PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 11, November 2016, Pages 4793–4805
http://dx.doi.org/10.1090/proc/13088
Article electronically published on April 13, 2016

ON THE REDUCIBILITY OF 2-DIMENSIONAL LINEAR QUASI-PERIODIC SYSTEMS WITH SMALL PARAMETERS

JUNXIANG XU, WANG KUN, AND ZHU MIN

(Communicated by Yingfei Yi)

ABSTRACT. In this paper we consider a real analytic linear quasi-periodic system of 2-dimension, whose coefficient matrix depends on a small parameter C^m-smoothly and closes to constant. Under some non-resonance conditions about the basic frequencies and the eigenvalues of the constant matrix and without any non-degeneracy assumption with respect to the small parameter, we prove that the system is reducible for many of the sufficiently small parameters.

1. Introduction and main result

1.1. **Problems.** The reducibility of linear differential systems has been studied widely by many authors. The earliest result in this field is the well-known Floquet's theory, which says that linear periodic systems are always reducible. This means that for linear periodic systems there exist periodic transformations which change the systems to constant ones.

However, quasi-periodic systems are more complicated. Consider a linear quasi-periodic system

$$\dot{x} = A(t)x,\ x \in \mathbb{R}^n,$$

where $A(t) \in gl(n, \mathbb{R})$ is an analytic quasi-periodic matrix or a $\mathbb{C}^r$-smooth one. It is said to be reducible if there exists a non-singular quasi-periodic $\Phi(t) \in GL(n, \mathbb{R})$, which is at least C^1-smooth, such that the transformation $x = \Phi(t)y$ changes it to the form $\dot{y} = By$, where B is a constant matrix. If $\Phi(t)$ is analytic (or $\mathbb{C}^r$-smooth), we say the quasi-periodic system is analytic (or $\mathbb{C}^r$) reducible. Here $gl(n, \mathbb{R})$ denotes the Lie algebra of $n \times n$ real matrices and $GL(n, \mathbb{R})$ the linear group of $n \times n$ real invertible matrices.

Johnson and Sell [10] first obtained that if the quasi-periodic matrix $A(t)$ is sufficiently smooth and satisfies the "full spectrum" condition, then the system $\dot{x} = A(t)x$ is reducible.

Received by the editors July 28, 2015 and, in revised form, December 21, 2015 and January 9, 2016.

2010 *Mathematics Subject Classification.* Primary 34D10, 34D23; Secondary 34C27.

The first author was supported by the National Natural Science Foundation of China (Grant No. 11371090).

The third author was partially supported by the National Natural Science Foundation of China (Grant. No. 11401309), the NSF of the Universities in Jiangsu Province in China (No. 13KJB110012), the start high-level personnel of scientific research funds of Nanjing Forestry University in China (No.GXL2014051) and the high level academic papers published aid funds of Nanjing Forestry University in China (No:163101613).

In [11] Jorba and Simó considered a class of linear quasi-periodic systems close to constant,

$$\dot{x} = (A + \epsilon Q(t, \epsilon))x, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $A \in gl(n, \mathbb{R})$ is a constant matrix with n different eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, ϵ is a small parameter and $Q(t, \epsilon) \in gl(n, \mathbb{R})$ is analytic quasi-periodic in time t and $\mathbb{C}^1$-smooth in the parameter ϵ. Suppose that the following non-resonance condition holds:

$$|\mathrm{i}\langle \omega, k\rangle + \lambda_i - \lambda_j| \geq \frac{\alpha}{|k|^\tau}, \forall k \in \mathbb{Z}^r \setminus \{0\}, \forall i, j = 1, 2, \cdots, n, \tag{1.2}$$

where $\mathrm{i} = \sqrt{-1}$, $\alpha > 0$ and $\tau > r - 1$ are constants. Let $\lambda_j^0(\epsilon)$ $(j = 1, 2, \cdots, n)$ be eigenvalues of $A + \epsilon[Q]$. The authors proved that if the non-degeneracy condition holds,

$$\frac{d}{d\epsilon}(\lambda_i^0(\epsilon) - \lambda_j^0(\epsilon))|_{\epsilon=0} \neq 0, \quad \forall\, i \neq j, \tag{1.3}$$

then, for sufficiently small $\epsilon_0 > 0$, there exists a non-empty Cantor subset $E \subset (0, \epsilon_0)$ such that the system (1.1) is reducible for $\epsilon \in E$.

The non-degeneracy condition (1.3) is important for the subset E to be non-empty. It is used for the measure estimate of the parameter ϵ of small divisor conditions, which shows that E occupies relatively larger Lebesgue measure in $(0, \epsilon_0)$.

More recently, Eliasson [4] obtained the almost reducibility for a class of quasi-periodic systems and the result implies that in [11]. Then Her and You [7] proved the full measure reducibility for a generic family of quasi-periodic systems. Moreover, an infinite-dimensional case is considered in [5]. In [17] the authors obtained a similar result on the reducibility of (1.1) under a weaker non-degeneracy condition. Furthermore, the result in [11] was generalized to a class of non-linear quasi-periodic systems [12] by the same authors.

The 2-dimensional quasi-periodic system is very important since it concerns the one-dimensional Schrödinger equation with quasi-periodic potentials

$$-x''(t) + q(t)x = Ex,$$

which is equivalent to the following two-dimensional quasi-periodic system:

$$\dot{x} = y, \quad \dot{y} = (q(t) - E)x, \tag{1.4}$$

where $q(t)$ is a quasi-periodic function and E is the energy parameter. Dinaburg and Sinai [2] proved that if the frequency vector is Diophantine, then the system (1.4) is reducible for most sufficiently large energy parameters E. Since the reducibility is related to the spectrum of linear Schrödinger operators, the problem becomes more important; see [3–5, 9, 14, 15].

In particular, there are some interesting aspects for 2-dimensional systems. In [16] Rüssmann considered an analytic Hamiltonian system with two degrees of freedom and obtained the convergence of the transformation into a norm form. Similarly, Fayad and Krikorian [6] proved the accumulation of invariant curves for a two-dimensional area preserving surface diffeomorphism without any non-degeneracy condition. More recently, Avila, Fayad and Krikorian [1] developed a new KAM method to consider the rotations reducibility of $SL(2, \mathbb{R})$ cocycles with Liouvillean frequencies. Their method is based on the continued fraction expansion of irrational numbers and the CD bridge. Then Hou and You [8] used a generalized KAM-type scheme and Floquet theory to prove the almost reducibility and non-perturbative reducibility for two-dimensional linear quasi-periodic Hamiltonian systems with Liouvillean frequencies. In particular, they obtained the rotations reducibility if the basic frequencies and the rotation number satisfy a non-resonance condition. Later, in [20] Zhou and Wang used the periodic approximation and a KAM scheme to extend the above results to quasi-periodic G-cocycles with one frequency.

Usually, some non-degeneracy condition is required for the reducibility of (1.1). However, in the case where $n = 2$, without any non-degeneracy assumption it is proved in [18] that if the system (1.1) depends analytically on the small parameter ϵ, it is reducible for most sufficiently small parameters. In this paper we will consider a 2-dimensional linear quasi-periodic Hamiltonian system, which depends on a small parameter C^m-smoothly. Without imposing any non-degeneracy condition and by the KAM technique developed in [19], we will prove that the system is reducible for many of the sufficiently small parameters. This is an interesting phenomena of 2-dimensional quasi-periodic systems.

1.2. **Definitions and notation.** Before stating our results, we first give some notation and definitions.

As usual, $\mathbb{R}^r$ ($\mathbb{C}^r$) denotes the r-dimensional real (complex) space. Let $\mathbb{Z}^r$ be the subset of $\mathbb{R}^r$, which consists of all the vectors whose components are integers. Let $\mathbb{T}^r = \mathbb{R}^r/2\pi\mathbb{Z}^r$ be the r-dimensional torus and

$$\mathbb{T}^r_s = \big\{\theta = (\theta_1, \theta_2, \cdots, \theta_r) \in \mathbb{C}^r/2\pi\mathbb{Z}^r \;\big|\; |\mathrm{Im}\theta_j| \le s,\ j = 1, 2, \cdots, r\big\}$$

be a strip domain in $\mathbb{C}^r$ with s-width. $sl(2,\mathbb{R})$ denotes the Lie subalgebra of $gl(2,\mathbb{R})$, which consists of 2×2 real Hamiltonian matrices, and $SL(2,\mathbb{R})$ the linear subgroup of $GL(2,\mathbb{R})$, which consists of 2×2 real symplectic matrices.

A function $f(t)$ is called quasi-periodic in time t with frequency vector $\omega = (\omega_1, \omega_2, \cdots, \omega_r)$ if there is a function $F(\theta)$ $\big(\theta = (\theta_1, \theta_2, \cdots, \theta_r)\big)$, which is 2π periodic in all its arguments θ_i, such that $f(t) = F(\omega t)$. If $F(\theta)$ is analytic on $\mathbb{T}^r_s$, we call $f(t)$ analytic quasi-periodic on $\mathbb{T}^r_s$. Expand $f(t)$ as a Fourier series with $f(t) = \sum_{k\in\mathbb{Z}^r} f_k e^{\mathrm{i}\langle \mathrm{k},\omega\rangle t}$, where $\mathrm{i} = \sqrt{-1}, \mathrm{k} = (\mathrm{k}_1, \mathrm{k}_2, \cdots, \mathrm{k}_r) \in \mathbb{Z}^r$ and $\langle k, \omega\rangle = \sum_{j=1}^r k_j\omega_j$. Define a norm by $\|f\|_s = \sum_{k\in\mathbb{Z}^r} |f_k|e^{s|k|}$, where $|k| = \sum_{j=1}^r |k_j|$. Denote by $[f] = f_0$ the average of f.

Consider a matrix function $Q(t) = (q_{ij}(t))_{1\le i,j\le 2}$. If all q_{ij} are analytic quasi-periodic with frequency vector ω, we call $Q(t)$ analytic quasi-periodic with the frequency vector ω. Denote by $C^a\big(\mathbb{T}^r_s, sl(2,\mathbb{R})\big)$ the set of all $Q(t) \in sl(2,\mathbb{R})$ which are real analytic quasi-periodic on $\mathbb{T}^r_s$ with the frequency vector ω. For $Q = (q_{ij})_{1\le i,j\le 2} \in C^a\big(\mathbb{T}^r_s, sl(2,\mathbb{R})\big)$, define $\|Q\|_s = 2 \times \max_{1\le i,j\le 2} \|q_{ij}\|_s$. It is easy to see that $\|Q_1 \cdot Q_2\|_s \le \|Q_1\|_s \cdot \|Q_2\|_s$. If Q is a constant matrix, denote $\|Q\| = \|Q\|_s$ for simplicity. Denote the average of Q by $[Q] = \big([q_{ij}]\big)_{1\le i,j\le 2}$.

Let $U_\delta = [0, \delta]$. Denote by $C^{a,m}\big(\mathbb{T}^r_s \times U_\delta, sl(2,\mathbb{R})\big)$ the set which consists of all $Q(t,\epsilon) \in sl(2,\mathbb{R})$ such that Q are analytic quasi-periodic in t on $\mathbb{T}^r_s$ with the frequency vector ω and C^m-smooth in ϵ on U_δ. Let $Q(t,\epsilon) = \sum_{k\in\mathbb{Z}^r} Q_k(\epsilon)e^{\mathrm{i}\langle\omega,k\rangle t}$ and define

$$\|Q\|_{s,\delta} = \sum_{k\in\mathbb{Z}^r} \sup_{0\le\epsilon\le\delta} \|Q_k(\epsilon)\|e^{s|k|}.$$

Similarly, we have the notation $C^a\big(\mathbb{T}^r_s, SL(n,\mathbb{R})\big)$ and $C^{a,m}\big(\mathbb{T}^r_s \times U_\delta, SL(n,\mathbb{R})\big)$.

1.3. **Main results.**

Theorem 1.1. *Consider the following real linear quasi-periodic Hamiltonian system:*

$$\dot{x} = (A + Q(t,\epsilon))x, \quad x \in \mathbb{R}^2, \tag{1.5}$$

where $A \in sl(2,\mathbb{R})$ is a constant matrix, ϵ is a small parameter and Q belongs to $C^{a,m}(\mathbb{T}^r_s \times U_\delta, sl(2,\mathbb{R}))$ ($m = 1$ or 0) with the frequency vector ω. Let $\pm\lambda_0$ be eigenvalues of A and (ω,λ_0) satisfy the non-resonance condition

$$\begin{cases} |\langle\omega,k\rangle| \geq \frac{\alpha_0}{|k|^\tau}, \\ |\mathrm{i}\langle\omega,k\rangle \pm 2\lambda_0| \geq \frac{\alpha_0}{|k|^\tau} \end{cases} \tag{1.6}$$

for all $0 \neq k \in \mathbb{Z}^r$, where $\alpha_0 > 0$ and $\tau > r-1$. Then the following conclusions hold true:

(i) Let $m = 1$. If $\|Q\|_{s,\delta} \to 0$ and $\|\partial_\epsilon Q\|_{s,\delta} \leq M$ as $\delta \to 0$, then there exists a sufficiently small $\epsilon_0 > 0$ and a non-empty subset $E \subset (0,\epsilon_0)$, such that for $\epsilon \in E$ the system (1.5) is reducible. Moreover, the set $E \cap (0,\delta)$ has positive Lebesgue measure for any $\delta > 0$.

(ii) Let $m = 0$. If $\|Q\|_{s,\delta} \to 0$ as $\delta \to 0$, there exists a sufficiently small $\epsilon_0 > 0$ and a non-empty subset $E \subset (0,\epsilon_0)$, such that for $\epsilon \in E$ the system (1.5) is reducible. Moreover, the set $E \cap (0,\delta)$ has cardinality number of the continuum for any $\delta > 0$.

Remark. As in the previous papers, when the system depends C^m-smoothly on the small parameter, it usually requires some non-degeneracy condition to guarantee the reducibility of (1.5) for most of the sufficiently small parameters. Our theorem says that in the C^1-smooth case, the system (1.5) can be reducible for many of the sufficiently small parameters with positive measure without imposing any non-degeneracy assumption. However, we cannot have some precise measure estimate of parameters as in [18]. In the continuous case, we can still obtain the reducibility of (1.5) for many of the sufficiently small parameters. In fact, these results are obvious if the rotation number of the system such as (1.4) has some monotonicity with respect to the parameter. This fact was first observed by Dinaburg-Sinai and later by Moser-Pöschel and also by Eliasson for the one-dimensional Schrödinger equation [2,3,13]. In this case, since the rotation numbers can take all numbers in an interval, there are many rotation numbers non-resonant with the basic frequencies, which implies the reducibility of the system.

2. Lemma

In this section we state a lemma, which is used in the proof of Theorem 1.1.

Lemma 2.1. *Consider the following Hamiltonian system:*

$$\dot{x} = (A + Q(t,\epsilon))x, \quad x \in \mathbb{R}^2. \tag{2.1}$$

Suppose $A \in sl(2,\mathbb{R})$ with the eigenvalues $\pm\lambda_0$ and $Q \in C^{a,m}\big(\mathbb{T}^r_s \times U_\delta, sl(2,\mathbb{R})\big)$ satisfying that $Q(t,0) \equiv 0$ and for all $\epsilon \in [0,\delta]$,

$$\|Q(\cdot,\epsilon)\|_s \leq \gamma \text{ and } \big\|\partial_\epsilon Q(\cdot,\epsilon)\big\|_s \leq M \quad \text{if } m = 1, \tag{2.2}$$

where $\partial_\epsilon Q = \frac{\partial Q}{\partial \epsilon}$ and $M > 0$ is a constant. Assume that the non-resonance condition (1.6) holds. Let $\alpha > 0$ and $\tau' > \tau$. Then, there exists a sufficiently small

$\gamma_0 > 0$, which only depends on $\alpha_0, \alpha, M, s, \tau, \tau'$, such that if $\gamma \le \gamma_0$, the following conclusions hold true:

(i) There exists a matrix function $\Phi(t,\epsilon) \in SL(2,\mathbb{R})$, which is analytic quasi-periodic in t and $\mathbb{C}^m$-smooth in ϵ, such that $\Phi \in C^{a,m}\big(\mathbb{T}^r_{s/2} \times U_\delta, SL(2,\mathbb{R})\big)$ with

$$\|\Phi(\cdot,\epsilon) - I\|_{\frac{s}{2}} \le c\gamma/\alpha_0^2\alpha s^v \text{ and } \|\partial_\epsilon \Phi(\cdot,\epsilon)\|_{\frac{s}{2}} \le cM/\alpha_0^2\alpha s^v,\ v = \tau' + 2\tau + r.$$

(ii) There exists an ϵ-dependent Hamiltonian matrix $A_\infty \in C^m\big([0,\delta], sl(2,\mathbb{R})\big)$ such that

$$A_\infty(0) = A \text{ and } \|A_\infty(\epsilon) - A\| \le 2\gamma, \quad \epsilon \in [0,\delta].$$

(iii) Let

$$\Pi_\infty = \Big\{\epsilon \in [0,\delta] \ \Big| \ \big| |\langle \omega, k\rangle|^2 - 4\det(A_\infty(\epsilon))\big| \ge \frac{\alpha}{|k|^{\tau'}},\ \forall k \ne 0\Big\}. \tag{2.3}$$

If $\epsilon \in \Pi_\infty$, then under the symplectic transformation $x = \Phi(t,\epsilon)y$, the system (2.1) is transformed into $\dot y = A_\infty(\epsilon)y$.

Below we first use Lemma 2.1 to prove Theorem 1.1 and delay the proof of Lemma 2.1 until later.

3. Proof of Theorem 1.1

By Lemma 2.1 we need to prove that Π_∞ always includes many sufficiently small parameters. The proof of Theorem 1.1 is divided into two parts: differentiable case and continuous case.

3.1. **Differentiable case.** We first prove a lemma about measure estimate.

Lemma 3.1. *Suppose $\omega \in \mathbb{R}^r$ and $\lambda_0 \in \mathbb{C}$ satisfy (1.6). Let $0 < \alpha \le \alpha_0^2/2, \tau' > 4\tau + r$ and*

$$\varphi_k(\epsilon) = \ |\langle \omega, k\rangle|^2 + 4\lambda_0^2 - \epsilon.$$

Let $\eta > 0$ be an arbitrarily small positive constant and set

$$E_\pm = \Big\{\epsilon \in (0, \pm\eta) \ \Big| \ |\varphi_k(\epsilon)| \ge \frac{\alpha}{|k|^{\tau'}},\ \forall k \ne 0\Big\}, \tag{3.1}$$

where the notation $(0,-\eta)$ denotes the interval $(-\eta, 0)$ for simplicity. Then

$$meas((0,\pm\eta) \setminus E_\pm) \le c\eta^2\frac{\alpha}{\alpha_0^4}, \tag{3.2}$$

where c is a constant.

Note that in our problem the number λ_0 is real or pure imaginary, since $\pm\lambda_0$ are the eigenvalues of a Hamiltonian matrix A. Here we are only interested in the pure imaginary case since it is simpler when λ_0 is a real number.

Proof. Let

$$O_k = \left\{\epsilon \in (0,\eta) \ \Big| \ |\varphi_k(\epsilon)| < \frac{\alpha}{|k|^{\tau'}}\right\}.$$

Now suppose $\epsilon \in (0,\eta)$. If

$$0 < \epsilon \le \frac{\alpha_0^2}{2|k|^{2\tau}},$$

by (1.6) it follows that

$$\big||\langle \omega, k\rangle|^2 + 4\lambda_0^2\big| \ge \frac{\alpha_0^2}{|k|^{2\tau}} \text{ and so } |\phi_k(\epsilon)| \ge \frac{\alpha_0^2}{2|k|^{2\tau}} \ge \frac{\alpha}{|k|^{\tau'}}. \tag{3.3}$$

Thus we only need to consider the ϵ belonging to $(\frac{\alpha_0^2}{2|k|^{2\tau}}, \eta)$. Then we have

$$\mathrm{meas}(O_k) \leq \frac{2\alpha}{|k|^{\tau'}} \leq (\frac{2|k|^{2\tau}\eta}{\alpha_0^2})^2 \frac{2\alpha}{|k|^{\tau'}} \leq 8 \cdot \eta^2 \cdot \frac{\alpha}{\alpha_0^4} \cdot \frac{1}{|k|^{\tau'-4\tau}}.$$

Since $\tau' - 4\tau > r$, it follows that

$$\mathrm{meas}((0,\eta) \setminus E_+) \leq \sum_{k\neq 0} \mathrm{meas}(O_k) \leq c\eta^2 \frac{\alpha}{\alpha_0^4},$$

where $c = 8\sum_{k\neq 0} \frac{1}{|k|^{\tau'-4\tau}}$.

E_- can be considered similarly and so we omit the details. □

Now we prove Theorem 1.1. At first, let $\alpha = \alpha_0^2/2, \tau' > 4\tau + r$. By assumption, there is a constant $M > 0$ such that $\|\partial_\epsilon Q\|_s \leq M$. Let $\gamma_0 > 0$ be the sufficiently small number as given in Lemma 2.1. By assumption, when δ is sufficiently small, we have $\|Q\|_s \leq \gamma_0$ for all $\epsilon \in [0,\delta]$. Moreover, we have $Q(t,0) \equiv 0$. Below we prove $\mathrm{meas}(\Pi_\infty) > 0$. Note $\det(A) = -\lambda_0^2$. Let $\det(A_\infty(\epsilon)) = -\lambda_0^2 + \mu(\epsilon)$. Noting that $A_\infty(0) = A$, then it is easy to see $\mu(\epsilon)$ is differentiable on $[0,\delta]$ with $\mu(0) = 0$. Let

$$\eta_0 = 4 \max_{0\leq\epsilon\leq\delta} |\mu(\epsilon)| = 4|\mu(\epsilon_0)|,\ 0 \leq \epsilon_0 \leq \delta.$$

Without loss of generality, we assume $\eta_0 \neq 0$, $\epsilon_0 \neq 0$ and $\eta_0 = 4\mu(\epsilon_0)$. Since μ is continuous, we have

$$\left\{4\mu(\epsilon) \ \mid \epsilon \in [0,\epsilon_0]\right\} \supset [0,\eta_0].$$

Recall $\alpha = \alpha_0^2/2$ and $\tau' > 4\tau + r$. By Lemma 3.1, $E_{\eta_0} = E_+$ with $\eta = \eta_0$ is non-empty and $\mathrm{meas}(E_{\eta_0}) > 0$. Let

$$\Pi_{\epsilon_0} = \{\epsilon \in [0,\epsilon_0] \ \mid 4\mu(\epsilon) \in E_{\eta_0}\}.$$

Then $4\mu(\Pi_{\epsilon_0}) = E_{\eta_0}$ and it follows that $\mathrm{meas}(\mu(\Pi_{\epsilon_0})) > 0$.

Since μ is differentiable and $|\mu'(\epsilon)| \leq c,\ \forall \epsilon \in [0,\delta]$, we have

$$\mathrm{meas}(\mu(\Pi_{\epsilon_0})) \leq \int_{\Pi_{\epsilon_0}} |\mu'(\epsilon)|\, d\epsilon \leq c \cdot \mathrm{meas}(\Pi_{\epsilon_0}),$$

hence $\mathrm{meas}(\Pi_{\epsilon_0}) > 0$. Obviously, $\Pi_{\epsilon_0} \subset \Pi_\infty$ and so $\mathrm{meas}(\Pi_\infty) > 0$. □

3.2. **Continuous case.** By the above discussion of the differentiable case, we have $4\mu(\Pi_{\epsilon_0}) = E_{\eta_0}$. Because the mapping $\mu : \Pi_{\epsilon_0} \to \frac{1}{4}E_{\eta_0}$ is continuous and surjective, the cardinality of the set Π_{ϵ_0} is not less than that of the set E_{η_0}. Since $\mathrm{meas}(E_{\eta_0}) > 0$, both E_{η_0} and Π_{ϵ_0} have cardinality number of the continuum. Noting that $\Pi_{\epsilon_0} \subset \Pi_\infty$, Π_∞ also has cardinality number of the continuum.

4. Proof of Lemma 2.1

In the same way as in [19], we can make a formal convergent KAM iteration without any non-degeneracy condition. Of course, the KAM iteration only makes sense for such parameters that the non-resonance conditions hold.

4.1. **Outline of the KAM step.** In the same way as in [11, 12, 17], we want to construct a linear quasi-periodic transformation of the form $x = e^{P(t,\epsilon)}x_+$, which transforms the system (1.5) into

$$\dot{x}_+ = (A - \dot{P} - PA + AP + Q + R)x_+,$$

where

$$\begin{aligned} R &= -PAP + (e^{-P} - I + P)Ae^P + (I - P)A(e^P - I - P) + (e^{-P} - I)Qe^P \\ &\quad + Q(e^P - I) - (e^{-P} - I)\dot{P} - e^{-P}\frac{d}{dt}(e^P - I - P). \end{aligned}$$

Let Q^K be the truncation of Q, which is defined by $Q^K = \sum_{|k|\le K} Q_k e^{\mathrm{i}\langle k,\omega\rangle t}$. Let $A_+ = A + [Q]$. If P solves the homological equation

$$\dot{P} + PA - AP = Q^K - [Q], \tag{4.1}$$

then the transformed system becomes

$$\dot{x}_+ = \big(A_+(\epsilon) + Q_+(t,\epsilon)\big)x_+, \tag{4.2}$$

where $A_+ = A + [Q]$, $Q_+ = (Q - Q^K) + R$.

Thus, if Q is sufficiently small, the new perturbation Q_+ is much smaller. If this step can iterate infinitely many times, then the new perturbation becomes smaller and smaller and the system finally approaches to a constant one.

4.2. **Solving the linear homological equation.** Solving the linear homological equation (4.1) is most important for our KAM step, from which arises the notorious small divisor problem.

Let $A \in sl(2,\mathbb{R})$ and define a linear operator $LX = XA - AX$, where $X = (X_{ij})_{1\le i,j\le 2}$. Correspond the matrix X to the 4-dimensional vector

$$(X)_v = (X_{11}, X_{12}, X_{21}, X_{22});$$

then L is a linear operator from $\mathbb{C}^4$ to itself. Denote it by $L : (X)_v \to (LX)_v$. Below, without confusion we use $\mathcal{L}$ to indicate the corresponding matrix of the linear operator L . Thus the linear equation is equivalent to

$$(\dot{P})_v + \mathcal{L}(P)_v = (Q^K)_v - ([Q])_v. \tag{4.3}$$

Since $A \in sl(2,\mathbb{R})$, it has two eigenvalues $\pm\sqrt{-\det(A)}$. Then the matrix $\mathcal{L}$ has four eigenvalues $0,\ 0,\ 2\sqrt{-\det(A)},\ -2\sqrt{-\det(A)}$, with characteristic polynomial

$$\det(\mathcal{L} - \lambda I_4) = \lambda^2(\lambda^2 + 4\det(A)),$$

where I_4 denotes the 4×4 unit matrix.

Let P_k and Q_k be the Fourier coefficients of P and Q, respectively. From (4.3) it follows that

$$(\mathrm{i}\langle\omega,k\rangle I_4 + \mathcal{L})(P_k)_v = (Q_k)_v, \quad \mathrm{i} = \sqrt{-1},\ 0 < |k| \le K.$$

It is easy to see that

$$\det(\mathrm{i}\langle\omega,\mathrm{k}\rangle\mathrm{I}_4 + \mathcal{L}) = |\langle\omega,\mathrm{k}\rangle|^2\big(|\langle\omega,\mathrm{k}\rangle|^2 - 4\det(\mathrm{A})\big). \tag{4.4}$$

If the non-resonance condition

$$\langle\omega,k\rangle \ne 0, \quad |\langle\omega,k\rangle|^2 - 4\det(A) \ne 0, \quad \forall k \ne 0, \tag{4.5}$$

holds, then the coefficient matrix $\mathrm{i}\langle\omega,\mathrm{k}\rangle\mathrm{I}_4+\mathcal{L}$ is non-singular and so we solve P_k with

$$(P_k)_v=\big(\mathrm{i}\langle\omega,k\rangle\ I_4+\mathcal{L}\big)^{-1}\ (Q_k)_v.$$

Moreover, $A(\epsilon)$ and $Q(t,\epsilon)$ both belong to $sl(2,\mathbb{R})$, and so does the solution $P(t,\epsilon)$, thus $e^{P(t,\epsilon)}\in SL(2,\mathbb{R})$ and $x=e^{P(t,\epsilon)}x_+$ is a symplectic transformation.

4.3. **Small divisors extension.** Let $A(\epsilon)\in sl(2,\mathbb{R})$ for $\epsilon\in[0,\delta]$. By the above discussion, our small divisors are given by (4.5). By the assumption (1.6), we only need to consider

$$|\langle\omega,k\rangle|^2-4\det(A(\epsilon)).$$

The non-resonance condition is given by

$$\big|\ |\langle\omega,k\rangle|^2-4\det(A(\epsilon))\big|\geq\frac{\alpha}{|k|^{\tau'}},\ \forall k\neq 0.$$

Now take a $C^\infty(\mathbb{R})$-smooth function $\varphi(t)$ such that

$$\varphi(t)=\begin{cases}0, & |t|\leq\dfrac{1}{2},\\ 1, & |t|\geq 1.\end{cases}$$

For $h>0$, let $\varphi_h(t)=\varphi(t/h)$. Then $\varphi_h(t)\in C^\infty(\mathbb{R})$ with

$$|\frac{d^\ell}{dt^\ell}\varphi_h(t)|\leq c_\ell/h^l,\quad\forall t\in\mathbb{R},\quad\forall\ell\geq 1,\tag{4.6}$$

where c_ℓ is a constant depending on ℓ.

Let

$$h=\frac{\alpha}{|k|^{\tau'}},\quad t_k(\epsilon)=|\langle\omega,k\rangle|^2-4\det(A(\epsilon)),\quad g_k(\epsilon)=\frac{\varphi_h(t_k(\epsilon))}{t_k(\epsilon)}.$$

Define

$$\Pi=\big\{\epsilon\in[0,\delta]\ \big|\ \big|\ |\langle\omega,k\rangle|^2-4\det(A(\epsilon))\big|\geq\frac{\alpha}{|k|^{\tau'}},\ \forall 0<|k|\leq K\big\}.$$

Then $g_k(\epsilon)$ is well defined on $[0,\delta]$ and

$$g_k(\epsilon)=\frac{1}{|\langle\omega,k\rangle|^2-4\det(A(\epsilon))},\quad\forall\epsilon\in\Pi.$$

Note that even if $\Pi=\emptyset$, the above extension is still valid. Furthermore, if $\det(A(\epsilon))\in C^m[0,\delta]$, by (4.6) it follows that $g_k(\epsilon)\in C^m[0,\delta]$ with

$$\big|\frac{d^m g_k}{d\epsilon^m}(\epsilon)\big|\leq ch^{-m-1},\quad\epsilon\in[0,\delta],$$

where c depends on $\|\det(A)\|_{C^m[0,\delta]}$.

Let $D=\mathrm{i}\langle\omega,k\rangle\ I_4+\mathcal{L}$. If $\det(D(\epsilon))\neq 0$, then we have the inverse matrix

$$D^{-1}(\epsilon)=(D(\epsilon))^*/\det(D(\epsilon)).$$

As usual, we should avoid some ϵ such that $\det(D(\epsilon))$ is too small. For this we define $\tilde{D}(\epsilon)=\frac{g_k(\epsilon)}{|\langle\omega,k\rangle|^2}(D(\epsilon))^*$, and then $\tilde{D}(\epsilon)$ is well defined for all ϵ. In particular, for $\epsilon\in\Pi$ we have $\tilde{D}(\epsilon)=D^{-1}(\epsilon)$.

Below we estimate $\tilde{D}$. Suppose $\sup_{0\leq\epsilon\leq\delta}\|A(\epsilon)\|\leq c_1$. It is easy to see that there exists a sufficiently large $c_2>c_1$ such that if $|\langle\omega,k\rangle|\geq c_2$, $\|\tilde{D}(\epsilon)\|=\|D^{-1}(\epsilon)\|\leq c$, where c is independent of k. Otherwise, if $|\langle\omega,k\rangle|<c_2$, we have

$\|(D(\epsilon))^*\| \leq c$ and so $\|D^{-1}(\epsilon)\| \leq \frac{c}{|\det(D(\epsilon))|}$, where c only depends on c_2. Thus, if $A \in C^1\big([0,\delta], sl(2,\mathbb{R})\big)$, we have $\tilde{D}(\epsilon) \in C^1[0,\delta]$ with

$$\|\tilde{D}(\epsilon)\| \leq \frac{c|k|^{\tau'+2\tau}}{\alpha_0^2\alpha}, \quad \|\tilde{D}'(\epsilon)\| \leq \frac{c|k|^{2\tau'+2\tau}}{\alpha_0^2\alpha^2}, \tag{4.7}$$

where c is independent of k. Note that the constant c in (4.7) can depend on c_1 and c_2, however, c_1 and c_2 are independent of KAM steps, and so is c.

Then for $\epsilon \in \Pi$ we solve the linear homological equation with

$$(P_k)_v = \big(\mathrm{i}\langle\omega,k\rangle\ I_4 + \mathcal{L}\big)^{-1}\ (Q_k)_v.$$

To extend P_k, we let

$$(\tilde{P}_k)_v = \tilde{D}(\epsilon)\ (Q_k)_v. \tag{4.8}$$

It is easy to see that $\{\tilde{P}_k\}$ are well defined on $[0,\delta]$, moreover, $\tilde{P}_k = P_k$ for $\epsilon \in \Pi$.

4.4. **KAM iteration.** Below we make our KAM iteration, which consists of several parts.

A. One KAM step. Consider the following quasi-periodic system:

$$\dot{x} = \big(A(\epsilon) + Q(t,\epsilon)\big)x,\ \ \epsilon \in [0,\delta]. \tag{4.9}$$

Let $A \in C^m\big([0,\delta], sl(2,\mathbb{R})\big)$ with $\|A\|_{C^m[0,\delta]} \leq c$. Suppose $Q \in C^{a,m}\big(\mathbb{T}^r_s \times U_\delta, sl(2,\mathbb{R})\big)$ with $Q(t,0) \equiv 0$, $\|Q\|_s \leq \gamma$ and $\|\partial_\epsilon Q\|_s \leq M$ for all $\epsilon \in [0,\delta]$ if $m=1$.

We first have a transformation $x = e^{P(t,\epsilon)}x_+$, where P satisfies (4.1). Let

$$\Pi = \big\{\epsilon \in [0,\delta]\ \ \big|\ \ \big|\,|\langle\omega,k\rangle|^2 - 4\det(A(\epsilon))\big| \geq \frac{\tilde{\alpha}}{|k|^{\tau'}},\ \forall 0 < |k| \leq K\big\}. \tag{4.10}$$

By (4.7) and (4.8), to extend P in ϵ from Π to $[0,\delta]$, we have $\tilde{P}_k$ with

$$\|\tilde{P}_k\| \leq \frac{c|k|^{\tau'+2\tau}}{\alpha_0^2\tilde{\alpha}}\|Q_k\|,\ \forall 0 < |k| \leq K, \forall \epsilon \in [0,\delta].$$

Let $\tilde{P}(t,\epsilon) = \sum_{|k|\leq K} \tilde{P}_k e^{\mathrm{i}\langle k,\omega\rangle t}$. Then for $0 < \rho < s$ and $\epsilon \in [0,\delta]$ it follows that

$$\|\tilde{P}(\cdot,\epsilon)\|_{s-\rho} \leq \frac{c\gamma}{\alpha_0^2\tilde{\alpha}\rho^v} \triangleq \Gamma,\ \ v = \tau' + 2\tau + r.$$

Moreover, it is easy to see that $\tilde{P}(t,0) \equiv 0$. In the differentiable case where $m=1$, by $\|\partial_\epsilon Q\|_s \leq M$ and (4.7), we have

$$\big\|\partial_\epsilon \tilde{P}(\cdot,\epsilon)\big\|_{s-\rho} \leq \frac{cM}{\alpha_0^2\tilde{\alpha}\rho^v} + \frac{c\gamma}{\alpha_0^2\tilde{\alpha}^2\rho^{v+\tau'}} \triangleq \Gamma',\ \ \epsilon \in [0,\delta].$$

Let

$$\begin{aligned} Q_+ \ \ &= \ \ (Q - Q^K) - \tilde{P}A\tilde{P} + (e^{-\tilde{P}} - I + \tilde{P})Ae^{\tilde{P}} + (I - \tilde{P})A(e^{\tilde{P}} - I - \tilde{P}) \\ &\quad + (e^{-\tilde{P}} - I)Qe^{\tilde{P}} + Q(e^{\tilde{P}} - I) - (e^{-\tilde{P}} - I)\dot{\tilde{P}} - e^{-\tilde{P}}\frac{d}{dt}(e^{\tilde{P}} - I - \tilde{P}). \end{aligned}$$

Then we have $Q_+ \in C^{a,m}\big(\mathbb{T}^r_{s-\rho} \times U_\delta, sl(2,\mathbb{R})\big)$. Moreover, we have $Q_+(t,0) \equiv 0$.

Note that for $\epsilon \in \Pi$, we have $\tilde{P} = P$; then the transformation $x = e^{\tilde{P}(t,\epsilon)}x_+$ changes the system (4.9) to

$$\dot{x}_+ = \big(A_+(t,\epsilon) + Q_+(t,\epsilon)\big)x_+.$$

B. Estimate for the new perturbation. Suppose $\|\tilde{P}(\cdot,\epsilon)\|_{s-\rho} \le \Gamma \le 1$. Note that $e^{\tilde{P}} = \sum_{n\ge 0} \frac{1}{n!}\tilde{P}^n$. It is easy to see $\big\|e^{\pm\tilde{P}(\cdot,\epsilon)}\big\|_{s_+} \le e$, where $s_+ = s-\rho$. Moreover, if $m=1$, we have

$$\big\|\partial_\epsilon e^{\pm\tilde{P}(\cdot,\epsilon)}\big\|_{s_+} \le e\Gamma'.$$

Let $K>0$ such that $e^{-K\rho} = \gamma/\alpha_0^2\tilde{\alpha}\rho^v$. If $\Gamma \le 1$, by the standard KAM estimate it follows that

$$\|Q_+(\cdot,\epsilon)\|_{s_+} \le \frac{c\gamma^2}{\alpha_0^2\tilde{\alpha}\rho^v} \triangleq \gamma_+,$$

$$\big\|\partial_\epsilon Q_+(\cdot,\epsilon)\big\|_{s_+} \le M_+ \triangleq \frac{cM\gamma}{\alpha_0^4\tilde{\alpha}^2\rho^{2v}} + \frac{c\gamma^2}{\alpha_0^4\tilde{\alpha}^3\rho^{2v+\tau'}} \text{ for } m=1.$$

C. KAM Iteration. Below we choose some suitable parameter sequences such that the KAM step can be iterated infinitely many times. Recall the assumption (2.2). Let $A_0 = A$ and $Q_0 = Q$. At the initial step $n=0$, we set

$$s_0 = s,\ \rho_0 = s/4,\ \tilde{\alpha}_0 = \alpha,\ \gamma_0 = \gamma,\ M_0 = M,$$

$$\Gamma_0 = \frac{c\gamma_0}{\alpha_0^2\tilde{\alpha}_0\rho_0^v},\quad \Gamma_0' = \frac{cM_0}{\alpha_0^2\tilde{\alpha}_0\rho_0^v} + \frac{c\gamma_0}{\alpha_0^2\tilde{\alpha}_0^2\rho_0^{v+\tau'}}.$$

If these parameters are well defined at the $(n-1)$-th step, at the n-th step, let

$$s_n = s_{n-1} - \rho_{n-1},\ \rho_n = \rho_{n-1}/2\,,\ \tilde{\alpha}_n = \alpha(1-1/2^n),\ e^{-K_n\rho_n} = \gamma_n/\alpha_0^2\tilde{\alpha}_n\rho_n^v,$$

$$\gamma_n = \frac{c\gamma_{n-1}^2}{\alpha_0^2\tilde{\alpha}_{n-1}\rho_{n-1}^v},\quad M_n = \frac{c\gamma_{n-1}M_{n-1}}{\alpha_0^4\tilde{\alpha}_{n-1}^2\rho_{n-1}^{2v}} + \frac{c\gamma_{n-1}^2}{\alpha_0^4\tilde{\alpha}_{n-1}^3\rho_{n-1}^{2v+\tau'}}, \tag{4.11}$$

$$\Gamma_n = \frac{c\gamma_n}{\alpha_0^2\tilde{\alpha}_n\rho_n^v},\quad \Gamma_n' = \frac{cM_n}{\alpha_0^2\tilde{\alpha}_n\rho_n^v} + \frac{c\gamma_n}{\alpha_0^2\tilde{\alpha}_n^2\rho_n^{v+\tau'}}. \tag{4.12}$$

Note that the constants c in (4.11) and (4.12) are independent of n.

Define Π_n by (4.10) with A_n, K_n and $\tilde{\alpha}_n$ in place of A, K and $\tilde{\alpha}$, respectively. By the above discussion, we have $\tilde{P}_n(t,\epsilon) \in C^{a,m}\big(\mathbb{T}^r_{s_{n+1}} \times U_\delta, sl(2,\mathbb{R})\big)$ with

$$\|\tilde{P}_n(\cdot,\epsilon)\|_{s_{n+1}} \le \Gamma_n,\quad \big\|e^{\pm\tilde{P}_n(\cdot,\epsilon)}\big\|_{s_{n+1}} \le e.$$

Moreover, if $m=1$, we have

$$\big\|\partial_\epsilon\tilde{P}_n(\cdot,\epsilon)\big\|_{s_{n+1}} \le \Gamma_n',\quad \big\|\partial_\epsilon e^{\pm\tilde{P}_n(\cdot,\epsilon)}\big\|_{s_{n+1}} \le e\Gamma_n'.$$

Let

$$\Phi_n = e^{\tilde{P}_0}\ e^{\tilde{P}_1}\cdots e^{\tilde{P}_{n-1}} \text{ with } \Phi_0 = I.$$

If $\Gamma_k \le 1$, for $\epsilon \in \bigcap_{k=0}^{n-1}\Pi_k$, by the transformation $x = \Phi_n(t,\epsilon)x_n$ with $x_0 = x$, the system (2.1) becomes

$$\dot{x}_n = \big(A_n(\epsilon) + Q_n(t,\epsilon)\big)x_n, \tag{4.13}$$

where $A_n = A_{n-1} + [Q_{n-1}]$. Moreover, $Q_n(t,\epsilon) \in C^{a,m}\big(\mathbb{T}^r_{s_n} \times U_\delta, sl(2,\mathbb{R})\big)$ with

$$\|Q_n(\cdot,\epsilon)\|_{s_n} \le \gamma_n,\quad \|[Q_n](\epsilon)\| \le \gamma_n. \tag{4.14}$$

Noting that $Q_0(t,0) = Q(t,0) \equiv 0$, we have $Q_n(t,0) \equiv 0$ and so $A_n(0) = A_0 = A$. Furthermore, if $m=1$, we have

$$\big\|\partial_\epsilon Q_n(\cdot,\epsilon)\big\|_{s_n} \le M_n,\quad \Big\|\frac{d}{d\epsilon}[Q_n](\epsilon)\Big\| \le M_n. \tag{4.15}$$

D. Convergence. Now we first verify $\Gamma_n \leq 1$. By (4.11), it follows that $c\Gamma_n = (c\Gamma_{n-1})^2 \leq (c\Gamma_0)^{2^n}$. It is easy to see that if γ is sufficiently small such that $c\Gamma_0 \leq 1$, we have $\Gamma_n \leq 1$.

Now we claim that if γ_0 is sufficiently small, we have $M_n \leq M_0, \forall n \geq 1$. In fact, we can choose γ_0 sufficiently small such that

$$\frac{c\gamma_{n-1}}{\alpha_0^4\tilde{\alpha}_{n-1}^2\rho_{n-1}^{2v}} = \frac{c\Gamma_{n-1}}{\alpha_0^2\tilde{\alpha}_{n-1}\rho_{n-1}^{v}} \leq 1/2, \quad \frac{c\gamma_{n-1}^2}{\alpha_0^4\tilde{\alpha}_{n-1}^3\rho_{n-1}^{2v+\tau'}} = \frac{c\Gamma_{n-1}^2}{\tilde{\alpha}_{n-1}\rho_{n-1}^{\tau'}} \leq M_0/2;$$

thus we have $M_n \leq M_{n-1}/2 + M_0/2$. Inductively it follows that $M_n \leq M_0, \forall n \geq 1$. Therefore, we have

$$M_n \leq \frac{c\Gamma_{n-1}}{\alpha_0^2\tilde{\alpha}_{n-1}\rho_{n-1}^{v}}.$$

By the estimate of Γ_n, it follows that for $\epsilon \in [0,\delta]$, $\Phi_n(t,\epsilon)$ is convergent to $\Phi(t,\epsilon)$ as $n \to \infty$. In the same way as [11] it follows that $\|\Phi(\cdot,\epsilon) - I\|_{\frac{s}{2}} \leq c\Gamma_0, \forall \epsilon \in [0,\delta]$.

Now we consider $\partial_\epsilon \Phi(t,\epsilon)$. By definition we have $\Phi_n = \Phi_{n-1}e^{\tilde{P}_{n-1}}$. It follows that

$$\partial_\epsilon\Phi_n - \partial_\epsilon\Phi_{n-1} = \partial_\epsilon\Phi_{n-1}(e^{\tilde{P}_{n-1}} - 1) + \partial_\epsilon\Phi_{n-1}\partial_\epsilon e^{\tilde{P}_{n-1}}.$$

It is easy to see that $\|\partial_\epsilon\Phi_{n-1}(\cdot,\epsilon)\|_{s_n} \leq c$ and $\Gamma_n \leq \Gamma_n', \forall n \geq 0$. Thus for $\epsilon \in [0,\delta]$ we have

$$\|\partial_\epsilon\Phi_n(\cdot,\epsilon) - \partial_\epsilon\Phi_{n-1}(\cdot,\epsilon)\|_{s_n} \leq c\Gamma_{n-1}'$$

and so

$$\|\partial_\epsilon\Phi(\cdot,\epsilon)\|_{s/2} \leq c\sum_{n\geq 0}\Gamma_n' \leq \frac{cM}{\alpha_0^2\alpha s^v}.$$

Thus we prove the conclusion (i).

Since $A_n = A + \sum_{j=0}^{n-1}[Q_j]$, by (4.14), (4.15) and the above estimates for Γ_n and M_n, if γ_0 is sufficiently small, we have that $\{A_n(\epsilon)\}$ is convergent in $C^1[0,\delta]$.

By (4.14) it is easy to see

$$\|A_n(\epsilon) - A(\epsilon)\| \leq \sum_{j=0}^{n-1}\gamma_j \leq 2\gamma_0. \tag{4.16}$$

Moreover, it is easy to see that for all $j \geq 0$, $[Q_j](0) = 0$, thus $A_n(0) = A,\ \forall n \geq 0$. Furthermore, if $m = 1$, by (4.15) it follows that

$$\|\frac{d}{d\epsilon}A_n(\epsilon) - \frac{d}{d\epsilon}A(\epsilon)\| \leq \sum_{j=0}^{n-1}M_j \leq 2M_0. \tag{4.17}$$

Let $A_n \to A_\infty$ as $n \to \infty$. Then (4.16) and (4.17) yield $A_\infty(0) = A$, $\|A_\infty(\epsilon) - A\| \leq 2\gamma_0$ and $\|A_\infty'(\epsilon)\| \leq 2M_0$ when $m = 1$. Thus the conclusion (ii) holds.

E. Reducibility. It is easy to see

$$|\det(A_n(\epsilon)) - \det(A_{n+1}(\epsilon))| \leq c\gamma_n.$$

Obviously, if $c\gamma_n \leq \alpha/K_n^{\tau'}2^{n+3}$, then $\Pi_{n+1} \subset \Pi_n$. Let $\Pi_* = \bigcap_{n\geq 0}\Pi_n$. It is easy to see that for $\epsilon \in \Pi_*$, the KAM iteration is convergent and the transformation $x = \Phi y$ changes the system (2.1) to $\dot{y} = A_\infty(\epsilon)y$; thus the system is reducible. Below we prove $\Pi_\infty \subset \Pi_*$, which implies the conclusion (iii).

Obviously, it is enough to verify that $\Pi_\infty \subset \Pi_n,\ \forall n \geq 1$. Suppose $\epsilon \in \Pi_\infty$. Now that

$$\|A_n(\epsilon) - A_\infty(\epsilon)\| \leq \sum_{j=n}^{\infty} \gamma_j \leq 2\gamma_n,$$

then

$$|\det(A_n(\epsilon)) - \det(A_\infty(\epsilon))| \leq c\gamma_n.$$

In the same way as the above, if $c\gamma_n \leq \frac{\alpha}{2^{n+2}K_n^{\tau'}}$, then for $|k| \leq K_n$ we have

$$\left||\langle\omega, k\rangle|^2 - 4\det(A_n(\epsilon))\right| \geq \frac{\alpha}{|k|^{\tau'}} - 4|\det(A_n(\epsilon)) - \det(A_\infty(\epsilon))| \geq \frac{\tilde{\alpha}_n}{|k|^{\tau'}},$$

therefore, $\epsilon \in \Pi_n$.

It remains to verify that if γ_0 is sufficiently small, there holds $c\gamma_n \leq \frac{\alpha}{2^{n+3}K_n^{\tau'}}$. By the definition of K_n and noting that $v = \tau' + 2\tau + r$, we have $K_n\rho_n = -\ln\Gamma_n$ and

$$\frac{2^{n+3}K_n^{\tau'}c\gamma_n}{\alpha} = \frac{c\alpha_0^2\tilde{\alpha}_n\rho_n^{v-\tau'}\Gamma_n(-\ln\Gamma_n)^{\tau'}2^{n+3}}{\alpha} \leq c\alpha_0^2\rho_0^{v-\tau'}\Gamma_n(-\ln\Gamma_n)^{\tau'}.$$

Thus, if γ_0 is sufficiently small, then Γ_n is sufficiently small such that $c\gamma_n \cdot 2^{n+3}K_n^{\tau'} \leq \alpha,\quad \forall n \geq 1$. Thus we finish the proof of Lemma 2.1. $\square$

Acknowledgement

The authors are very grateful for the reviewer's valuable suggestions, which have helped to improve this paper greatly.

References

[1] Artur Avila, Bassam Fayad, and Raphaël Krikorian, *A KAM scheme for* $\mathrm{SL}(2,\mathbb{R})$ *cocycles with Liouvillean frequencies*, Geom. Funct. Anal. **21** (2011), no. 5, 1001–1019, DOI 10.1007/s00039-011-0135-6. MR2846380 (2012i:37102)

[2] E. Dingaburg and Ya Sinai, *The one-dimensional Schrödinger equation with a quasi-periodic potential*, Funct. Anal. Appl. **9** (1975), 279-289.

[3] L. H. Eliasson, *Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation*, Comm. Math. Phys. **146** (1992), no. 3, 447–482. MR1167299 (93d:34141)

[4] L. H. Eliasson, *Almost reducibility of linear quasi-periodic systems*, Smooth ergodic theory and its applications (Seattle, WA, 1999), Proc. Sympos. Pure Math., vol. 69, Amer. Math. Soc., Providence, RI, 2001, pp. 679–705, DOI 10.1090/pspum/069/1858550. MR1858550 (2003a:34064)

[5] Håkan L. Eliasson and Sergei B. Kuksin, *On reducibility of Schrödinger equations with quasiperiodic in time potentials*, Comm. Math. Phys. **286** (2009), no. 1, 125–135, DOI 10.1007/s00220-008-0683-2. MR2470926 (2009j:35286)

[6] Bassam Fayad and Raphaël Krikorian, *Herman's last geometric theorem* (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 2, 193–219. MR2518076 (2011a:37092)

[7] Hai-Long Her and Jiangong You, *Full measure reducibility for generic one-parameter family of quasi-periodic linear systems*, J. Dynam. Differential Equations **20** (2008), no. 4, 831–866, DOI 10.1007/s10884-008-9113-6. MR2448214 (2009k:34090)

[8] Xuanji Hou and Jiangong You, *Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems*, Invent. Math. **190** (2012), no. 1, 209–260, DOI 10.1007/s00222-012-0379-2. MR2969277

[9] Svetlana Jitomirskaya, *Ergodic Schrödinger operators (on one foot)*, Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday, Proc. Sympos. Pure Math., vol. 76, Amer. Math. Soc., Providence, RI, 2007, pp. 613–647, DOI 10.1090/pspum/076.2/2307750. MR2307750 (2008g:82055)

[10] Russell A. Johnson and George R. Sell, *Smoothness of spectral subbundles and reducibility of quasiperiodic linear differential systems*, J. Differential Equations **41** (1981), no. 2, 262–288, DOI 10.1016/0022-0396(81)90062-0. MR630994 (83a:58072)

[11] Àngel Jorba and Carles Simó, *On the reducibility of linear differential equations with quasiperiodic coefficients*, J. Differential Equations **98** (1992), no. 1, 111–124, DOI 10.1016/0022-0396(92)90107-X. MR1168974 (94f:34024)

[12] Àngel Jorba and Carles Simó, *On quasi-periodic perturbations of elliptic equilibrium points*, SIAM J. Math. Anal. **27** (1996), no. 6, 1704–1737, DOI 10.1137/S0036141094276913. MR1416515 (97j:34055)

[13] Jürgen Moser and Jürgen Pöschel, *An extension of a result by Dinaburg and Sinaĭ on quasiperiodic potentials*, Comment. Math. Helv. **59** (1984), no. 1, 39–85, DOI 10.1007/BF02566337. MR743943 (85m:34064)

[14] Joaquim Puig i Sadurni, *Reducibility of Quasi-Periodic Skew-Products and the Spectrum of Schrödinger Operators*, Programa de doctrat de Mathemátiques Bienni, (1999-2001).

[15] Helmut Rüssmann, *On the one-dimensional Schrödinger equation with a quasiperiodic potential*, Nonlinear dynamics (Internat. Conf., New York, 1979), Ann. New York Acad. Sci., vol. 357, New York Acad. Sci., New York, 1980, pp. 90–107. MR612810 (83f:34032)

[16] Helmut Rüssmann, *Convergent transformations into a normal form in analytic Hamiltonian systems with two degrees of freedom on the zero energy surface near degenerate elliptic singularities*, Ergodic Theory Dynam. Systems **24** (2004), no. 5, 1787–1832, DOI 10.1017/S0143385703000774. MR2104603 (2005k:37133)

[17] Junxiang Xu and Qin Zheng, *On the reducibility of linear differential equations with quasiperiodic coefficients which are degenerate*, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1445–1451, DOI 10.1090/S0002-9939-98-04523-7. MR1458272 (98j:34086)

[18] Junxiang Xu and Xuezhu Lu, *On the reducibility of two-dimensional linear quasi-periodic systems with small parameter*, Ergodic Theory Dynam. Systems **35** (2015), no. 7, 2334–2352, DOI 10.1017/etds.2014.31. MR3394121

[19] Junxiang Xu and Xuezhu Lu, *General KAM theorems and their applications to invariant tori with prescribed frequencies*, Regul. Chaotic Dyn. **21** (2016), no. 1, 107–125, DOI 10.1134/S1560354716010068. MR3457078

[20] Qi Zhou and Jing Wang, *Reducibility results for quasiperiodic cocycles with Liouvillean frequency*, J. Dynam. Differential Equations **24** (2012), no. 1, 61–83, DOI 10.1007/s10884-011-9235-0. MR2890337

Department of Mathematics, Southeast University, Nanjing 210096, People's Republic of China

E-mail address: xujun@seu.edu.cn

Department of Mathematics, Southeast University, Nanjing 210096, People's Republic of China

E-mail address: wangkun880304@163.com

Department of Mathematics, Nanjing Forestry University, Nanjing 210037, People's Republic of China

E-mail address: zhumin@njfu.edu.cn