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GLOBAL AND BLOW UP SOLUTIONS TO CROSS DIFFUSION SYSTEMS ON 3D DOMAINS

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Abstract. Necessary and sufficient conditions for global existence of classical solutions to a class of cross diffusion systems on 3-dimensional domains are studied. Examples of blow up solutions are also given.

1. Introduction and main results

We consider in this paper the following cross diffusion system of m equations:

$$\left\{\begin{array}{ll} u_t = \operatorname{div}(A(x,t)Du) + f(x,t,u), & \text{in } \Omega \times (0,T), \\ u(x,t) = 0, & \text{on } \partial\Omega_1 \times [0,T), \\ Du \cdot \nu = 0, & \text{on } \partial\Omega_2 \times [0,T), \\ u(x,0) = U_0(x), & x \in \Omega. \end{array}\right. \tag{1.1}$$

Here, Ω is a smooth and bounded domain in $\mathbb{R}^n$ with $n \leq 3$ and $T > 0$; the boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ with $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$. The outward pointing unit normal to the boundary $\partial\Omega$ is denoted by ν. The solution $u = (u_i)_{i=1}^m$ is a vector valued function on $\Omega \times [0,T)$ and takes the given vector valued function U_0 as its initial value. Notation u_t, Du denote respectively the temporal, spatial derivatives of u. Thus, the components of u assume the Dirichlet and Neumann boundary conditions on the disjoint portions $\partial\Omega_1$ and $\partial\Omega_2$ of $\partial\Omega$. The matrix $A(x,t) = (a_{ij}(x,t))$ is a *full* matrix $m \times m$ whose entries are smooth functions a_{ij} on $\Omega \times (0,\infty)$ and $f(x,t,u)$ is a vector valued function on $\Omega \times (0,\infty) \times \mathbb{R}^m$.

In applications to biology and ecology phenomena, the unknown $u = (u_1, \ldots, u_m)$ is the vector of concentrations or population densities, $A(x,t)Du$ is the flux vector and $f(x,t,u) = (f_i(x,t,u))_{i=1}^m$ with $f_i(x,t,u)$ being the production/reaction rate for the ith component u_i. There is a large body in literature devoted to the Lotka-Volterra systems where only random diffusion is considered so that $A(x,t)$ is a *diagonal* matrix, i.e., $a_{ij}(x,t) \equiv 0$ for $i \neq j$, and $f(u)$ is the following:

$$f_i(x,t,u) = u_i(b_i(x,t) - \sum_{j=1}^m c_{ij}(x,t)u_j), \quad u = (u_i)_{i=1}^m \quad \text{and} \quad i = 1, \ldots, m. \tag{1.2}$$

The values of b_i and c_{ii} are the intrinsic birth or death rates of the ith species. The value of c_{ij}, when $i \neq j$, represents the effect that jth species has upon ith species and the effect is proportional to the populations of both species.

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Recently, there has been great interest in cross diffusion effects in such models to take into account the movement influence between the species. In this case, the diffusion matrix A becomes a *full* matrix. Many efforts have been made to investigate the global existence of solutions for (1.1) assuming restrictive structural conditions on the size of the systems or the dimension of the domains (see [8,11] and the references therein). Whether this full system possesses global time solutions or finite time blow up solutions remains challenging and vastly open. In this paper, we will study these questions for (1.1) when the dimension of the domain Ω is at most 3 and the reaction terms have quadratic growth.

A much more interesting and harder problem is the SKT system ([12]) and its generalized versions, where A is allowed to depend on u and f has a polynomial growth in u. We obtained some global existence results for such systems given on planar domains ($n = 2$). The case of $n = 3$, which is of course more relevant in applications, is extremely hard and still under investigation. This paper is thus our first attempt to gain some insight into the 3-dimensional and nonlinear cases.

We consider the following assumptions:

A): $A(x,t)$ is $C^{1+\theta}$ continuous on $\Omega \times [0,\infty)$ for some $\theta > 0$ and there is positive real λ such that

$$\lambda|\zeta|^2 \le \langle A(x,t)\zeta,\zeta\rangle, \quad \forall x \in \Omega, t \in [0,\infty).\zeta \in \mathbb{R}^{mn}. \tag{1.3}$$

Moreover, there is a continuous function $C(t)$ on $(0,\infty)$ such that

$$\|A(x,t)\|,\ \|DA(x,t)\|,\ \|A_t(x,t)\| \le C(t). \tag{1.4}$$

F1): There is a continuous function $C_1(t)$ on $(0,\infty)$ such that

$$|f_u(x,t,u)| \le C_1(t)(|u|+1) \quad \text{and} \quad |f(x,t,u)|,\ |f_t(x,t,u)| \le C_1(t)(|u|^2+1). \tag{1.5}$$

We first recall the local existence result of classical solutions for (1.1) with such reactions and full matrix $A(x,t)$ from the seminal papers [2,3] by H. Amann.

Theorem 1.1 ([2,3])**.** *Suppose $\Omega \subset \mathbb{R}^n$, $n \ge 2$, with $\partial\Omega$ being smooth. Let $p_0 \in (n,\infty)$ and U_0 be in $W^{1,p_0}(\Omega)$. Then there exists a maximal time $T_0 \in (0,\infty]$ such that the system* (1.1) *has a unique classical solution in* $(0,T_0)$ *with*

$$u \in C([0,T_0), W^{1,p_0}(\Omega)) \cap C^{1,2}((0,T_0)\times\bar{\Omega}).$$

Moreover, if $T_0 < \infty$, then

$$\lim_{t\to T_0^-} \|u(\cdot,t)\|_{W^{1,p_0}(\Omega)} = \infty. \tag{1.6}$$

We will always assume that the initial data U_0 of the system (1.1) is given in $W^{1,p_0}(\Omega)$ for some $p_0 > n$ and the dimension $n \le 3$ in our results below. In fact, the above theorem applies to much more general situations allowing A depends also on the unknown u; and the global existence of solutions is closely related to the Hölder regularity of bounded weak solutions, a very hard property to verify. Here, under A), we present a weaker condition for classical solutions to exist globally in the following theorem.

Theorem 1.2. *Suppose that* A) *and* F1) *are satisfied. Let u be a classical solution to* (1.1) *on some interval* $(0,T_0)$*. Then u exists globally on* $(0,\infty)$ *if and only if there is a continuous function $\Phi(t)$ on* $(0,\infty)$ *such that*

$$\sup_{t\in(0,T)} \int_0^t \int_\Omega \langle f(x,s,u(x,s)),u(x,s)\rangle\, dxds \le \Phi(T) \quad \forall T \in (0,T_0). \tag{1.7}$$

We can refer to the integral of $\langle f(x,s,u),u\rangle$ over $\Omega\times(0,t)$ on the left of (1.7) as the total reaction energy of the solution u. The above theorem thus shows that a solution u exists globally if and only if its total reaction energy does not blow up in finite time. On the other hand, if $f(u)$ is of quadratic growth as in F1), then one can easily see that the condition (1.7) is equivalent to the boundedness of the kinetic energy of u,

$$\sup_{t\in(0,T)}\int_\Omega |u|^2\,dx+\int_0^T\int_\Omega |Du(x,t)|^2\,dxdt, \text{or that of } \sup_{t\in(0,T)}\int_\Omega |u|^2\,dx.$$

In fact, we will also show that

Corollary 1.3. *Suppose that* A) *and* F1) *are satisfied. Let u be a classical solution to* (1.1) *on some interval* $(0,T_0)$. *Suppose that there is a continuous function* $\breve{\Phi}(t)$ *on* $(0,\infty)$ *such that for some* $p>\frac{3}{2}$,

$$\sup_{t\in(0,T)}\int_\Omega |u(x,t)|^p\,dx\le\breve{\Phi}(T),\quad \forall T\in(0,T_0). \tag{1.8}$$

Then u exists globally on $(0,\infty)$.

As an application of Theorem 1.2, we consider the following form of $f(x,t,u)$:

$$f_i(x,t,u)=u_i(b_i(x,t)-\sum_{j=1}^m c_{ij}(x,t)|u_j|),\quad u=(u_i)_{i=1}^m\quad\text{and}\quad i=1,\dots,m. \tag{1.9}$$

We will show that the total reaction energy of u does not blow up and have the following result.

Theorem 1.4. *Assume* A) *and that $f(x,t,u)$ is given by (*1.9*) for some nonnegative C^1 functions b_i,c_{ij}. For any initial data $U_0\in W^{1,p_0}(\Omega,\mathbb{R}^m)$ with $p_0>2$ the following Cauchy problem:*

$$\begin{cases} u_t=div(A(x,t)Du)+f(x,t,u), & in\ \Omega\times(0,\infty),\\ u(x,t)=0, & on\ \partial\Omega_1\times[0,T),\\ Du\cdot\nu=0, & on\ \partial\Omega_2\times[0,T),\\ u(x,0)=U_0(x), & x\in\Omega,\end{cases} \tag{1.10}$$

has a classical solution that exists globally.

Concerning applications in biology and ecology, the components u_i are usually population densities of the species under consideration. Therefore, it may be more desirable if one assumes positive initial data U_0 and drop the absolutes in (1.9) to consider the standard Lotka-Volterra setting

$$f_i(x,t,u)=u_i(b_i-\sum_{j=1}^m c_{ij}u_j),\quad u=(u_i)_{i=1}^m\quad\text{and}\quad i=1,\dots,m. \tag{1.11}$$

Of course, Theorem 1.4 can immediately apply to this case if one can establish that u stays nonnegative for all time. This is a well-known result for diagonal systems via a simple use of invariant principles (e.g. see [6]). However, it is not the case for cross diffusion systems as we will show that

Theorem 1.5. *There are a full matrix A and positive reals b_i,c_{ij} such that the system (*1.10*) with $f(u)$ being given by (*1.11*) has a unique classical solution u that takes positive initial values, exists globally and changes sign on* $(0,\infty)$.

Moreover, it is also well known that the solutions with positive initial data for the standard competitive Lotka-Volterra systems (i.e. A is diagonal and the coefficients b_i, c_{ij} in (1.11) are positive) will stay positive and exist globally. Again, we will present a counterexample for cross diffusion systems in the following result.

Theorem 1.6. *There are a matrix A and nonnegative reals b_i, c_{ij} such that the system (*1.10*) has a unique classical solution u which takes positive initial data, changes sign and blows up in finite time.*

Our paper is then organized as follows: Section 2 is devoted to the technical lemmas as well as the proof of our theorems; the last section includes the proofs of the existence of sign changing and blow up solutions.

2. The proof

In this section, we provide some technical lemmas and the proof of our main theorem for the dimension $n = 3$ only because the similar arguments apply to the case $n = 2$. In the rest of this paper, when there is no ambiguity C will denote a universal constant that can change from line to line in our argument. Furthermore, $C(\dots)$ is used to denote quantities which are bounded in terms of their parameters.

We first recall the following Gagliardo-Nirenberg inequality for any bounded smooth domain $\Omega \subset \mathbb{R}^3$ (see [13], Theorem 1.4.5) and $q, p \geq 1$ and $r \in [1, 3)$:

(2.1)
$$\left(\int_\Omega |\phi|^q\, dx\right)^{\frac{1}{q}} \leq C\left(\int_\Omega |\phi|^p\, dx\right)^{\frac{1-\alpha}{p}} \left(\int_\Omega |D\phi|^r\, dx\right)^{\frac{\alpha}{r}} + C\left(\int_\Omega |\phi|^p\, dx\right)^{\frac{1}{p}}$$

for every $\phi \in W^{1,r}(\Omega) \cap L^p(\Omega)$. Here, $\alpha \in [0, 1]$ is determined by $\frac{1}{q} = \frac{1-\alpha}{p} + (\frac{1}{r} - \frac{1}{3})\alpha$. In the proof we will frequently make use of this inequality for $q = 4, p = r = 2$ and $\alpha = \frac{3}{4}$ and the above becomes

$$\left(\int_\Omega |\phi|^4\, dx\right)^{\frac{1}{4}} \leq C\left(\int_\Omega |\phi|^2\, dx\right)^{\frac{1}{8}} \left(\int_\Omega |D\phi|^2\, dx\right)^{\frac{3}{8}} + C\left(\int_\Omega |\phi|^2\, dx\right)^{\frac{1}{2}} \tag{2.2}$$

for every $\phi \in W^{1,2}(\Omega)$.

Our two main technical lemmas show that if a classical solution u to (1.1) exists in some time interval, then its temporal and spatial derivatives can be controlled by the rate of change of the reaction with respect to u. For simplicity, we will present the proof for the case when f is independent of x, t but u. That is, $f(x, t, u) = f(u)$. The general case can be treated with minor changes and we refer the reader to Remark 2.3.

Lemma 2.1. *Let u be a classical solution to* (1.1) *in some interval* $(0, T_0)$*. Assume that there are continuous functions $\Phi_1(t)$, $\Phi_2(t)$ on $(0, \infty)$ such that*

$$\int_0^t \int_\Omega |Du|^2\, dxdt \leq \Phi_1(t), \quad \forall t \in (0, T_0), \tag{2.3}$$

$$\sup_{s \in (0,t]} \int_\Omega |f_u(u(x, s))|^2\, dx \leq \Phi_2(t), \quad \forall t \in (0, T_0). \tag{2.4}$$

Then there is a continuous function $\bar{\Phi}(t)$ on $(0,\infty)$ depending on Φ_1, Φ_2 such that

$$\int_\Omega |u_t(x,t)|^2\, dx \le \bar{\Phi}(t)\left(\int_\Omega |u_t(x,t_0)|^2\, dx + 1\right), \quad 0 < t_0 < t < T_0. \tag{2.5}$$

Proof. Because u is smooth in the interior of $Q = \Omega \times (0, T_0)$, we can differentiate the system of u with respect to t and get

$$u_{tt} = \operatorname{div}(ADu_t) + \operatorname{div}(A_t Du) + f_u(u)u_t, \quad \forall t \in (0,T_0). \tag{2.6}$$

Multiply u_t to the above system and integrate the result in x over Ω to obtain

$$\int_\Omega \langle u_{tt}, u_t\rangle dx = \int_\Omega [\langle \operatorname{div}(A(x,t)Du_t), u_t\rangle + \langle \operatorname{div}(A_t(x,t)Du), u_t\rangle + \langle f_u(u)u_t, u_t\rangle] dx \tag{2.7}$$

for $t \in (0,T_0)$.

The mixed boundary condition in (1.1) implies either u_t or $Du_t \cdot \nu = 0$ on the lateral boundary $\partial\Omega \times [0,T_0)$, so that boundary integral resulting in the integration by parts applying to the first term on the right is zero. We then have

$$\begin{aligned}&\int_\Omega \langle u_{tt}, u_t\rangle dx\\ &\quad = -\int_\Omega \langle A(x,t)Du_t, Du_t\rangle dx - \int_\Omega \langle A_t(x,t)Du, Du_t\rangle dx + \int_\Omega \langle f_u(u)u_t, u_t\rangle dx.\end{aligned}$$

By the condition A), we get $ADu_tDu_t \ge \lambda|Du_t|^2$. Meanwhile, by Young's inequality

$$\int_\Omega |A_t Du Du_t| dx \le \int_\Omega a(t)|Du||Du_t| dx \le \frac{\lambda}{4}\int_\Omega |Du_t|^2 dx + \frac{1}{\lambda}a^2(t)\int_\Omega |Du|^2 dx,$$

where $a(t) := \sup_{x\in\Omega} \|A_t(x,t)\|$.

The above estimates yield

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |u_t|^2 dx\right) + \frac{3\lambda}{4}\int_\Omega |Du_t|^2 dx \le \frac{1}{\lambda}a^2(t)\int_\Omega |Du|^2 dx + \int_\Omega |f_u(u)||u_t|^2 dx. \tag{2.8}$$

Applying Hölder's inequality to the last integral on the right of (2.8) and using the assumption (2.4), for $0 < t_0 < t < T_0$ we can estimate it by

$$\left(\int_\Omega |f_u(u)|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |u_t|^4\, dx\right)^{\frac{1}{2}} \le (\Phi_2(t))^{\frac{1}{2}}\left(\int_\Omega |u_t|^4\, dx\right)^{\frac{1}{2}}. \tag{2.9}$$

Using (2.2) with $\phi = u_t$ to estimate the last integral on the right of (2.9) we obtain

$$\begin{aligned}&\int_\Omega |f_u(u)||u_t|^2 dx\\ &\qquad \le (\Phi_2(t))^{\frac{1}{2}}\left[C\left(\int_\Omega |Du_t|^2 dx\right)^{\frac{3}{8}}\left(\int_\Omega |u_t|^2 dx\right)^{\frac{1}{8}} + C\left(\int_\Omega |u_t|^2 dx\right)^{\frac{1}{2}}\right]^2\\ &\qquad \le (\Phi_2(t))^{\frac{1}{2}}\, C\left(\int_\Omega |Du_t|^2 dx\right)^{\frac{3}{4}}\left(\int_\Omega |u_t|^2 dx\right)^{\frac{1}{4}} + (\Phi_2(t))^{\frac{1}{2}}\, C\int_\Omega |u_t|^2 dx.\end{aligned}$$

By Young's inequality, we can find a function Φ_3 depending on Φ_2 and λ such that

$$\int_\Omega |f_u(u)||u_t|^2 dx \le \frac{\lambda}{4}\int_\Omega |Du_t|^2 dx + \frac{\Phi_3(t)}{2}\int_\Omega |u_t|^2 dx.$$

Substituting this inequality into (2.8) and rearranging, we have

$$\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega |u_t|^2 dx\Big) + \frac{\lambda}{2}\int_\Omega |Du_t|^2 dx \le \frac{\Phi_3(t)}{2}\int_\Omega |u_t|^2 dx + \frac{a^2(t)}{\lambda}\int_\Omega |Du|^2 dx. \tag{2.10}$$

Let

$$y(t) = \int_\Omega |u_t(x,t)|^2\, dx, \quad \hat{\Phi}_3(t) = \frac{2a^2(t)}{\lambda}\int_\Omega |Du(x,t)|^2 dx.$$

The above (2.10) then implies for all $t \in (0, T_0)$,

$$y'(t) \le \Phi_3(t) y(t) + \hat{\Phi}_3(t).$$

This is a Gronwall inequality. Because $\Phi_3(t) \ge 0$ we easily deduce for all $t_0 > 0$ and $t \in (t_0, T_0)$ the following:

$$y(t) \le e^{\breve{\Phi}_3(t)}\left(y(t_0) + \int_{t_0}^t \hat{\Phi}_3(s) ds\right), \quad \breve{\Phi}_3(t) := \int_{t_0}^t \Phi_3(s)\, ds.$$

Therefore, by the assumption (2.3) and the definition of $\hat{\Phi}_3$, we get

$$\int_\Omega |u_t(x,t)|^2\, dx \le e^{\breve{\Phi}_3(t)} \max\{1, \int_{t_0}^t \hat{\Phi}_3(s) ds\}\left(\int_\Omega |u_t(x,t_0)|^2\, dx + 1\right)$$

for $t_0 > 0$ and $t \in (t_0, T_0)$. The above clearly gives the assertion of the lemma. □

We now estimate the integral of Du to provide the main vehicle of the proof of our theorems.

Lemma 2.2. *Suppose that there is a constant C such that*

$$|f_u(u)| \le C(|u| + 1), \tag{2.11}$$

and that

$$\sup_{t\in(0,T)} \int_0^t \int_\Omega \langle f(u(x,s)), u(x,s)\rangle\, dxds \le \Phi_*(T) \quad \forall T \in (0, T_0) \tag{2.12}$$

for some continuous function Φ_* on $(0,\infty)$.

Then there is a continuous function Φ^* on $(0,\infty)$ depending on Φ_* such that

$$\sup_{t\in[0,T]} \int_\Omega |Du(x,t)|^3 dx \le \Phi^*(T)\left[\int_\Omega |u_t(x,t_0)|^2\, dx + 1\right]^{\frac{9}{8}} \tag{2.13}$$

for all $t_0 \in (0, T_0)$ and $T \in (t_0, T_0)$.

Proof. Testing the system for u with u and using (2.12), we easily get

$$\int_\Omega |u|^2\, dx + 2\lambda \int_0^t \int_\Omega |Du|^2\, dxdt \le \int_\Omega |u(x,0)|^2\, dx + 2\int_0^t \int_\Omega \langle f(u), u\rangle\, dxdt.$$

Therefore, for $\Phi_4(t) = 2\int_\Omega |u(x,0)|^2\, dx + 3\Phi_*(t)$ the above and (2.12) imply

$$\sup_{t\in(0,T)} \int_\Omega |u|^2\, dx + \lambda \int_0^T \int_\Omega |Du|^2\, dxdt \le \Phi_4(T). \tag{2.14}$$

Hence

$$\int_0^T \int_\Omega |Du|^2 dxdt \le \lambda^{-1}\Phi_4(T). \tag{2.15}$$

We now let

$$F(t) := \int_\Omega \langle f(u(x,t)), u(x,t)\rangle \, dx, \quad t \in (0, T_0).$$

For any $t_0 \in (0, T_0)$ we have

$$F(t) = F(t_0) + \int_{t_0}^t \int_\Omega \frac{\partial}{\partial s} \langle f(u(x,s)), u(x,s)\rangle \; dxds \quad \forall t \in (t_0, T_0). \tag{2.16}$$

Now, $\frac{\partial}{\partial s}\langle f(u(x,s)), u(x,s)\rangle \le |f_u||u_s||u| + |f(u)||u_s|$ and the growth condition (1.5) on f_u gives some constant C such that $|f(u)| \le C(|u|^2 + 1)$. Therefore,

$$F(t) \le F(t_0) + C\int_{t_0}^T \int_\Omega (|u_s||u|^2 + |u_s|) \; dxds, \quad 0 < t_0 < t < T_0. \tag{2.17}$$

We estimate the integral of $|u_s||u|^2$ via the Hölder inequality in x by

$$\begin{aligned}&\int_{t_0}^T \left(\int_\Omega |u_s|^2 \; dx\right)^{\frac{1}{2}} \left(\int_\Omega |u|^4 \; dx\right)^{\frac{1}{2}} ds \\ &\qquad \le \sup_t \left(\int_\Omega |u_s|^2 \; dx\right)^{\frac{1}{2}} \int_{t_0}^T \left(\int_\Omega |u|^4 \; dx\right)^{\frac{1}{2}} ds.\end{aligned}$$

By (2.2), we have

$$\begin{aligned}\left(\int_\Omega |u|^4 dx\right)^{\frac{1}{2}} &\le C\left[\left(\int_\Omega |Du|^2 dx\right)^{\frac{3}{8}} \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{8}} + \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}}\right]^2 \\ \left(\int_\Omega |u|^4 dx\right)^{\frac{1}{2}} &\le C\left(\int_\Omega |Du|^2 dx\right)^{\frac{3}{4}} \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{4}} + C\int_\Omega |u|^2 dx,\end{aligned} \tag{2.18}$$

so that

$$\begin{aligned}&\int_0^T \left(\int_\Omega |u|^4 dx\right)^{\frac{1}{2}} dt \\ &\quad \le C \sup_{t\in(0,T)} \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{4}} \int_0^T \left(\int_\Omega |Du|^2 dx\right)^{\frac{3}{4}} dt + CT \sup_{t\in(0,T)} \int_\Omega |u|^2 dx.\end{aligned}$$

Applying Hölder's inequality to the last integral, we get

$$\begin{aligned}&\int_0^T \left(\int_\Omega |u|^4 dx\right)^{\frac{1}{2}} dt \\ &\quad \le CT^{\frac{1}{4}} \left(\sup_{t\in(0,T)} \int_\Omega |u|^2 dx\right)^{\frac{1}{4}} \left(\int_0^T \int_\Omega |Du|^2 dxdt\right)^{\frac{3}{4}} + CT \sup_{t\in(0,T)} \int_\Omega |u|^2 dx.\end{aligned}$$

On the other hand, by (2.11) and (2.14), we have

$$\int_\Omega |f_u(u(x,s))|^2 \; dx \le C\int_\Omega (|u|^2+1) \; dx \le C(\Phi_4(t) + |\Omega|), \quad 0 < s < t \le T_0.$$

Thus, by (2.15) and the above, the conditions (2.3) and (2.4) of the previous lemma holds here and we have the estimate (2.5) for the integral of $|u_t|^2$ over Ω. This estimate and (2.14) provide a continuous function $\Phi_5(t)$ on $(0,\infty)$ such that

$$\sup_{t\in(t_0,T]} F(t) \le F(t_0) + \Phi_5(T)\left[\int_\Omega |u_t(x,t_0)|^2\, dx + 1\right]^{\frac{1}{2}}, \quad t_0 < s < t < T_0. \tag{2.19}$$

Finally, rewriting the system for u as $-\mathrm{div}(A(x)Du) = f(u) - u_t$ and testing it with u and using Hölder's inequality, we obtain

$$\begin{aligned}\lambda\int_\Omega |Du|^2 dx &\le \int_\Omega \langle f(u), u\rangle\, dx + \int_\Omega |u_t||u|\, dx\\ &\le F(t) + \left(\int_\Omega |u_t|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |u|^2\, dx\right)^{\frac{1}{2}}.\end{aligned}$$

Combining the above with (2.5), (2.14) and (2.19), we have a continuous function Φ_6 on $(0,\infty)$ depending on Φ_* such that

$$\sup_{t\in[0,T]}\int_\Omega |Du(x,t)|^2 dx \le \Phi_6(T)\left[\int_\Omega |u_t(x,t_0)|^2\, dx + 1\right]^{\frac{1}{2}} \tag{2.20}$$

for all $t_0 \in (0,T_0)$ and $T \in (t_0,T_0)$.

By the above, (2.14) and (2.18), there is a continuous function Φ_7 on $(0,\infty)$ such that

$$\int_\Omega |u|^4 dx \le \Phi_7(T)\left(\int_\Omega |u_t(x,t_0)|^2\, dx + 1\right)^{\frac{3}{4}}. \tag{2.21}$$

By this and the quadratic growth of $f(u)$, there is a continous function Φ_8 on $(0,\infty)$ such that

$$\int_\Omega |f(u)|^2 dx \le \Phi_8(t)\left(\int_\Omega |u_t(x,t_0)|^2\, dx + 1\right)^{\frac{3}{4}}, \quad \forall t \in (t_0,T_0). \tag{2.22}$$

We now rewrite our system as $\mathrm{div}(ADu) = u_t - f(u)$ or $A\Delta u + DADu = u_t - f(u)$. Thanks to A), $A(x,t)$ is invertible and we can solve for Δu to get

$$\Delta u = A^{-1}(-DADu + u_t - f(u)).$$

By F1), the norms $\|A(x,t)^{-1}\|$, $\|DA(x,t)\|$ are bounded so that we can find a function $\Phi_9(t)$ such that

$$\int_\Omega |\Delta u|^2 dx \le \Phi_9(t)\left(\int_\Omega |Du|^2 dx + \int_\Omega |u_t|^2 dx + \int_\Omega |f(u)|^2 dx\right).$$

Combining the above estimates for the integrals of $|Du|^2, |u_t|^2$ and $|f(u)|^2$ we see that there is a continuous function Φ_{10} on $(0,\infty)$ such that

$$\int_\Omega |\Delta u|^2 dx \le \Phi_{10}(t)\left(\int_\Omega |u_t(x,t_0)|^2\, dx + 1\right), \quad \forall t \in (t_0,T_0).$$

By Schauder's estimates for elliptic systems, we get

$$\int_\Omega |D^2u|^2 dx \le C\Phi_{10}(t)\left(\int_\Omega |u_t(x,t_0)|^2\, dx + 1\right), \quad \forall t \in (t_0,T_0). \tag{2.23}$$

Finally, using (2.1) with $q = 3$, $p = r = 2$, $\alpha = \frac{1}{2}$ for $\phi = Du$ and (2.20) and the above, we derive

$$\left(\int_\Omega |Du|^3\,dx\right)^{\frac{1}{3}} \le \Phi_{11}(t)\left(\int_\Omega |u_t(x,t_0)|^2\,dx + 1\right)^{\frac{3}{8}}, \quad \forall t \in (t_0, T_0).$$

This gives the lemma. □

Remark 2.3. If f also depends on x, t, then we will have extra terms like $|f_t(x,t,u)||u_t|$ in the estimates of Lemma 2.1 (see (2.7)) and $|f_t(x,t,u)||u|$ in Lemma 2.2 (see (2.16)). By the growth conditions assumed on f and f_t in (F_1), we have

$$|f_t(x,t,u)||u_t| \le C(t)|u|^4 + C(t)|u_t|^2 \quad \text{and} \quad |f_t(x,t,u)||u| \le C(t)|u|^3.$$

The integrals of $|u|^3$ and $|u|^4$ can be treated by (2.18) and (2.12). We then see that our lemmas continue to hold when f also depends on x, t.

We are now ready to give the proof of our main theorems.

Proof of Theorem 1.2. If a classical solution u exists globally, then its total energy on the left hand side of (1.7) is obviously continuous on $(0,\infty)$. We need only show that (1.7) is sufficient for the norm $\|u\|_{W^{1,p_0}(\Omega)}$, $p_0 > 3$, not to blow up in finite time.

It is clear that the assumption of the theorem allows us to apply Lemma 2.2 here so that if u exists on $(0,T_0)$, then for any $t_0 \in (0,T_0)$ we can find a continuous function $H_{t_0}(t)$ on (t_0,∞) such that

$$\int_\Omega |Du(x,t)|^3\,dx \le H_{t_0}(t), \quad \forall t \in (t_0, T_0).$$

This and (2.1) imply that $u(\cdot,t) \in L^p(\Omega,\mathbb{R}^m)$ for all $p > 1$ if $t > t_0$. Since u is smooth in $\Omega \times (0,t_0]$, $u(\cdot,t)$ is also in $L^p(\Omega)$ for $t \in (0,t_0]$. Therefore, there exists a continuous function $C_{f,p}(t)$ on $(0,\infty)$ such that

$$\|f(\cdot,t,u(\cdot,t))\|_{L^p(\Omega)} \le C_{f,p}(t), \quad \forall t \in (0,T_0). \tag{2.24}$$

We now follow the argument in [7, 9] and present some details here for the convenience of the readers. Let us fix a $p > 2$ and consider $X = L^p(\Omega,\mathbb{R}^m)$. Let $\mathcal{A}(t)$ be the realization of the operator $\mathrm{div}(A(x,t)Dv)$ acting on functions in X with mixed boundary conditions

$$\begin{cases} u(x,t) = 0, & \text{on } \partial\Omega_1 \times [0,T), \\ Du\cdot\nu = 0, & \text{on } \partial\Omega_2 \times [0,T). \end{cases} \tag{2.25}$$

That is

$$\mathcal{A}(t)v = \mathrm{div}(A(x,t)Dv) \text{ with } \mathrm{dom}(\mathcal{A}(t)) = \{v \in W^{2,p}(\Omega,\mathbb{R}^m) : v \text{ satisfies } (2.25)\}.$$

For initial data $u_0 \in W^{1,p}(\Omega,\mathbb{R}^m)$ we can abstractly write (1.1) as

$$u_t = \mathcal{A}(t)u + F(t,u), \quad u(0) = u_0,$$

where $F(t,u)(x) = f(x,t,u(x,t))$ for $x \in \Omega$.

Under the smoothness assumptions A) of $A(x,t)$ we easily see that $\mathcal{A}(t)$ satisfies all the conditions in [4] to ensure the existence of evolution operators

$$\mathcal{U}(t,s) \in \mathcal{L}(X), \quad 0 \le s \le t < \infty,$$

such that

$$u(t) = \mathcal{U}(t,0)u_0 + \int_0^t \mathcal{U}(t,s)F(s,u(s))ds. \tag{2.26}$$

We have the following estimate concerning the operator $\mathcal{U}(t,s)$. There exist positive numbers ω, C_γ such that for any $\gamma \in [0,1]$ and $0 \le s \le t < \infty$ (see (16.38) of [4])

$$\|\mathcal{A}^\gamma(t)\mathcal{U}(t,s)\|_{\mathcal{L}(X)} \le \frac{C_\gamma e^{-\omega(t-s)}}{(t-s)^\gamma}. \tag{2.27}$$

We now apply $\mathcal{A}^\gamma(t)$ to (2.26) to have

$$\mathcal{A}^\gamma(t)u(t) = \mathcal{A}^\gamma(t)\mathcal{U}(t,0)u_0 + \int_0^t \mathcal{A}^\gamma(t)\mathcal{U}(t,s)F(s,u(s))ds.$$

Therefore, using (2.27) and (2.24) with $q = p$,

$$\begin{aligned}\|\mathcal{A}^\gamma(t)u(t)\|_X &\le \|\mathcal{A}^\gamma(t)\mathcal{U}(t,0)u_0\|_X + \int_0^t \|\mathcal{A}^\gamma(t)\mathcal{U}(t,s)F(s,u(s))\|_X ds \\ &\le C_\gamma t^{-\gamma}e^{-\omega t}\|u_0\|_X + \int_0^t C_\gamma (t-s)^{-\gamma}e^{-\omega(t-s)}\|F(s,u(s))\|_X ds \\ &\le C_\gamma t^{-\gamma}e^{-\omega t}\|u_0\|_X + \max_{0\le s\le t} C_{f,p}(s)\int_0^t C_\gamma (t-s)^{-\gamma}e^{-\omega(t-s)}ds.\end{aligned}$$

By the smoothness and ellipticity conditions in A), we can find a continuous function $C(s,t)$ such that $\|A(s)A^{-1}(t)\|_{\mathcal{L}(X)} \le C(s,t)$ for all $s,t > 0$. This also implies (see [4]) the existence of a continuous function $C(s,t,\gamma)$ such that $\|A^\gamma(s)A^{-\gamma}(t)\|_{\mathcal{L}(X)} \le C(s,t,\gamma)$ for all $s,t > 0$ and $\gamma \in (0,1)$. Therefore, for any fixed t_0 in $(0,T_0)$ the above estimate also gives

$$\begin{aligned}&\|\mathcal{A}^\gamma(t_0)u(t)\|_X \\ &\quad \le C(t_0,t,\gamma)\left[C_\gamma t^{-\gamma}e^{-\omega t}\|u_0\|_X + \max_{0\le s\le t} C_{f,p}(s)\int_0^t C_\gamma (t-s)^{-\gamma}e^{-\omega(t-s)}ds\right].\end{aligned}$$

It is clear that the last quantity is a continuous function of $t \in (0,\infty)$. We just showed that the norm $\|\mathcal{A}^\gamma(t_0)u(t)\|_X$ does not blow up in finite time. It is well known that the space $Y_\gamma = \mathrm{dom}(A^\gamma(t_0))$, with the graph norm $\|v\|_{Y^\gamma} = \|A^\gamma(t_0)v\|_X$, is continuously embedded in $C^\alpha(\Omega,\mathbb{R}^m)$ if $0 \le \alpha < 2\gamma - 2/p$. By (2.24), for any given $\gamma \in (0,1]$ we can choose p large enough so that $\alpha > 0$. Thus, the C^α norm of u does not blow up in finite time and the global existence of u then follows. The proof is complete. □

The proof of Corollary 1.3 then follows.

Proof of Corollary 1.3. We need only show that the assumption (1.8),

$$\sup_{t\in(0,T)} \int_\Omega |u(x,t)|^p \, dx \le \check{\Phi}(T), \quad \forall T \in (0,T_0), p > 1, \tag{2.28}$$

implies (1.7) of Theorem 1.2. Since $f(x,t,u)$ has quadratic growth in u, one has from F1) the following:

$$\int_\Omega \langle f(x,t,u),u\rangle \, dx \le C(t)\int_\Omega (|u|^3 + 1)\, dx, \quad \forall t \in (0,T_0). \tag{2.29}$$

If $p \ge 3$, then (1.7) holds trivially. Thus we only consider $p \in (\frac{3}{2}, 3)$. Taking $q = 3$ and $\alpha = \frac{2(3-p)}{6-p}$ in (2.1), we have

$$\int_\Omega |u|^3 \, dx \le C \left(\int_\Omega |u|^p \, dx \right)^{\frac{3}{6-p}} \left(\int_\Omega |Du|^2 \, dx \right)^{\frac{3(3-p)}{6-p}} + C \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}}.$$

By the assumption (2.28),

$$\begin{aligned} &\int_0^T \int_\Omega |u|^3 \, dxdt \\ &\le C \left(\int_\Omega |u|^p \, dx \right)^{\frac{3}{6-p}} \int_0^T \left(\int_\Omega |Du|^2 \, dx \right)^{\frac{3(3-p)}{6-p}} dt + C \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}} \int_0^T dt \\ &\le C \left(\breve{\Phi}(T) \right)^{\frac{3}{6-p}} \int_0^T \left(\int_\Omega |Du|^2 \, dx \right)^{\frac{3(3-p)}{6-p}} dt + C \left(\breve{\Phi}(T) \right)^{\frac{1}{p}} T. \end{aligned}$$

For $\gamma = \frac{3(3-p)}{6-p}$, using the above and (2.29), we get

$$\begin{aligned} &\int_0^T \int_\Omega \langle f(x,t,u), u \rangle \, dxdt \\ &\quad \le C(T) \left(\breve{\Phi}(T) \right)^{\frac{3}{6-p}} \int_0^T \left(\int_\Omega |Du|^2 \, dx \right)^\gamma dt \\ &\qquad + C(T) \left(\breve{\Phi}(T) \right)^{\frac{1}{p}} T + C(T)|\Omega| T \\ &\quad \le \breve{\Phi}_*(T) \int_0^T \left(\int_\Omega |Du|^2 \, dx \right)^\gamma dt + \breve{\Phi}_*(T), \end{aligned}$$

where $\bar{\Phi}_*(T) = C(T) \left(\breve{\Phi}(T) \right)^{\frac{3}{6-p}} + C(T) \left(\breve{\Phi}(T) \right)^{\frac{1}{p}} T + C(T)|\Omega| T.$

Because $p \in (\frac{3}{2}, 3)$ we have $\gamma \in (0,1)$. A simple use of Hölder's inequality applied to the first term on the right gives

$$\int_0^T \int_\Omega \langle f(x,t,u), u \rangle \, dxdt \le \breve{\Phi}_*(T) T^{1-\gamma} \left(\int_0^T \int_\Omega |Du|^2 \, dxdt \right)^\gamma + \breve{\Phi}_*(T).$$

Using Young's inequality gives

(2.30)
$$\int_0^T \int_\Omega \langle f(x,t,u), u \rangle \, dxdt \le \frac{\lambda}{2} \int_0^T \int_\Omega |Du|^2 \, dxdt + C(\lambda) \breve{\Phi}_*(T)^{\frac{1}{1-\gamma}} T + \breve{\Phi}_*(T).$$

As before, we test the system for u with u to get

$$\lambda \int_0^T \int_\Omega |Du|^2 \, dxdt \le \int_0^T \int_\Omega \langle f(x,t,u), u \rangle \, dxdt + \frac{1}{2} \int_\Omega |u(x,0)|^2 \, dx.$$

This and (2.30) yield

$$\int_0^T \int_\Omega \langle f(x,t,u), u \rangle \, dxdt \le 2C(\lambda) \breve{\Phi}_*(T)^{\frac{1}{1-\gamma}} T + 2\breve{\Phi}_*(T) + \frac{1}{2} \int_\Omega |u(x,0)|^2 \, dx.$$

The right hand side is a continuous function on $(0, \infty)$ so that the condition (1.7) is verified. Our proof is then complete. □

We end this section by giving the proof of Theorem 1.4.

Proof of Theorem 1.4. Since $f(x,t,u)$ is of quadratic growth the assumption F1) is verified. We need only to check the condition (1.7) of Theorem 1.2 here. We see that

$$\langle f(x,t,u),u\rangle = \sum_{i=1}^m b_i u_i^2 - \sum_{i,j=1}^m c_{ij}u_i^2|u_j| \le \sum_{i=1}^m b_i u_i^2.$$

Hence, by testing the system of u by u and using the ellipticity condition A), we easily obtain

$$\frac{d}{dt}\int_\Omega |u|^2\,dx \le 2\int_\Omega \langle f(x,t,u),u\rangle\,dx \le C_0(t)\int_\Omega |u|^2\,dx,$$

where $C_0(t) = 2\max_i \|b_i\|_{L^\infty(\Omega\times(0,t)}$ is a continuous function on $(0,\infty)$ by our assumptions. The above Gronwall inequality gives

$$\int_\Omega |u(x,t)|^2\,dx \le e^{\Phi_0(t)}\int_\Omega |u(x,0)|^2\,dx, \quad \Phi_0(t) = \int_0^t C_0(s)ds.$$

Therefore,

$$\begin{aligned}\int_0^T\int_\Omega \langle f(x,t,u),u\rangle\,dxdt &\le \frac{1}{2}C_0(T)\int_0^T\int_\Omega |u|^2\,dxdt\\ &\le \frac{1}{2}C_0(T)\int_0^T \Phi_0(s)ds\int_\Omega |u(x,0)|^2\,dx.\end{aligned}$$

Thus, (1.7) of Theorem 1.2 is established and our theorem then follows. □

3. Sign-changing and blow up solutions

Proof of Theorem 1.5. We present an example of a sign-changing solution to the following system of three equations given on $\Omega = (0,\pi)\times(0,\pi)\times(0,\pi)$:

$$\begin{cases} u_t = \operatorname{div}(A(x,t)Du) + f(u), & \text{in } \Omega\times(0,T),\\ Du\cdot\nu = 0, & \text{on } \partial\Omega\times[0,T],\\ u(x,0) = U_0(x), & x\in\Omega.\end{cases} \tag{3.1}$$

Here for $h(x) := \cos(x_1)\cos(x_2)\cos(x_3)$

$$A = \begin{bmatrix} 1 & -1 & -1\\ 1 & \frac{2}{3} & 0\\ 1 & 0 & 1\end{bmatrix}, \quad f(u) = \begin{bmatrix} u_1(4-u_1-u_2)\\ u_2(4-u_1-u_2)\\ u_3(6-u_1-u_3)\end{bmatrix}, \quad U_0(x) = \begin{bmatrix} 2+h(x)\\ 2-h(x)\\ 4-h(x)\end{bmatrix}.$$

The matrix $A = (a_{ij})$ satisfies the condition A) because for any three vectors η_1,η_2,η_3 and $\eta = (\eta_1,\eta_2,\eta_3)^T$,

$$\langle A\eta,\eta\rangle = |\eta_1|^2 - \langle\eta_1,\eta_2\rangle - \langle\eta_1,\eta_3\rangle + \frac{1}{2}\langle\eta_2,\eta_1\rangle + \frac{2}{3}|\eta_2|^2 + \langle\eta_3,\eta_1\rangle + |\eta_3|^2 \ge \frac{2}{3}|\eta|^2.$$

We will show that

$$v(x,t) = \begin{bmatrix} v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix} 2+e^t h(x)\\ 2-e^t h(x)\\ 4-e^t h(x)\end{bmatrix}$$

is the solution to the problem (3.1). Clearly, v takes positive initial data U_0 and changes sign in times.

We have $\Delta v_i = -3v_i$ for $i = 1, 2, 3$, so that v satisfies the following equations:

$$\begin{cases} (v_1)_t = & \Delta v_1 & - & \Delta v_2 & - & \Delta v_3 & +v_1(4 - v_1 - v_2), \\ (v_2)_t = & \Delta v_1 & + & \frac{2}{3}\Delta v_2 & + & 0 & +v_2(4 - v_1 - v_2), \\ (v_3)_t = & \Delta v_1 & + & 0 & + & \frac{2}{3}\Delta v_3 & +v_3(6 - v_1 - v_3). \end{cases}$$

One can see that $Dh \cdot \nu = 0$ on the boundary of Ω so that v satisfies the Neumann boundary condition. Our proof is complete. □

Proof of Theorem 1.6. We now construct a finite time blow up solution to a cross diffusion system of two equations. To this end, let us denote by ϕ the normalized eigenfunction associated to the first eigenvalue λ_1 of the Dirichlet-Laplace operator on Ω. That is,

$$\Delta\phi = -\lambda_1\phi,\ x \in \Omega\ ;\ \phi = 0,\ x \in \partial\Omega\ ;\ \int_\Omega \phi dx = 1. \tag{3.2}$$

It is well known that ϕ is positive on Ω. We then consider the following system:

$$\begin{cases} u_t = \operatorname{div}(A(x,t)Du) + f(u), & \text{in } \Omega \times (0,T), \\ u = 0, & \text{on } \partial\Omega \times [0,T], \\ u(x,0) = U_0(x), & x \in \Omega, \end{cases} \tag{3.3}$$

where $u = (u_1, u_2)^T$ and

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}, \quad f(u) = \begin{bmatrix} u_1(2\lambda_1 - u_1) \\ (\lambda_1 + k)u_2 \end{bmatrix}, \quad U_0(x) = \begin{bmatrix} u_0(x) \\ \phi(x) \end{bmatrix}.$$

Here, $k > \lambda_1$ can be any constant and $u_0(x)$ is a smooth positive function on Ω.

The matrix A satisfies the ellipticity condition A) because for any two vectors η_1, η_2 and $\eta = (\eta_1, \eta_2)^T$,

$$\langle A\eta, \eta\rangle = |\eta_1|^2 + \frac{1}{2}\langle \eta_1, \eta_2\rangle + |\eta_2|^2 = \frac{1}{4}|\eta_1 + \eta_2|^2 + \frac{3}{4}|\eta|^2 \geq \frac{3}{4}|\eta|^2.$$

We can rewrite the system (3.3) as follows:

$$\begin{cases} (u_1)_t & = & \Delta u_1 & +\frac{1}{2}\Delta u_2 & +u_1(2\lambda_1 - u_1), \\ (u_2)_t & = & & \Delta u_2 & +(\lambda_1 + k)u_2. \end{cases} \tag{3.4}$$

We can easily check that the solution to the second equation is $u_2(x,t) = e^{kt}\phi(x)$ so that the first equation becomes

$$(u_1)_t = \Delta u_1 - \frac{\lambda_1}{2}e^{kt}\phi + 2\lambda_1 u_1 - u_1^2. \tag{3.5}$$

We will prove that u_1 blows up in finite time. Assume by contradiction that u_1 is a global solution so that the quantity

$$y(t) := \int_\Omega u_1(x,t)\phi(x)dx \text{ is finite for every } t \in (0, \infty). \tag{3.6}$$

For $t > 0$, multiply the equation (3.5) by ϕ_1 and integrate

$$\left(\int_\Omega u_1\phi dx\right)_t = \int_\Omega \Delta u_1\phi dx - \frac{\lambda_1}{2}e^{kt}\int_\Omega \phi^2 dx + 2\lambda_1\int_\Omega u_1\phi dx - \int_\Omega u_1^2\phi dx. \tag{3.7}$$

Here, integration by parts implies $\int_\Omega \Delta u_1 \phi dx = \int_\Omega u_1 \Delta\phi dx = -\lambda_1 \int_\Omega u_1\phi dx$, so that, with the notation (3.6), the above can be written as

$$y_t = \lambda_1 y - \frac{\lambda_1}{2} e^{kt} \int_\Omega \phi^2 dx - \int_\Omega u_1^2 \phi dx. \tag{3.8}$$

The above then implies

$$y_t \le \lambda_1 y - \frac{\lambda_1}{2} e^{kt} \int_\Omega \phi^2 dx.$$

This Gronwall inequality yields

$$y(t) \le e^{\lambda_1 t}\left(y(0) - \frac{\lambda_1}{2(k-\lambda_1)} \int_\Omega \phi^2 dx \left[e^{(k-\lambda_1)t} - 1 \right] \right) \quad \text{for} \quad t > 0.$$

Because $k > \lambda_1$, the above implies that there is $t_1 > 0$ such that $y(t) < 0$ for $t \ge t_1$.

By Hölder's inequality and the normalization of ϕ,

$$y^2(t) = \left(\int_\Omega u_1 \sqrt{\phi}\sqrt{\phi} dx \right)^2 \le \int_\Omega u_1^2 \phi dx \int_\Omega \phi dx = \int_\Omega u_1^2 \phi dx.$$

So, for $t \ge t_1$ one has $y(t) < 0$ and (3.8) yields

$$y_t \le \lambda_1 y - y^2 \Rightarrow \frac{y_t}{y^2} + \lambda_1 \frac{-1}{y} \le -1.$$

Denote $w = \frac{-1}{y}$. The above gives $w(t) > 0$ and $w_t + \lambda_1 w \le -1$ for $t \ge t_1$, so that, for $t \ge t_1$ we have

$$e^{\lambda_1 t} w_t + e^{\lambda_1 t} \lambda_1 w \le -e^{\lambda_1 t} \le -e^{\lambda_1 t_1} \Rightarrow \left(e^{\lambda_1 t} w \right)_t \le -e^{\lambda_1 t_1}.$$

Integrating the last inequality over $[t_1, t]$, we get

$$e^{\lambda_1 t} w(t) - e^{\lambda_1 t_1} w(t_1) \le -e^{\lambda_1 t_1}(t - t_1).$$

Thus, $0 \le e^{\lambda_1 t} w(t) \le e^{\lambda_1 t_1}[t_1 + w(t_1) - t]$. This shows that $w(t) \to 0$ when $t \to t_2^-$ with $t_2 = t_1 + w(t_1) > 0$. Hence, $y(t) \to -\infty$ when $t \to t_2^-$. This is a contradiction to (3.6). Thus, the solution to (3.5), and therefore (3.3), blows up in finite time and cannot stay positive. ☐

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