PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 11, November 2016, Pages 4709–4714
http://dx.doi.org/10.1090/proc/12476
Article electronically published on July 21, 2016

NON-AUTONOMOUS BASINS WITH UNIFORM BOUNDS ARE ELLIPTIC

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(Communicated by Franc Forstneric)

ABSTRACT. We prove that a non-autonomous basin with bounds is an Oka manifold. A consequence is that it has an abundance of holomorphic maps from $\mathbb{C}^m$ into it, and in particular it does not carry a non-constant bounded plurisubharmonic function.

1. Introduction

For $j = 1, 2, 3, \dots$ we let $f_j \in \mathrm{Aut}_{hol}\,\mathbb{C}^n$ be a sequence which we denote by f, where each map satisfies $f_j(0) = 0$. We will denote by Ω_f the non-autonomous basin of attraction to the origin, *i.e.*, the set

$$\Omega_f := \{z \in \mathbb{C}^n : \lim_{j\to\infty} f_j \circ \cdots \circ f_1(z) = 0\}.$$

If there exist $0 < a < b < 1$ such that

$$a\|z\| \leq \|f_j(z)\| \leq b\|z\| \tag{1.1}$$

for all $z \in \mathbb{B}^n$ and for all $j \in \mathbb{N}$, we will say that Ω_f is a non-autonomous basin with uniform bounds. The main purpose of this note is to prove the following:

Theorem 1.1. *A non-autonomous basin with uniform bounds is elliptic, hence an Oka-manifold.*

We recall the definitions of elliptic (in the sense of Gromov [6]) and Oka-manifolds (see *e.g.* [3], Ch. 5) A complex manifold X is *elliptic* if it admits a dominating holomorphic spray, *i.e.*, a holomorphic vector bundle $E \to X$ with a holomorphic map $s : E \to X$ with the properties that $s(0_x) = x$ for all $x \in X$ and such that $ds_{0_x} : T_{0_x}E \to T_xX$ maps the vertical subspace E_x of $T_{0_x}E$ surjectively onto T_xX. In our case $E \to \Omega_f$ will be the trivial bundle $\Omega_f \times \mathbb{C}^n$. Recall further that X is an *Oka manifold* if any holomorphic map $f : K \to X$ from a compact convex set $K \subset \mathbb{C}^n$ can be approximated uniformly on K by entire maps from $\mathbb{C}^n$ to X.

The motivation for proving this theorem is the very natural problem of trying to construct "many" holomorphic maps into Ω_f from Stein manifolds. In fact it is a long-standing conjecture, the so-called Bedford conjecture, that Ω_f is biholomorphic to $\mathbb{C}^n$, which is the "best" target manifold for maps from Stein manifolds that one can think of (see the recent survey [1] on the Bedford conjecture). A consequence of Theorem 1.1 is the following (see the monograph [3], Corollary 5.4.7).

Received by the editors January 15, 2014 and, in revised form, January 30, 2014.
2010 *Mathematics Subject Classification.* Primary 32E30, 32H50.
The second author is supported by the NRC-grant 209751.

Theorem 1.2. *Let Y be a Stein manifold. Then if $K \subset Y$ is any holomorphically convex compact set, and if $f : K \to \Omega_f$ is a holomorphic map, then f is approximable uniformly on K by entire maps from Y to Ω_f.*

An immediate corollary is the following.

Corollary 1.3. *Ω_f does not carry a non-constant bounded plurisubharmonic function.*

For a direct proof of this result see [1], Lemma 25. We can further use Theorem 1.2 to prove the following.

Theorem 1.4. *For any Stein manifold Y there exists a holomorphic map $f : Y \to \Omega_f$ whose image is dense. If $\dim(Y) \geq n$ we may achieve that $f(Y) = \Omega_f$, and if $2\dim(Y) + 1 \leq n$ we may achieve that f is an embedding.*

That Theorem 1.4 is a consequence of Theorem 1.2 is presumably well known to those familiar with Oka theory: density of the image follows immediately from jet-interpolation for maps from Stein manifolds to elliptic manifolds (stated by Gromov [6] and proved by Forstnerič-Prezelj [4]), and the embedding part follows from jet-transversality. We give a direct simple proof in Section 3.

The subtlety regarding these things is perhaps illustrated by the following example due to the first author [2]: if we allow the lower bound a to be equal to zero, there exists a sequence $f_j \in \mathrm{Aut}_{hol}\,\mathbb{C}^n$ such that the inequalities (1.1) are satisfied, and such that

(i) Ω_f is an increasing sequence of balls,
(ii) the Kobayashi metric vanishes identically on Ω_f, and
(iii) Ω_f carries a non-constant bounded plurisubharmonic function ρ.

It follows immediately that none of the above theorems can hold for such a basin, since the holomorphic image of any $\mathbb{C}^m$ must be contained in a level set of ρ.

Our main tool for proving Theorem 1.1 is a method by the first author of constructing maps into Ω_f from [1].

Lemma 1.5. *Fix constants $R > 1, c < 1, r < 1$. Then there exist $L > 0, \delta > 0$ so that for every $\epsilon > 0$ and all large enough N we have uniformly for any analytic function $g(z) = \sum a_n z^n : \Delta \to \Delta$ that*

$$|\sum_{n\leq LN} a_n z^n| \leq R^N \qquad \textit{if } |z| < 1+\delta, \quad \textit{and} \tag{1.2}$$

$$|\sum_{n>LN} a_n z^n| \leq c^N \epsilon \qquad \textit{if } |z| < r. \tag{1.3}$$

We include the proof for the benefit of the reader.

Proof. From Cauchy estimates we know that $|a_n| \leq 1$ for all n. Hence the estimates (1.2) and (1.3) follow if

$$\frac{(1+\delta)^{LN+1} - 1}{\delta} < R^N, \text{ and} \tag{1.4}$$

$$\frac{r^{LN}}{1-r} < c^N \epsilon. \tag{1.5}$$

Next choose first L so that $r^L < c$ and then choose δ so that $(1+\delta)^L < R$. Then both (1.4) and (1.5) hold for all large enough N independent of g. ☐

2. Proof of Theorem 1.1

We start by giving a multivariable version of Lemma 1.5. Let $f \in \mathcal{O}(t\mathbb{B}^n)$ and expand f as $f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$. For any $N \in \mathbb{N}$ we set $f_N := \sum_{|\alpha| \le N} a_\alpha z^\alpha$, and $\widehat{f}_N := f - f_N$.

Lemma 2.1. *Let $R > 0$ and $0 < c, r < 1$. There exist $\delta > 0$ and $L > 0$ such that the following hold: for any $\epsilon > 0$, any large enough N, any $t_1, t_2 > 0$, and any holomorphic function $f : t_1\mathbb{B}^n \to t_2\mathbb{D}, f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$, we have that*

(1) $|f_{LN}(z)| < t_2 \cdot R^N$ *for all* $\|z\| \le t_1(1 + \delta)$, *and*
(2) $|\widehat{f}_{LN}(z)| < t_2 \cdot c^N \cdot \epsilon$ *for all* $\|z\| \le t_1(1 - r)$.

Proof. By scaling it is enough to prove it for $t_1 = t_2 = 1$ and by slicing it is enough to prove it for $n = 1$. This is Lemma 1.5 above. □

To prove Theorem 1.1 we do the following: Let $E \to \Omega$ be the trivial bundle $E = \Omega \times \mathbb{C}^n$, with coordinates (z, w). Set $s_0(z, w) := z + w$. Let $K_0 \subset K_1 \subset K_2 \subset \ldots$ be the normal exhaustion of Ω defined by $K_j := (f^j)^{-1}(\overline{\mathbb{B}^n})$, where $f^j := f_j \circ \cdots \circ f_1$, and $f^0 = \mathrm{id}$.

We inductively construct sprays $s_k : K_k \times k \cdot \mathbb{B}^N \to \Omega$, each spray approximating the previous one well enough to get convergent limits.

The following is the main lemma in the inductive construction.

Lemma 2.2. *There exist constants $\delta, r > 0$ such that the following hold: for any strictly positive continuous function $t \in \mathcal{C}(K_{k+1})$, any holomorphic map $s_k : K_{k+1} \times \mathbb{C}^n \to \mathbb{C}^n$ with $s_k(z, 0) = z$ for $z \in K_{k+1}$ and $s_k(\{z\} \times t(z) \cdot \overline{\mathbb{B}^n}) \subset \Omega$ for $z \in K_{j+1}$, and any $\epsilon > 0$ (small), there exists a holomorphic map $s_{k+1} : K_{k+1} \times \mathbb{C}^n \to \mathbb{C}^n$ such that the following hold:*

(1) $s_{k+1}(z, 0) = z$ *for all* $z \in K_{k+1}$,
(2) $s_{k+1}(\{z\} \times t(z)(1 + \delta) \cdot \overline{\mathbb{B}^n}) \subset \Omega$ *for* $z \in K_{k+1}$, *and*
(3) $\|s_k(z, \cdot) - s_{k+1}(z, \cdot)\|_{\{z\} \times t(z)(1-r) \cdot \overline{\mathbb{B}^n}} < \epsilon$ *for* $z \in K_{k+1}$.

Proof. Let a and b be the constants from (1.1). By possibly having to decrease a we may assume that

$$\|f_j^{-1}(z) - f_j^{-1}(w)\| \le 1/a\|z - w\| \text{ for all } z, w \in (a/2) \cdot \mathbb{B}^n. \tag{2.1}$$

Furthermore, by scaling, we may assume that $2\triangle^n \subset \Omega_f$ (here $\triangle^n$ denotes the unit polydisk). We set $c = a$ and $R = 1/b$, choose $0 < r < 1$, and let L, δ be as in Lemma 2.1. Let $t_0 = \inf\{t(z) : z \in K_{k+1}\}$ and assume that $t_0 < 1$. By possibly having to compose with some iterate f^M we may assume that

$$s_k(\{z\} \times t(z) \cdot \overline{\mathbb{B}^n}) \subset \frac{t_0 a}{4} \cdot \mathbb{B}^n, \tag{2.2}$$

for all $z \in K_{k+1}$. This can be compensated for at the end by composing with f^{-M}. For large enough N we define a sequence of maps $s_{k+1,N}$ as follows: first we set $s^1_{k+1,N}(z, w) := f^N \circ s_k(z, w)$. We may expand this map as a power series in w with coefficients in $\mathcal{O}(K_{k+1})$

$$s^1_{k+1,N}(z, w) = \sum_\alpha a_\alpha(z) w^\alpha. \tag{2.3}$$

Next define a map $s^2_{k+1,N}$ by

$$s^2_{k+1,N}(z,w) = \sum_{|\alpha|\leq LN} a_\alpha(z)w^\alpha. \tag{2.4}$$

Note that since $s^1_{k+1,N}(z,w) \in b^N \cdot t_0 \cdot \overline{\mathbb{B}^n}$ for all $|w| \leq t(z)$ we have by Lemma 2.1 that $s^2_{k+1}(z,w) \in \triangle^n$ for all $|w| \leq (1+\delta)t(z)$. Further we note that

$$\|s^1_{k+1,N}(z,w) - s^2_{k+1,N}(z,w)\| < a^N \cdot \epsilon, \tag{2.5}$$

for all $\|w\| \leq (1-r)\cdot t(z)$ for N sufficiently large. Finally we define

$$s_{k+1}(z,w) := f^{-N}(s^2_{k+1,N}(z,w)) \tag{2.6}$$

for large N. By (2.2) and (2.5) and repeated use of (2.1) we get that $\|s_{k+1}(z,w) - s_k(z,w)\| < \epsilon$ for all $z \in K_{j+1}$ and $|w| \leq (1-r)t(z)\mathbb{B}^n$. □

We now explain the inductive construction of a dominating holomorphic spray. We keep δ and r as in the previous lemma fixed.

The induction hypothesis is the following: we have constructed a holomorphic spray $s_k : K_k \times k\cdot\overline{\mathbb{B}^n} \to \Omega$ which is dominating near the zero section. We will show how to construct a spray $s_{k+1} : K_{k+1} \times (k+1)\cdot\overline{\mathbb{B}^n} \to \Omega$ that approximates s_k as well as we want on $K_k \times (1-r)k\cdot\overline{\mathbb{B}^n}$, and is dominating along the zero section. Since r is fixed it is clear that this will allow us to construct a sequence of sprays s_k that converges to a dominating spray $s : \Omega \times \mathbb{C}^N \to \Omega$.

Write s_k as

$$s_k(z,w) = z + \sum_{\alpha\in\mathbb{N}^n\setminus\{0\}} p_\alpha(z)\cdot w^\alpha \tag{2.7}$$

where each p_α is a (vector valued) holomorphic function on K_k.

Now the degree one part of the sum above is a map $A : K_k \to Gl_n(\mathbb{C}^n_w)$, and since K_k is diffeomorphic to a ball, the map A is null-homotopic. Since $Gl_n(\mathbb{C}^n)$ is an Oka-manifold (see Grauert [5] or Forstnerič, Proposition 5.5.1, [3]), we may assume that A is holomorphic on K_{k+1}. The rest of the sum is approximable by polynomials, and so we may assume that s_k is defined on $K_{k+1}\times\mathbb{C}^n$ and is dominating near the zero section.

Choose $t(z) > 0$ such that $s_k(z,w) \in \Omega$ for all $z \in K_{k+1}$ and $\|w\| \leq t(z)$. If we choose m such that $t(z)(1+\delta)^m \geq k+1$, it is clear that repeated use of Lemma 2.2 will give us a spray $s_{k+1} : K_{k+1}\times(k+1)\mathbb{B}^n \to \Omega$, and if each approximation is good enough we keep the domination condition.

3. Proof of Theorem 1.4

Since by Theorem 1.1 we have that Ω_f is an Oka-manifold it is known that the approximation result, Theorem 1.2, holds with exact interpolation on a finite set of points [3] (in this setting it is not hard to prove that it follows directly from Theorem 1.2; we leave this to the reader).

To construct a holomorphic map $f : Y \to \Omega_f$ with dense image we choose a dense set of points $E = \{b_j\}_{j\in\mathbb{N}}$ in Ω_f, and we choose a normal exhaustion $K_j \subset K^\circ_{j+1}$ of Y by holomorphically convex compact sets. We may assume that K_1 is a single point a_1, we let $g_1 : K_1 \to \Omega_f$ be the constant map $g_1(a_1) = b_1$, and we set $\epsilon_1 = \frac{1}{2}\text{dist}(b_1, b\Omega_f)$. We construct by induction a sequence of maps $g_j : K_j \to \Omega_{f_j}$, a sequence of points $a_j \in K_j \setminus K_{j-1}$ (we set $K_0 = \emptyset$), and a decreasing sequence ϵ_j of positive numbers such that the following hold for $j \geq 2$:

(1_j) $\|g_j - g_{j-1}\|_{K_{j-1}} < \epsilon_{j-1}2^{-j}$,
(2_j) $g_j(a_i) = b_i$ for $i = 1, 2, \ldots, j$, and
(3_j) $\epsilon_j < \min\{\epsilon_{j-1}, \text{dist}(g_j(K_j), b\Omega_f)\}$.

For the induction step we assume that we have constructed maps $g_1, \ldots, g_m$ for $m \geq 1$ satisfying (1_m)-(3_m) for $m \geq 2$ (unless $m = 1$). Choose any point $a_{m+1} \in K_{m+1} \setminus K_m$. By the approximation theorem with interpolation there exists a map $g_{m+1} : K_{m+1} \to \Omega_f$ such that (1_{m+1}) and (2_{m+1}) are satisfied. To obtain (3_{m+1}) we simply set $\epsilon_{m+1} := \frac{1}{2}\min\{\epsilon_m, \text{dist}(g_m(K_m), b\Omega_f)\}$.

Now set $f := \lim_{j\to\infty} g_j$. By (1_j) the sequence converges uniformly to a holomorphic map $f : Y \to \mathbb{C}^n$. By (1_j) and (3_j) we have that $f(Y) \subset \Omega_f$, and by (2_j) we have that $f(a_j) = b_j$ for all $j \in \mathbb{N}$, which implies that $f(Y)$ is dense.

Next we indicate the modification needed to get that f is surjective if $k = \dim(Y) \geq n$. In this case we also choose, before we start the induction process, a normal exhaustion $L_j \subset L_{j+1}^\circ$ of Ω_f by compact sets. Then we replace the construction of a sequence a_j of points, by the construction of a sequence of balls $B_{\delta_j}(a_j) \subset K_j \setminus K_{j-1}$, and then we replace (2_j) and (3_j) by

(2_j) $L_j \subset g_j(B_{\delta_j}(a_j))$, and
(3_j) $\epsilon_j < \min\{\epsilon_{j-1}, \text{dist}(g_j(K_j), b\Omega_f), \text{dist}(g_j(bB_\delta(a_j)), L_j)\}$.

Since Ω_f is an increasing sequence of balls, it is easy to achieve this inductive construction using Theorem 1.2.

Finally, to get an embedding provided the co-dimension is big enough we do the following. Choose compact sets $K'_j \subset K_j$ such that $\{K'_j\}$ exhausts Y. By transversality it is then possible to achieve in the inductive construction that $g_j : K_j \to \Omega_f$ is an embedding. Furthermore, if the limit map f is sufficiently close to g_j on K_j, then $g : K'_j \to \Omega_f$ is still an embedding. So by possibly having to let the sequence ϵ_j decrease even faster to zero, we may assume that $f : K'_j \to \Omega_f$ is an embedding for all j.

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