PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 11, November 2016, Pages 4743–4752
http://dx.doi.org/10.1090/proc/13032
Article electronically published on July 21, 2016

ON THE ABSOLUTELY CONTINUOUS COMPONENT OF A WEAK LIMIT OF MEASURES ON $\mathbb{R}$ SUPPORTED ON DISCRETE SETS

ALEXANDER Y. GORDON

(Communicated by Michael Hitrik)

ABSTRACT. Let $\mu_1, \mu_2, \ldots$ be a sequence of positive Borel measures on $\mathbb{R}$ each of which is supported on a set having no finite limit points. Suppose the sequence μ_n weakly converges to a Borel measure ν. Let ν_{ac} be the absolutely continuous component of ν, and $X \subset \mathbb{R}$ the essential support of ν_{ac}. We characterize the set X in terms of the limiting behavior of the Hilbert transforms of the measures μ_n. Potential applications include those in spectral theory.

1. Statements

Let $\mu_1, \mu_2, \ldots$ be a sequence of positive Borel measures on $\mathbb{R}$ such that for each $n \in \mathbb{N}$ the measure μ_n is supported on a set D_n having no finite limit points. Suppose μ_n weakly converges to a Borel measure ν (notation: $\mu_n \overset{w}{\to} \nu$) in the following sense:

$$\int_{\mathbb{R}} f(x)\mu_n(dx) \to \int_{\mathbb{R}} f(x)\nu(dx) \tag{1}$$

for any continuous compactly supported function $f\colon \mathbb{R} \to \mathbb{R}$ (or equivalently, $\mu_n(I) \to \nu(I)$ for any bounded interval I whose endpoints have zero ν-measure).

The Lebesgue decomposition of the measure ν represents it as a sum $\nu = \nu_{\mathrm{ac}} + \nu_s$, where ν_{ac} is absolutely continuous with respect to the Lebesgue measure λ while ν_s and λ are mutually singular. By the Radon-Nikodým theorem, there is a Borel measurable function $\rho(\cdot) \geq 0$ on $\mathbb{R}$ (the density of ν_{ac}) such that $\nu_{\mathrm{ac}}(A) = \int_A \rho(x)dx$ for any Borel set $A \subset \mathbb{R}$.

The essential support of the measure ν_{ac} is defined as

$$X = \{x \in \mathbb{R}\colon \rho(x) > 0\}.$$

The set X is determined up to a zero Lebesgue measure set, as is the function $\rho(\cdot)$.

The goal of the present paper is to characterize the set X in terms of the limiting behavior of the sequence of functions

$$f_1(x), f_2(x), \ldots, \tag{2}$$

Received by the editors October 28, 2015.
2010 *Mathematics Subject Classification.* Primary 28A33, 44A15.
Key words and phrases. Weak convergence of measures, Hilbert transform of a measure, absolutely continuous component of a measure, essential support of an absolutely continuous measure.

where $f_n(x)$ is the Hilbert transform of the measure μ_n:

$$f_n(x) = \int_{\mathbb{R}} \frac{\mu_n(dt)}{x-t} \equiv \sum_i \frac{\alpha_i^{(n)}}{x - x_i^{(n)}}; \tag{3}$$

here $x_i^{(n)}$ is the ith atom of μ_n and $\alpha_i^{(n)}$ is its weight. Under the condition

$$\int_0^\infty \frac{\mu_n([-t,t])}{1+t^2}\, dt < \infty, \tag{4}$$

which we will assume to hold for all n, the function (3) is well defined on $\mathbb{R} \setminus D_n$.

Remark. In this paper we deal with functions on $\mathbb{R}$ that are undefined on certain at most countable sets. In order to simplify formulations, we introduce the following convention: the statement $f(x) > A$ will mean that the function f is defined at x and $f(x) > A$; similarly, the statement $\sup_n g_n(x) = B$ will be understood as "all functions g_n are defined at x and $\sup_n g_n(x) = B$". By writing "$h_n \to C$ as $n \to \infty$" we will mean that all functions h_n are defined at x and $\lim_{n\to\infty} h_n(x) = C$, etc.

Our first result is the following statement, which was conjectured by S. Molchanov [3].

Theorem 1. *For Lebesgue almost every (a.e.) $x \in X$*

$$\sup_{n\in\mathbb{N}} |f_n(x)| = \infty.$$

For our second statement the condition (4) should be strengthened—made uniform in n: now we require that

$$\mu_n([-t,t]) \le \psi(t) \quad \text{for all } n \in \mathbb{N} \quad \text{and all} \quad t \ge 0, \tag{5}$$

where $\psi\colon [0,\infty) \to [0,\infty)$ is a nondecreasing function such that

$$\int_0^\infty \frac{\psi(t)}{1+t^2}\, dt < \infty. \tag{6}$$

Theorem 2. *Under conditions* (5) *and* (6)*, there exists a sequence $n_k \nearrow \infty$ in $\mathbb{N}$ such that for a.e. $x \in \mathbb{R} \setminus X$ the sequence $\{f_{n_k}(x)\}_{k=1}^\infty$ has a finite limit.*

Therefore, under conditions (5) and (6), X is the smallest Borel set (up to a zero Lebesgue measure set) outside of which some subsequence of the sequence $f_n(x)$ converges almost everywhere. Also, X is the largest Borel set (also up to a zero Lebesgue measure set) on which any subsequence of the sequence $f_n(x)$ is unbounded almost everywhere.

Potential applications of the above results include those in spectral theory, such as proofs of the singularity of spectra of Schrödinger operators on graphs. One such application can be found in [4].

2. Proofs

In the sequel we will need the following at most countable sets:

$$D = \bigcup_{n=1}^\infty D_n,$$

$$D_\nu = \{x \in \mathbb{R} \colon \nu(\{x\}) > 0\},$$

$$D^* = D \cup D_\nu.$$

Proof of Theorem 1. **1.** Define a Borel set B by

$$B = \{x \in \mathbb{R} \setminus D^*\colon \ \sup_{n\in\mathbb{N}} |f_n(x)| < \infty\}.$$

Assume that the statement of the theorem does not hold, so the Borel set $B \cap X$ has a positive Lebesgue measure: $|B \cap X| > 0$. It follows then by the σ-additivity of the Lebesgue measure that setting for $M \in \mathbb{N}$

$$B_M = \{x \in \mathbb{R} \setminus D^*\colon \ \sup_{n\in\mathbb{N}} |f_n(x)| \le M\}, \tag{7}$$

we have for some M (which will be fixed from now on),

$$|B_M \cap X| > 0. \tag{8}$$

2. Fix an arbitrary $x^* \in \mathbb{R}$ and define a function $F\colon \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \begin{cases} \nu((x^*, x]) & \text{if } x > x^*, \\ -\nu((x, x^*]) & \text{if } x \le x^*, \end{cases}$$

so that $\nu((a, b]) = F(b) - F(a)$ for any a, b ($a < b$). By Lebesgue's differentiation theorem, the function $F(\cdot)$ is differentiable almost everywhere and its derivative is the density of ν_{ac},

$$F'(x) = \rho(x) \ \text{ for a.e. } x \in \mathbb{R}.$$

Define a Borel set V by

$$V = \{x \in \mathbb{R} : \ F'(x) \text{ exists and equals } \rho(x)\}.$$

As was stated, $|\mathbb{R} \setminus V| = 0$. Finally, let

$$W = X \cap B_M \cap V. \tag{9}$$

The Borel set $W \subset \mathbb{R}$ has the following properties:

(a) $W \cap D^* = \varnothing$ (because $W \subset B_M$);
(b) $|f_n(x)| \le M$ for all $x \in W$, $n \in \mathbb{N}$ (again because $W \subset B_M$);
(c) for every $x \in W$ there exists $F'(x) > 0$ (since $W \subset X \cap V$);
(d) $|W| = |X \cap B_M| > 0$.

3. By Lebesgue's density theorem, almost every point of the set W is its density point. Fix such a point x_0. We may assume that $x_0 = 0$ (to achieve this, we shift all the measures μ_n and ν by $-x_0$).

Let

$$G(x) = F(x) - F(0) \equiv \begin{cases} \nu((0, x]) & \text{if } x > 0, \\ -\nu((x, 0]) & \text{if } x \le 0. \end{cases}$$

We have $G(0) = 0$ and $G'(0) > 0$. Let K and L be two constants such that

$$0 < K < G'(0) < L/2.$$

Let, furthermore, $a_0 > 0$ be so small that

$$K < \frac{G(x)}{x} < L/2 \quad \text{for all } \ x \in [-a_0, a_0] \setminus \{0\}.$$

Fix any $a \in \mathbb{R}$ such that

$$0 < a \le a_0 \quad \text{and} \quad \pm\, a \notin D^*. \tag{10}$$

In addition to this, a should be small enough, which will be made precise later.

4. For each $n \in \mathbb{N}$, decompose the function (3) as follows:

$$f_n = f_n^l + f_n^r + f_n^o,$$

where

$$f_n^l(x) = \int_{[-a,0)} \frac{\mu_n(dt)}{x-t}, \qquad f_n^r(x) = \int_{(0,a]} \frac{\mu_n(dt)}{x-t}, \qquad f_n^o(x) = \int_{\mathbb{R}\setminus[-a,a]} \frac{\mu_n(dt)}{x-t}$$

(we use the fact that $\mu_n(\{0\}) = 0$ since $0 \in W$ and $W \cap D^* = \varnothing$).

Given $C > 0$, define two sets $H_n^l(C)$ and $H_n^r(C)$ by

$$H_n^l(C) = \{x \in \mathbb{R} \setminus [-a,0]\colon\ f_n^l(x) > C\},$$
$$H_n^r(C) = \{x \in \mathbb{R} \setminus [0,a]\colon\ f_n^r(x) < -C\}.$$

Clearly, $H_n^l(C) \subset (0,\infty)$ and $H_n^r(C) \subset (-\infty,0)$.

Lemma 1. *For any $\gamma > 0$ there are $C_\gamma > 0$ and $n_{a,\gamma} \in \mathbb{N}$ such that for all $C \geq C_\gamma$ and all $n \geq n_{a,\gamma}$*

$$|H_n^l(C)| < \frac{\gamma a}{C} \qquad \text{and} \qquad |H_n^r(C)| < \frac{\gamma a}{C}.$$

Proof. We will prove the part of the lemma pertaining to the first inequality. The similar fact for the second inequality can be proved in the same way.

Let us estimate $f_n^l(x)$ $(x > 0)$ from above. Set

$$b_{a,\gamma} = \min\left(\frac{\gamma a}{8L}, \frac{a}{2}\right). \tag{11}$$

Pick any $b \in \mathbb{R}$ such that

$$b_{a,\gamma} \leq b < 2b_{a,\gamma} \tag{12}$$

and $-b \notin D$. Note that $0 < b < a$.

For $x \in \mathbb{R}$, let

$$G_n(x) = \begin{cases} \mu_n((0,x]) & \text{if } x > 0, \\ -\mu_n((x,0]) & \text{if } x \leq 0. \end{cases}$$

Then

$$f_n^l(x) = \int_{-a}^0 \frac{dG_n(t)}{x-t} = \int_{-a}^{-b} \frac{dG_n(t)}{x-t} + \int_{-b}^0 \frac{dG_n(t)}{x-t} \equiv \widetilde{f_n^l}(x) + \widetilde{\widetilde{f_n^l}}(x).$$

The functions $G_n(\cdot)$ $(n \in \mathbb{N})$ are nondecreasing and $G_n(t) \to G(t)$ on the (dense) set of points of continuity of the nondecreasing function $G(\cdot)$. Since $G(t) \geq (L/2)t$ for $t \in [-a,0]$ and $G_n(0) = 0$, we have $Lt < G_n(t) \leq 0$ for all $t \in [-a,-b_{a,\gamma}]$ and all large enough n $(n \geq n(a,b_{a,\gamma}))$. Therefore, for any such n and any $x > 0$

$$\begin{aligned}
\widetilde{f_n^l}(x) &= \int_{-a}^{-b} \frac{dG_n(t)}{x-t} \\
&= \frac{G_n(-b)}{x+b} - \frac{G_n(-a)}{x+a} - \int_{-a}^{-b} \frac{G_n(t)}{(x-t)^2}dt \\
&< \frac{La}{x+a} + \int_{-a}^{-b} \frac{L|t|}{(x-t)^2}dt \\
&< L + \int_0^a \frac{Lu\,du}{(x+u)^2} \\
&< L + \int_0^a \frac{L\,du}{x+u} \\
&= L\left(1 + \ln\left(1 + \frac{a}{x}\right)\right)
\end{aligned}$$

and

$$\widetilde{\widetilde{f_n^l}}(x) \le \int_{-b}^{0} \frac{dG_n(t)}{x} = \frac{-G_n(-b)}{x} < \frac{Lb}{x} < \frac{\gamma a}{4x}$$

(the last inequality uses (11) and (12)). Therefore, for any $x > 0$ and all $n \geq n(a, b_{a,\gamma})$

$$f_n^l(x) < \frac{\gamma a}{4x} + L\left(1 + \ln\left(1 + \frac{a}{x}\right)\right).$$

Consequently,

$$\begin{aligned}|H_n^l(C)| &\le \left|\left\{x \in (0,\infty)\colon\ \frac{\gamma a}{4x} > \frac{C}{2}\right\}\right| \\ &\qquad + \left|\left\{x \in (0,\infty)\colon L\left(1 + \ln\left(1 + \frac{a}{x}\right)\right) > \frac{C}{2}\right\}\right| \\ &= \frac{\gamma a}{2C} + \frac{a}{e^{\frac{C}{2L}-1} - 1}.\end{aligned}$$

Let $C_\gamma > 0$ be so large that

$$\frac{e^{\frac{C}{2L}-1} - 1}{C} > \frac{2}{\gamma} \quad \text{for all} \ \ C \ge C_\gamma;$$

also set $n_{a,\gamma} = n(a, b_{a,\gamma})$. Then we have

$$|H_n^l(C)| < \frac{\gamma a}{C}$$

if $C \ge C_\gamma$ and $n \ge n_{a,\gamma}$. This completes the proof of Lemma 1. $\square$

5. In the rest of the proof, by writing $u_n \overset{\text{ult}}{<} A$ we will mean that $u_n < A$ for all large enough n; the symbol $\overset{\text{ult}}{>}$ will have similar meaning.

We will use the following result about Hilbert transforms of measures supported on finite subsets of $\mathbb{R}$.

Proposition 1 (G. Boole [1])**.** *If $p_i > 0$ $(i = 1, \ldots, m)$ and*

$$g(x) = \sum_{i=1}^{m} \frac{p_i}{x - a_i},$$

then for any $C > 0$

$$|\{x \in \mathbb{R}\colon g(x) > C\}| = |\{x \in \mathbb{R}\colon g(x) < -C\}| = \frac{\sum_{i=1}^m p_i}{C}.$$

In other words, if μ is a positive measure supported on a finite subset of $\mathbb{R}$ and $g(x) = \int_{\mathbb{R}} (x-t)^{-1} \mu(dt)$, then for $C > 0$

$$|\{x \in \mathbb{R}\colon\ \pm g(x) > C\}| = \frac{\|\mu\|}{C},$$

where $\|\mu\| = \mu(\mathbb{R})$.

This result of G. Boole, having a short and beautiful proof, was rediscovered by L. H. Loomis [2].

Applying Proposition 1 to the function $f_n^l(\cdot)$, we derive that for any $C > 0$

$$|\{x \in \mathbb{R}\ :\ f_n^l(x) > C\}| = \frac{\mu_n([-a, 0))}{C}.$$

We have $-a, 0 \notin D^*$ so that, as $n \to \infty$, the right-hand side converges to $\nu([-a,0))/C$, which is $> Ka/C$. Therefore, for any $C > 0$

$$|\{x \in \mathbb{R} : f_n^l(x) > C\}| \overset{\text{ult}}{>} \frac{Ka}{C}. \tag{13}$$

6. Pick any $\gamma \in \mathbb{R}$ such that

$$0 < \gamma < \frac{K}{2}.$$

According to Lemma 1, for any $C \geq C_\gamma$

$$|\{x \in \mathbb{R} \setminus [-a,0] : f_n^l(x) > C\}| \equiv |H_n^l(C)| \overset{\text{ult}}{<} \frac{\gamma a}{C}. \tag{14}$$

It follows from (13) and (14) that for $C \geq C_\gamma$

$$|\{x \in [-a,0) : f_n^l(x) > C\}| \overset{\text{ult}}{>} \frac{(K-\gamma)a}{C}.$$

Using Lemma 1 again, we see that if $C \geq C_\gamma + M$, then

$$|\{x \in [-a,0) : f_n^r(x) < -C + M\}| \leq |H_n^r(C-M)| \overset{\text{ult}}{<} \frac{\gamma a}{C-M}.$$

Therefore, setting

$$Q_n^l = \{x \in [-a,0) \colon f_n^l(x) + f_n^r(x) > M\},$$

we have

$$|Q_n^l| \overset{\text{ult}}{>} \frac{(K-\gamma)a}{C} - \frac{\gamma a}{C-M} \tag{15}$$

for any $C \geq C_\gamma + M$. Similarly, for any such C

$$|Q_n^r| \overset{\text{ult}}{>} \frac{(K-\gamma)a}{C} - \frac{\gamma a}{C-M}, \tag{16}$$

where

$$Q_n^r = \{x \in (0,a] \colon f_n^l(x) + f_n^r(x) < -M\}.$$

7. Fix $\beta \in \mathbb{R}$ such that

$$0 < \beta < K - 2\gamma.$$

The common right-hand side of inequalities (15) and (16) is $\geq \beta a/C$ if

$$C \geq M\left(1 + \frac{\gamma}{K - 2\gamma - \beta}\right).$$

Setting

$$C^* = \max\left(C_\gamma + M,\ M\left(1 + \frac{\gamma}{K-2\gamma-\beta}\right)\right),$$

we have

$$|Q_n^l| \overset{\text{ult}}{>} \beta a/C^* \quad \text{and} \quad |Q_n^r| \overset{\text{ult}}{>} \beta a/C^*. \tag{17}$$

8. Note that the function $f_n^o(\cdot)$ is nonincreasing on $[-a,a]$ and hence for each n either $f_n^o(x) \geq 0$ on $[-a,0]$ or $f_n^o(x) \leq 0$ on $[0,a]$. In the former case we have $f_n(x) > M$ on Q_n^l, and in the latter case $f_n(x) < -M$ on Q_n^r. Therefore, for any n the set $\{x \in [-a,a] \colon |f_n(x)| > M\}$ contains either Q_n^l or Q_n^r. In view of (17),

$$|\{x \in [-a,a] : |f_n(x)| > M\}| \overset{\text{ult}}{>} \frac{\beta a}{C^*}. \tag{18}$$

9. Until now a was an arbitrary positive number satisfying (10). Now we will use the fact that 0 is a density point of the set W defined by (9) and impose an additional requirement: a should be so small that

$$\frac{|W \cap [-a,a]|}{2a} \geq 1 - \frac{\beta}{2C^*},$$

or, equivalently,

$$|W \cap [-a,a]| \geq 2a - \frac{\beta a}{C^*}.$$

Together with (18) this shows that for large n the intersection of the sets $\{x \in [-a,a] : |f_n(x)| > M\}$ and $W \cap [-a,a]$ has a positive Lebesgue measure. This contradicts the fact that $W \subset B_M$ (see (7), (9)), which completes the proof of Theorem 1. □

Proof of Theorem 2. **1.** It suffices to show that for any $l \in \mathbb{N}$ there is a subsequence $f_{n_1}, f_{n_2}, \ldots$ of the sequence (2) that converges Lebesgue almost everywhere on the set $[-l,l] \setminus X$. (Once this is done, it remains to use Cantor's diagonal process to obtain a subsequence converging almost everywhere on the set $\mathbb{R} \setminus X$.)

Let $I \subset \mathbb{R}$ be a compact interval whose interior contains $[-l,l]$ and whose endpoints have zero ν-measure. Represent the measures μ_n $(n \in \mathbb{N})$ and ν as $\mu_n = \tilde{\mu}_n + \tilde{\tilde{\mu}}_n$ and $\nu = \tilde{\nu} + \tilde{\tilde{\nu}}$, where $\tilde{\mu}_n = 1_I\, \mu_n$, $\tilde{\tilde{\mu}}_n = 1_{\mathbb{R}\setminus I}\, \mu_n$, $\tilde{\nu} = 1_I\, \nu$, and $\tilde{\tilde{\nu}} = 1_{\mathbb{R}\setminus I}\, \nu$. We have $\tilde{\mu}_n \xrightarrow{\text{w}} \tilde{\nu}$ and $\tilde{\tilde{\mu}}_n \xrightarrow{\text{w}} \tilde{\tilde{\nu}}$.

The function (3) gets decomposed into a sum $f_n = \tilde{f}_n + \tilde{\tilde{f}}_n$, where

$$\tilde{f}_n(x) = \int_{\mathbb{R}} \frac{\tilde{\mu}_n(dt)}{x-t} \equiv \int_I \frac{\mu_n(dt)}{x-t} \quad \text{and} \quad \tilde{\tilde{f}}_n(x) = \int_{\mathbb{R}} \frac{\tilde{\tilde{\mu}}_n(dt)}{x-t} \equiv \int_{\mathbb{R}\setminus I} \frac{\mu_n(dt)}{x-t}.$$

It is easy to derive from (5) and (6) that the functions $\tilde{\tilde{f}}_n(\cdot)$, $n \in \mathbb{N}$, are uniformly bounded and uniformly equicontinuous on $[-l,l]$. Consequently, due to the Arzela-Ascoli theorem, any subsequence of the sequence $\tilde{\tilde{f}}_n(\cdot)$ contains a subsubsequence that converges uniformly on $[-l,l]$, and it suffices to prove that some subsequence of the sequence $\tilde{f}_n$ converges almost everywhere on $I \setminus X$ (and hence almost everywhere on $[-l,l] \setminus X$). Therefore, the proof of Theorem 2 reduces to proving its special case: *Let $\mu_1, \mu_2, \ldots$ be a sequence of measures supported on finite subsets of a compact interval $I \subset \mathbb{R}$; suppose μ_n weakly converges to a measure ν such that the endpoints of I have zero ν-measure. Let $\nu = \nu_{ac} + \nu_s$ be the Lebesgue decomposition of ν, and X the essential support of ν_{ac}. Then there is a subsequence $f_{n_1}, f_{n_2}, \ldots$ of the sequence* (2) *that converges almost everywhere on $I \setminus X$.*

2. Choosing f in (1) so that $f \equiv 1$ on I, we see that $\|\mu_n\| \to \|\nu\|$ as $n \to \infty$ and hence

$$\sup_n \|\mu_n\| < \infty. \tag{19}$$

There is a Borel set $Y \subset I$ such that $|Y| = 0$ and $\nu_s(I \setminus Y) = 0$; denote the set $X \cup Y \cup D$ by A. Note that $\nu(I \setminus A) = 0$. We want to prove that some subsequence of the sequence (2) converges Lebesgue almost everywhere on $I \setminus A$.

Fix any $\varepsilon > 0$; let $U \subset \mathbb{R}$ be an open set such that $U \supset A$ and $|U \setminus A| < \varepsilon$.

The open set U is the union of an at most countable set of disjoint nonempty open intervals I_k; we may assume that the set is countable and the endpoints of each interval I_k have zero ν-measure.

Since $\nu(I \setminus U) = 0$, there is a sequence of integers $0 < k_1 < k_2 < \dots$ such that, setting

$$B_j = \left(\bigcup_{k=1}^{k_j} \overline{I}_k \right) \cap I,$$

we have

$$\nu(I \setminus B_j) < 4^{-j}. \tag{20}$$

The set B_j is a compact subset of I. Let

$$V_j = \{x \in I \,:\, \mathrm{dist}(x, B_j) < \gamma_j\},$$

where $\gamma_j > 0$ is so small that $|V_j \setminus B_j| < \varepsilon/2^j$. Setting $V_\infty = \bigcup_{j=1}^{\infty} V_j$, we have $|V_\infty \setminus \bigcup_j B_j| < \varepsilon$ and hence

$$|V_\infty \setminus A| < 2\varepsilon.$$

We are going to show that some subsequence of the sequence (2) converges almost everywhere on $I \setminus V_\infty$.

3. For each $j \in \mathbb{N}$, every measure μ_n can be represented as a sum:

$$\mu_n = \sigma_n^j + \tau_n^j,$$

where $\sigma_n^j = 1_{B_j}\, \mu_n$ and $\tau_n^j = 1_{I \setminus B_j}\, \mu_n$. Each function f_n will also be decomposed:

$$f_n = g_n^j + h_n^j,$$

where

$$g_n^j(x) = \int_I \frac{\sigma_n^j(dt)}{x - t} \equiv \int_{B_j} \frac{\mu_n(dt)}{x - t}, \quad h_n^j(x) = \int_I \frac{\tau_n^j(dt)}{x - t} \equiv \int_{I \setminus B_j} \frac{\mu_n(dt)}{x - t}.$$

4. For each j we have a sequence of functions

$$g_n^j(\cdot), \quad n = 1, 2, \dots. \tag{21}$$

On the compact set $I \setminus V_j$ these functions are uniformly bounded and uniformly equicontinuous (indeed, each of them is, up to a bounded factor (see (19)), a convex combination of functions $(x - t)^{-1}$, where $t \in B_j$ so that $\mathrm{dist}(t, I \setminus V_j) \geq \gamma_j > 0$). Consequently, there is a subsequence of the sequence (21) that is uniformly convergent on $I \setminus V_j$. Using Cantor's diagonal process, we can construct a sequence $n_k \nearrow \infty$ such that for any j the sequence of functions $g_{n_k}^j(\cdot)$, $k = 1, 2, \dots$, converges uniformly on the set $I \setminus V_j$ and hence on the intersection of these sets:

$$\bigcap_{j=1}^{\infty} (I \setminus V_j) = I \setminus \bigcup_{j=1}^{\infty} V_j = I \setminus V_\infty.$$

We will assume that the sequence of measures μ_n is already thinned out so that for each j the sequence $g_n^j(\cdot)$ itself converges uniformly on the set $I \setminus V_\infty$ to some limit $g^j(\cdot)$:

$$g_n^j(\cdot) \rightrightarrows g^j(\cdot) \;\text{ on } I \setminus V_\infty \;\text{ as }\; n \to \infty \quad (j = 1, 2, \dots).$$

5. We have

$$g_n^{j+1}(x) - g_n^j(x) = \int_I \frac{\Delta\sigma_n^{j+1}(dt)}{x - t}, \tag{22}$$

where

$$\Delta\sigma_n^{j+1} = \sigma_n^{j+1} - \sigma_n^j = 1_{B_{j+1} \setminus B_j}\, \mu_n.$$

As $n \to \infty$,

$$\|\Delta\sigma_n^{j+1}\| = \Delta\sigma_n^{j+1}(I) = \mu_n(B_{j+1} \setminus B_j) \to \nu(B_{j+1} \setminus B_j) \leq \nu(I \setminus B_j) < 4^{-j} \tag{23}$$

(we used the fact that $B_{j+1} \setminus B_j$ is a finite union of disjoint intervals whose endpoints have ν-measure zero).

It follows from Proposition 1 that for any finitely supported measure μ on $\mathbb{R}$ the function $f(x) = \int_{\mathbb{R}} (x-t)^{-1}\mu(dt)$ satisfies the equality

$$|\{x \in \mathbb{R}\colon |f(x)| > C\}| = \frac{2\|\mu\|}{C}$$

for all $C > 0$. Consequently, by (22) and (23), setting

$$\delta_j = 2 \cdot 4^{-j},$$

we have for any $j \in \mathbb{N}$ and all large enough n

$$g_n^{j+1} \overset{\delta_j}{\sim} g_n^j,$$

by which we mean that

$$|\{x \in \mathbb{R} : |g_n^{j+1}(x) - g_n^j(x)| > C\}| \leq \frac{\delta_j}{C} \tag{24}$$

for all $C > 0$.

6. As $n \to \infty$, we have $g_n^j(\cdot) \rightrightarrows g^j(\cdot)$ and $g_n^{j+1}(\cdot) \rightrightarrows g^{j+1}(\cdot)$ on the set

$$Z = I \setminus V_\infty$$

and hence $g_n^{j+1}(\cdot) - g_n^j(\cdot) \rightrightarrows g^{j+1}(\cdot) - g^j(\cdot)$ on Z. It follows that for any $j \in \mathbb{N}$ and all $C > 0$

$$|\{x \in Z : |g^{j+1}(x) - g^j(x)| > C\}| \leq \frac{\delta_j}{C}. \tag{25}$$

(Indeed, let $\Delta g_n^{j+1} = g_n^{j+1} - g_n^j$ and $\Delta g^{j+1} = g^{j+1} - g^j$. Suppose that for some $C > 0$ the inequality $|\Delta g^{j+1}(x)| > C$ holds on a subset of Z whose measure is $> \delta_j/C$. Then for a small enough $h > 0$ the measure of the set $\{x \in Z\colon |\Delta g^{j+1}(x)| > C + h\}$ is still greater than δ_j/C; since $\Delta g_n^{j+1}(\cdot) \rightrightarrows \Delta g^{j+1}(\cdot)$ on Z, for large n we have $|\Delta g_n^{j+1}(x)| > C$ on the same set, which is impossible because of (24).)

7. Let $X_j = \{x \in Z : |g^{j+1}(x) - g^j(x)| > 2^{-j}\}$, $j = 1, 2, \ldots$. In view of (25), $|X_j| \leq 2^{-j+1}$ so that $\sum_{j=1}^\infty |X_j| < \infty$, and by the Borel-Cantelli lemma for Lebesgue a.e. $x \in Z$ we have $|g^{j+1}(x) - g^j(x)| \leq 2^{-j}$ for all large enough j. Consequently, there is a Borel measurable function $g(\cdot)$ on Z such that $g^j(x) \to g(x)$ for a.e. $x \in Z$.

8. For each j, we have $g_n^j \rightrightarrows g^j$ on Z as $n \to \infty$ so that for large enough n_j (which can be chosen so that $n_j > n_{j-1}$ if $j \geq 2$)

$$|g_{n_j}^j(x) - g^j(x)| < \frac{1}{j} \quad \text{for all} \quad x \in Z. \tag{26}$$

Since $g^j(x) \to g(x)$ for a.e. $x \in Z$, we conclude that *there is a sequence $n_j \nearrow \infty$ such that*

$$g_{n_j}^j(x) \to g(x) \quad as \quad j \to \infty \qquad for \quad a.e. \quad x \in Z. \tag{27}$$

9. The second summand in the decomposition $f_{n_j} = g^j_{n_j} + h^j_{n_j}$ is

$$h^j_{n_j}(x) = \int_I \frac{\tau^j_{n_j}(dt)}{x-t} \equiv \int_{I\setminus B_j} \frac{\mu_{n_j}(dt)}{x-t}. \tag{28}$$

We have $\mu_n \xrightarrow{\mathrm{w}} \nu$ as $n \to \infty$, and since the set $I \setminus B_j$ is a finite union of disjoint intervals whose endpoints have zero ν-measure,

$$\|\tau^j_n\| = \mu_n(I \setminus B_j) \to \nu(I \setminus B_j) \quad \text{as} \quad n \to \infty.$$

In view of (20), $\|\tau^j_n\| < 4^{-j}$ for all large enough n; therefore, for each j we can increase n_j so that (26) and hence (27) are still true, the sequence n_j is still increasing and, in addition, $\|\tau^j_{n_j}\| < 4^{-j}$ for all j.

Applying Proposition 1 to the measure $\tau^j_{n_j}$, we derive from the last inequality and (28) that

$$|\{x \in I : |h^j_{n_j}(x)| > 2^{-j}\}| < 2^{-j+1}$$

for all j. By the Borel-Cantelli lemma, for a.e. $x \in I$

$$|h^j_{n_j}(x)| \le 2^{-j} \quad \text{for all large enough} \quad j.$$

10. Therefore, for each $j \in \mathbb{N}$ there is a decomposition

$$f_{n_j} = g^j_{n_j} + h^j_{n_j},$$

where, as $j \to \infty$, we have $g^j_{n_j}(x) \to g(x)$ and $h^j_{n_j}(x) \to 0$ for a.e. $x \in Z = I \setminus V_\infty$.

11. We have shown that *for any $\varepsilon > 0$ there are a sequence $n_j \nearrow \infty$ and a Borel set $V_\infty \subset I$ such that $V_\infty \supset A$, $|V_\infty \setminus A| < 2\varepsilon$, and the sequence $f_{n_j}(x)$ has a finite limit for a.e. $x \in I \setminus V_\infty$.*

We can use this statement iteratively, with ε replaced by $\varepsilon_k = k^{-1}$, $k = 1, 2, \ldots$, to extract thinner and thinner subsequences of the sequence $f_n(\cdot)$; denoting the corresponding sets V_∞ as $V^{(k)}_\infty$, we have $|V^{(k)}_\infty \setminus A| < 2k^{-1}$. Cantor's diagonal subsequence will converge almost everywhere on the set $I \setminus \bigcap_{k\ge 1} V^{(k)}_\infty$ and hence almost everywhere on $I \setminus A$. $\square$

Acknowledgment

The author is grateful to S. Molchanov for useful conversations.

References

[1] G. Boole, *On the comparison of transcendents, with certain applications to the theory of definite integrals*, Philos. Trans. Roy. Soc. London **147** (1857), 745-803.

[2] Lynn H. Loomis, *A note on the Hilbert transform*, Bull. Amer. Math. Soc. **52** (1946), 1082–1086. MR0019155 (8,377d)

[3] S. Molchanov, Private communication.

[4] S. Molchanov and Lukun Zheng, *Cluster expansion of the resolvent for the Schrödinger operator on non-percolating graphs with applications to Simon-Spencer type theorems and localization.* Preprint.

Department of Mathematics and Statistics, University of North Carolina at Charlotte, 9201 University City Blvd., Charlotte, North Carolina 28223

E-mail address: aygordon@uncc.edu