r_1^\alpha - r^\alpha}{r_1^\alpha - r_2^\alpha} \leq M(r_2) \qquad \text{for} \quad r < r_2.\end{aligned}$$

This implies that u is bounded above in the vicinity of 0 and the proof of the corollary is completed.

Case 2 ($p > Q$). In this case α as defined in the beginning of the proof is positive. It holds that

$$\lim_{r_1\to 0^+}\frac{r^\alpha - r_2^\alpha}{r_1^\alpha - r_2^\alpha} = 1 - \left(\frac{r}{r_2}\right)^\alpha \qquad \text{and} \qquad \lim_{r_1\to 0^+}\frac{r_1^\alpha - r^\alpha}{r_1^\alpha - r_2^\alpha} = \left(\frac{r}{r_2}\right)^\alpha < 1.$$

Similarly to estimate (12) we now arrive at the following inequality for $r_1 \to 0^+$:

$$M(r) \leq M(r_1)r_1^{-\alpha}\frac{r^\alpha - r_2^\alpha}{1-\left(\frac{r_2}{r_1}\right)^{\alpha}} + M(r_2)\left(\frac{r}{r_2}\right)^\alpha \leq M(r_2) \qquad \text{for} \quad r < r_2.$$

From this, the assertion follows and the proof of Corollary 1 is completed. ☐

A similar approach gives us the next observation.

Corollary 2. *Under the assumptions of Theorem 2, let $1 < p < Q$ and suppose additionally that*

$$(13)\qquad \limsup_{r_2\to\infty} M(r_2) \leq 0.$$

Then $r^{\frac{Q-p}{p-1}}M(r)$ is a nondecreasing function of r and hence $r^{\frac{Q-p}{p-1}}u$ is bounded above.

Proof. As in the proof of Corollary 1, we employ inequality (9) and get

$$r^{\frac{Q-p}{p-1}}M(r) \leq r_1^{\frac{Q-p}{p-1}}M(r_1)\frac{1-\left(\frac{r}{r_2}\right)^{\frac{Q-p}{p-1}}}{1-\left(\frac{r_1}{r_2}\right)^{\frac{Q-p}{p-1}}} + (\limsup_{r_2\to\infty} M(r_2))\frac{\left(\frac{r}{r_1}\right)^{\frac{Q-p}{p-1}}-1}{1-\left(\frac{r_1}{r_2}\right)^{\frac{Q-p}{p-1}}}.$$

Thus for $r_2 \to \infty$ we obtain that under assumption (13), the second term in the above sum becomes negative. Hence $r^{\frac{Q-p}{p-1}}M(r) \leq r_1^{\frac{Q-p}{p-1}}M(r_1)$ for $r_1 < r$, and the assertion follows immediately. ☐

Since the case of the subelliptic Laplace operator is of special interest in the theory of PDEs, we gathered together the results of Theorem 2 and Corollaries 1 and 2 as a separate result for $p = 2$.

Theorem 4. *Let G be an H-type group and let $U \subset G$ be a domain containing the identity element of G. Assume that $u \in HW^{1,2}_{loc}(U)$ is a subsolution of the harmonic equation (6). Moreover, let us consider three concentric gauge-norm spheres with radii $r_1 < r < r_2$ contained in U. Then*

$$(14)\qquad M(r) \leq M(r_1)\frac{r^{2-Q} - r_2^{2-Q}}{r_1^{2-Q} - r_2^{2-Q}} + M(r_2)\frac{r_1^{2-Q} - r^{2-Q}}{r_1^{2-Q} - r_2^{2-Q}}.$$

Equality holds if and only if $u(X) \equiv \phi(r)$*, where* $r = |X|$ *and* ϕ *is a function on the right-hand side of* (14)*. Moreover, suppose additionally that*

$$\liminf_{r_1 \to 0^+} r_1^{Q-2} M(r_1) \leq 0. \tag{15}$$

Then $M(r)$ *is a nondecreasing function of* r *and hence* u *is bounded above in some neighborhood of* 0*. If, instead of* (15)*, we additionally assume that*

$$\limsup_{r_2 \to \infty} M(r_2) \leq 0,$$

then $r^{Q-2}M(r)$ *is a nondecreasing function of* r *and hence* $r^{Q-2}u$ *is bounded above.*

Proof. The proof follows immediately from the proofs of Theorem 2 and Corollaries 1 and 2 by taking $p = 2$ and noticing that the homogeneous dimension $Q > 4 \neq p$. □

3.2. **The Q-harmonic equations.** The proof of the three-spheres theorem for Q-harmonic equations proceeds in exactly the same way as the case $p \neq Q$. The only difference is in the form of the fundamental solution given in Lemma 1.

Theorem 5. *Let* G *be an H-type group and let* $U \subset G$ *be a domain containing the identity element of* G*. Let* $u \in HW^{1,Q}_{loc}(U)$ *be a* Q*-subsolution. Moreover, let us consider three concentric gauge-norm spheres with radii* $r_1 < r < r_2$ *contained in* U*. Then*

$$M(r) \leq M(r_1)\frac{\log \frac{r_2}{r}}{\log \frac{r_2}{r_1}} + M(r_2)\frac{\log \frac{r}{r_1}}{\log \frac{r_2}{r_1}}. \tag{16}$$

Equality holds if and only if $u(X) \equiv \phi(r)$*, where* $r = |X|$ *and* ϕ *is a function on the right-hand side of* (16)*.*

Proof. The proof is similar to the one for Theorem 2. By Lemma 1 and the homogeneity of the Q-harmonic equation (7), we have that a function $\phi(|X|) = a + bc_1(Q)\log|X|$ with $a, b \in \mathbb{R}$ is a solution to the Q-harmonic equation for $X \neq 0$. Choose a and b such that

$$\begin{cases} \phi(r_1) = M(r_1), \\ \phi(r_2) = M(r_2). \end{cases}$$

By solving this linear system of equations for a and b, we find that

$$\phi(|X|) = M(r_1)\frac{\log|X| - \log r_2}{\log r_1 - \log r_2} + M(r_2)\frac{\log r_1 - \log|X|}{\log r_1 - \log r_2}.$$

The comparison principle in Lemma 2 together with the reasoning similar to the proof of Theorem 2 gives us that $M(r) \leq \phi(|X|)$ in the annuli $r_1 < |X| < r_2$. Thus the proof is completed. □

Remark 2. We note that Theorem 5 also holds for a general Carnot group G. Indeed, by the second part of Lemma 1 we have that $\phi(N(X)) = c(G)\log N(X)$ is a solution to the Q-harmonic equation for $X \neq 0$ and a homogeneous norm N in G. Moreover, Lemmas 2 and 3 also hold in the general setting; see the discussion following Lemma 1. Hence we can reason in the same way as above and obtain the three-spheres theorem for any Carnot group, but without an explicit expression for N. As explained in Remark 1, meaningful estimates can still be obtained from such a theorem if we employ Proposition 4.16 in [17].

In order to generalize Theorem 5 to spheres not centered at the identity, one makes the same modifications used to obtain Theorem 3.

As a corollary we obtain the following Liouville-type result.

Corollary 3. *Let u be a solution of the Q-harmonic equation* (7) *in the whole H-type group G except possibly at the origin. If u is bounded above, then u is constant.*

Proof. Suppose that $u \leq M$. We use (16) and let $r_2 \to \infty$. Then $\lim_{r_2\to\infty} \frac{\log \frac{r}{r_1}}{\log \frac{r_2}{r_1}} = 0$ and since $M(r_2) \leq M$, we get that the second term in (16) goes to 0. Also, $\lim_{r_2\to\infty} \frac{\log \frac{r_2}{r}}{\log \frac{r_2}{r_1}} = 1$. Thus $M(r) \leq M(r_1)$ for $r \geq r_1$.

Similarly, we let $r_1 \to 0^+$ and obtain that $M(r) \leq M(r_2)$ for $r \leq r_2$. Since r_1 and r_2 are chosen arbitrarily, we conclude that $M(r)$ is constant for all r. By the strong maximum principle Lemma 3 we obtain that u must be constant as well. □

The next result refines Corollary 3 and generalizes an observation known in the setting of subharmonic functions in $\mathbb{R}^2$; see Theorem 2.2 in Ramankutty [26] and Remark 1 on p. 130 in Protter–Weinberger [25]. Let us note that, while the result of [26] fails to hold beyond the plane, our result is true for all H-type groups. This is due to a remarkable fact that $c_1(Q)\log|X|$ remains to be a fundamental solution to the Q-harmonic equation in H-type groups for all homogeneous dimensions Q; cf. Lemma 1.

Corollary 4. *Let u be a nonconstant solution of the Q-harmonic equation* (7) *in the whole H-type group $G \setminus \{0\}$. Then, either*

$$\liminf_{r\to 0} \frac{M(r)}{|\log r|} > 0 \qquad \text{or} \qquad \liminf_{r\to\infty} \frac{M(r)}{|\log r|} > 0. \tag{17}$$

Proof. On the contrary, let us assume that both limit infima in (17) are nonpositive for $r \to 0$ and $r \to \infty$. As in Theorem 5, consider radii $0 < r_1 < r < r_2$. Then for $r_2 \to \infty$, we have that

$$\lim_{r_2\to\infty} \frac{\log r_2 - \log r}{\log r_2 - \log r_1} = 1 \qquad \text{and} \qquad \lim_{r_2\to\infty} \frac{\log \frac{r}{r_1}}{1 - \frac{\log r_1}{\log r_2}} = \log \frac{r}{r_1} > 0.$$

Thus, inequality (16) and our assumption on limit infima in (17) result in the following estimate for $r_2 \to \infty$:

$$M(r) \leq M(r_1)\frac{\log r_2 - \log r}{\log r_2 - \log r_1} + \frac{M(r_2)}{\log r_2}\,\frac{(\log \frac{r}{r_1})\log r_2}{\log r_2 - \log r_1} \leq M(r_1).$$

Now let $r_1 \to 0^+$. By reasoning similar to the above, we obtain that

$$M(r) \leq \frac{M(r_1)}{-\log r_1}\,\frac{(\log \frac{r_2}{r})(-\log r_1)}{\log r_2 - \log r_1} + M(r_2)\frac{\log \frac{r}{r_1}}{\log \frac{r_2}{r_1}} \leq M(r_2).$$

As in Corollary 3, we observe that, since r_1 and r_2 are chosen arbitrarily, then $M(r)$ is constant for all r. The strong maximum principle in Lemma 3 implies that u must be constant as well, leading us to the contradiction. □

By Remark 2 we know that Theorem 5 holds for any Carnot groups G and, therefore, Corollaries 3 and 4 hold in this general setting as well. Moreover, by Remark 1 explicit estimates can be formulated.

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