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ISOMETRIC EQUIVALENCE OF ISOMETRIES ON H^p

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ABSTRACT. We consider a natural notion of equivalence for bounded linear operators on H^p, for $1 \leq p < \infty, p \neq 2$. We study the structure of isometries on H^p of finite codimension and we determine when two such isometries are equivalent. Among these isometries, we determine which operators S satisfy $\bigcap_1^\infty S^n H^p = (0)$.

1. Introduction

In this note, we study a class of isometric operators on the Hardy space $H^p, 1 \leq p < \infty, p \neq 2$. Forelli determined all such isometries S. If ϕ is an automorphism of the unit disc, then $S = M_{(\phi')^{1/p}} C_\phi$ is an onto isometry, where C_ϕ is composition with ϕ, and $M_{(\phi')^{1/p}}$ is multiplication by $(\phi')^{1/p}$. With a certain normalization this operator is written as U_ϕ, and every onto isometry has the form ρU_ϕ where ϕ is a disc automorphism and $|\rho| = 1$ (see Theorem A). For isometries S of finite codimension $d > 0$, Forelli's result (Theorem B) yields that $S = M_\Psi U_\phi$ where Ψ is a d-fold Blaschke product.

We say that operators S, T on H^p are isometrically equivalent if there is an onto isometry U such that $US = TU$. This notion can also be studied for general Banach spaces. For operators on Hilbert spaces, isometric equivalence is unitary equivalence. We show in Proposition 2 that S is equivalent to an onto isometry U_ϕ iff $S = \rho U_\psi$, where $\psi = \eta \circ \phi \circ \eta^{-1}$ for some disc automorphism η and some $\rho, |\rho| = 1$, which depends on ϕ and ψ.

To motivate the results for the codimension $d > 0$ case, we recall the Wold decomposition for an isometry S on a Hilbert space $\mathbb{H}$ (see [9]). S is a direct sum of a unitary and a shift of multiplicity d. $(S\mathbb{H})^\perp = \mathbb{H} \ominus S\mathbb{H}$ is the wandering subspace of the shift part of S. The domain of the unitary part is $\mathbb{H} \ominus \bigcap_1^\infty S^n\mathbb{H}$, so S is a pure shift iff $\bigcap_1^\infty S^n\mathbb{H} = (0)$.

We now outline the main results of this paper. Suppose $S = M_\Psi U_\phi$ is an isometry of codimension $d > 0$ as described above. Just as for the case of an onto isometry, the isometric equivalence class of S is determined in an explicit sense by the equivalence class of the automorphism ϕ. We show that $\bigcap_1^\infty S^n H^p = (0)$ iff ϕ is an elliptic automorphism. Refer to Propositions 3, 4 and to Theorem 4. This easily answers a question of J. Jamison, which motivated this study. Namely, we determine which isometries of H^p are equivalent to the shift M_z.

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If ϕ is not elliptic (that is, ϕ is hyperbolic or parabolic), we show that $\bigcap_1^\infty S^n H^p = BH^p$ for a certain Blaschke product B constructed from ϕ and Ψ. Further, $S|_{BH^p}$ is equivalent to U_ϕ.

2. Preliminaries

In this paper we consider the Banach spaces H^p of the unit disc D, for $1 \leq p < \infty$, $p \neq 2$. Recall that H^p consists of the analytic functions f on D for which

$$Sup_{0<r<1} \int_T |f(r\zeta)|^p dm(\zeta)$$

is finite, with dm the usual Lebesgue measure on the unit circle T (see [3]).

Definition 1. $\mathbb{I}(H^p)$ will denote the onto isometries of H^p, $p \neq 2$.

$\mathbb{I}(H^p)$ is a group under the usual operator multiplication. The description of $\mathbb{I}(H^p)$ for $p = 1$ is due to de Leeuw, et al. [4], and for $1 < p < \infty$, $p \neq 2$, to Forelli [6] and is given in Theorem A below.

Definition 2. Let $\mathbb{A}$ be the collection of holomorphic automorphisms of the unit disc D. That is,

$$\mathbb{A} = \{\phi(z) = \frac{\lambda(z-a)}{1-\overline{a}z} : \quad a \in D, \quad |\lambda| = 1\}.$$

$\mathbb{A}$ is a group under composition with identity e, where $e(z) = z$. Following [8], we write ϕ_n for the n-fold composition of ϕ with itself. In addition, we denote the (compositional) inverse of ϕ by ϕ_{-1}. Note that $(\phi_n)_{-1}$, the inverse of ϕ_n, is just $(\phi_{-1})_n$, the n-fold composition of ϕ_{-1} with itself, which we denote by ϕ_{-n}.

Definition 3. For ϕ and ψ in $\mathbb{A}$ we say that ϕ is conjugate to ψ if there is an $\eta \in \mathbb{A}$ with $\phi = \eta \circ \psi \circ \eta_{-1}$. The conjugacy class of ϕ is denoted by $\mathbb{C}(\phi)$ and so $\mathbb{C}(\phi) = \{\eta \circ \phi \circ \eta_{-1} : \eta \in \mathbb{A}.\}$

The following proposition is well known and we include it for notational purposes.

Proposition 1. *Suppose* (b_k) *is a sequence of functions in* $\mathbb{A}$ *and that* $b_k(a_k) = 0\,,\ \forall\ k \geq 1$.

(i) *If* $\prod_{k=1}^\infty b_k(z)$ *is a convergent Blaschke product, then* (a_k) *is a Blaschke sequence. That is,* $\sum_{k=1}^\infty (1-|a_k|) < \infty$.

(ii) *Conversely, if* (a_k) *is a Blaschke sequence, then there exists* $(\lambda_k)_{k=1}^\infty \in T$ *so that* $\prod_{k=1}^\infty \lambda_k b_k(z)$ *converges.*

3. Isometric equivalence

A theorem of Forelli [6] describes all isometries of H^p onto H^p. For $\phi \in \mathbb{A}$ and $d \in T$ Forelli showed that the map

$$f \mapsto d(\phi'(z))^{1/p} f \circ \phi$$

is in $\mathbb{I}(H^p)$ and that all onto isometries have this form. Theorem A below is a restatement of this result.

If

$$\phi(z) = \frac{\lambda(a-z)}{1-\overline{a}z},$$

then

$$\phi'(z) = \frac{\lambda(1-|a|^2)}{(1-\overline{a}z)^2},$$

so that the choice of a branch of the pth root function that will make $(\phi'(z))^{1/p}$ analytic will depend on λ. It is useful to set our notation so that we always use the principal branch given by $(r\exp(\imath\theta))^{1/p} = r^{1/p}\exp(\imath\theta/p)$, $-\pi < \theta < \pi$, $r > 0$. The range of $(\overline{\lambda}\phi'(z)) = \frac{(1-|a|^2)}{(1-\overline{a}z)^2}$ does not meet the negative real axis, so $(\overline{\lambda}\phi'(z))^{1/p}$ is analytic on D.

Let $C_\phi f = f\circ\phi$ denote composition by ϕ and for $F \in H^\infty$, M_F denotes multiplication by F. Finally, let

$$U_\phi = M_{((\overline{\lambda})\phi')^{1/p}} C_\phi.$$

Forelli's result can be stated as

Theorem A. $\mathbb{I}(H^p) = \{\rho U_\phi : \phi \in \mathbb{A}\, , |\rho| = 1\}$.

From this result we see that $\mathbb{I}(H^p)$ is determined by $\mathbb{A}$. We will examine the relation between the group structures of $\mathbb{A}$ and $\mathbb{I}(H^p)$.

We note that for $p = 2$, the operators of the form ρU_ϕ in Theorem A are of course unitary operators on H^2 which are tied to the analytic structure of H^2. These unitaries are a small subgroup of the full unitary group on H^2.

Lemma 1. *Let ϕ and $\psi \in \mathbb{A}$. Then*

(a) $U_\phi U_\psi = \rho U_{\psi\circ\phi}$ *for some $\rho \in T$, which depends on ϕ and ψ.*

(b) $U_\phi^{-1} = U_{\phi_{-1}}$.

Proof. Suppose that $\phi(z) = \frac{\lambda_1(z-a_1)}{1-\overline{a_1}z}$ and $\psi(z) = \frac{\lambda_2(z-a_2)}{1-\overline{a_2}z}$. Then

$$\psi\circ\phi(z) = \frac{\lambda_3(z-a_3)}{1-\overline{a_3}z}$$

for some $\lambda_3 \in T, a_3 \in D$. Note that $C_\psi C_\phi = C_{\phi\circ\psi}$. Also if $F \in H^\infty$, then $C_\phi M_F = M_{F\circ\phi} C_\phi$. Thus

$$U_\phi U_\psi = M_{(\overline{\lambda_1}\ \phi')^{1/p}} C_\phi M_{(\overline{\lambda_2}\ \psi')^{1/p}} C_{\psi\circ\phi} = (\overline{\lambda_1}\phi')^{1/p}(\overline{\lambda_2}\psi'\circ\phi)^{1/p} C_{\psi\circ\phi}.$$

But

$$U_{\psi\circ\phi} = (\overline{\lambda_3}(\psi\circ\phi)')^{1/p} C_{\psi\circ\phi} = (\overline{\lambda_3}(\psi'\circ\phi)\phi')^{1/p} C_{\psi\circ\phi}.$$

So one sees that $U_{\psi\circ\phi}$ is a unimodular multiple of $U_\phi U_\psi$, and (a) is proven. For part (b) we recall that

$$\phi_{-1}(z) = \frac{\overline{\lambda_1}(z+\lambda_1 a_1)}{(1+\overline{\lambda_1 a_1}z)}.$$

Take $\psi = \phi_{-1}$ in the last proof. Thus

$$U_\phi U_{\phi_{-1}} = (\overline{\lambda_1}\ \phi')^{1/p}(\lambda_1\phi'_{-1}\circ\phi)^{1/p} C_{\phi_{-1}\circ\phi} = I. \qquad \square$$

Remark. The value of the constant ρ in Lemma 1 (a) will not be needed in our work, but it can of course be explicitly computed. For ϕ and ψ as in the proof of Lemma 1, one can show that $\rho = \exp i\theta$, where $\theta = arg(1+\overline{\lambda_1 a_1}a_2)^{2/p}$.

For A and B in $\mathbb{B}(H^p)$ we say that A is isometrically equivalent to B if there is a $C \in \mathbb{I}(H^p)$ with $B = CAC^{-1}$, and write this equality as $A \approx B$. We now describe all $S \in \mathbb{I}(H^p)$ which are isometrically equivalent to a fixed $U_\phi \in \mathbb{I}(H^p)$.

Proposition 2. *For $S \in \mathbb{B}(H^p)$ we have that $S \approx U_\phi \Leftrightarrow$ there exists $\eta \in \mathbb{A}$ and $\rho \in T$ so that*

$$S = \rho U_{\eta \circ \phi \circ \eta_{-1}}.$$

Proof. $S \approx U_\phi \Leftrightarrow$ there exists $\eta \in \mathbb{A}$ so that

$$U_{\eta_{-1}} U_\phi U_\eta = S.$$

But

$$U_{\eta_{-1}} U_\phi U_\eta = U_{\eta_{-1}}(\rho_1 U_{\eta \circ \phi}) = \rho_2 \rho_1 U_{\eta \circ \phi \circ \eta_{-1}},$$

where $\rho_1, \rho_2 \in T$ as in Lemma 1 (a). □

So Proposition 2 states that

$$\widetilde{\phi} \in \mathbb{C}(\phi) \Leftrightarrow U_\phi = \rho U_{\widetilde{\phi}}$$

for some $\rho \in T$.

We focus on the isometries of H^p into H^p. The most familiar example is the shift M_z on H^p. The range of M_z is zH^p, so is of codimension one. Motivated by a question of J. Jameson [oral communication], we consider several related questions. See [2, 5, 7] for motivation. For example, which $S \in \mathbb{B}(H^2)$ satisfy $S \approx M_z$? We will in fact classify all finite codimension isometries up to isometric equivalence. We give the details of our results for the case for codimension one isometries and the codimension n case follows similarly.

Note that M_z has the additional property that

$$\bigcap_{n=1}^{\infty} (M_z)^n H^p = (0).$$

Definition 4. A codimension one isometry S on H^p is called Crownover (see [2], [7]) if $\bigcap_{n=1}^{\infty} S^n H^p = (0)$.

We will also classify such isometries up to isometric equivalence.

4. Finite codimensional isometries

We will now state Forelli's theorem [6, Theorem 1] describing all isometries of H^p, $p \neq 2$.

Theorem B. *The operator S is an isometry of H^p, $1 \leq p < \infty$, $p \neq 2$, iff $\forall f \in H^p$, $Sf = Ff(\phi)$ for some ϕ inner and an $F \in H^p$ which is related to ϕ.*

The precise relationship between F and ϕ can be found in [6] and is not needed in the work that follows. We will provide a simpler description of the isometries of finite codimension, which follows from the next lemma.

Lemma 2. *For $1 \leq p < \infty$, suppose that $F \in H^p$, that $\phi : D \to D$ is holomorphic on D, and that $T = M_F C_\phi$ is bounded on H^p. If ϕ is not univalent on D, then the closure of the range of T has infinite codimension.*

Proof. We will modify the argument of [1, Lemma 3.26]. ϕ is not univalent, so choose distinct points $a, b \in D$ with $\phi(a) = \phi(b)$. Next choose disjoint open sets $\mathbb{O}_a$ and $\mathbb{O}_b$ so that $a \in \mathbb{O}_a, b \in \mathbb{O}_b$, and so that $\mathbb{O}_a \cup \mathbb{O}_b$ has compact closure in D. ϕ is an open map, so $\phi(\mathbb{O}_a) \cap \phi(\mathbb{O}_b)$ is open and nonempty. Then we choose a sequence (c_n) of distinct points in $\phi(\mathbb{O}_a) \cap \phi(\mathbb{O}_b)$. For every n we select $a_n \in \mathbb{O}_a, b_n \in \mathbb{O}_b$ with $\phi(a_n) = \phi(b_n) = c_n$.

It suffices to assume that F is not the zero function. It follows that F has finitely many zeros in $\mathbb{O}_a \cup \mathbb{O}_b$ and that we may assume that $F(a_n) \neq 0$ and that $F(b_n) \neq 0$ for every n. For $w \in D$, let $K_w(z) = \frac{1}{1-\overline{w}z}$ be the kernel for evaluation at w. The set $\{K_w : w \in D\}$ is a linearly independent set. For every n, let

$$g_n(z) = \overline{F(b_n)}K_{a_n} - \overline{F(a_n)}K_{b_n}.$$

$\{g_n\}$ is a linearly independent set, and each g_n induces a bounded linear functional Λ_n on H^p. For $f \in H^P$,

$$\Lambda_n(Tf) = \Lambda_n(Ff \circ \phi) = F(b_n)(F(a_n)f(c_n)) - F(a_n)(F(b_n)f(c_n)) = 0.$$

Hence,

$$\bigcap Ker(\Lambda_j) \supset T(H^p),$$

and the proof is complete. □

It follows immediately that if $T = M_F C_\phi$ as in Theorem B and if T has finite codimension, then $\phi \in \mathbb{A}$.

Hence we need only consider isometries of the form $M_F C_\phi$ where $\phi \in \mathbb{A}$ and $F \in H^p$. If $\phi(z) = \frac{\lambda(z-c)}{1-\bar{c}z}$, then $M_F C_\phi = M_{\frac{F}{(\overline{\lambda}\,\phi')^{1/p}}} U_\phi$. Since $U_\phi \in \mathbb{I}(H^p)$, it follows that $M_{\frac{F}{(\overline{\lambda}\,\phi')^{1/p}}}$ must be isometric. This means that $\frac{F}{(\overline{\lambda}\,\phi')^{1/p}}$ is an inner function, which we label as Ψ.

Clearly $M_\Psi U_\phi$ has the codimension of M_Ψ. The codimension is $n < \infty \Leftrightarrow \Psi$ is an n-fold Blaschke product. In this case we write $\Psi \in \mathbb{A}_n$. In particular $\mathbb{A}_1 = \mathbb{A}$.

We have shown that the set of isometries of codimension n is given by

$$\mathbb{I}_n(H^p) = \{M_\Psi U_\phi \ : \ \phi \in \mathbb{A}\,, \Psi \ \in \mathbb{A}_n \}.$$

In most of what follows, we focus on the isometries

$$\mathbb{I}_1(H^p) = \{M_\psi U_\phi \ : \phi, \psi \in \mathbb{A}\}$$

of codimension one.

Theorem 1. *Let* $S_1 = M_\psi U_\phi \in \mathbb{I}_1(H^p)$. *If* $S_2 \in \mathbb{I}_1(H^p)$, *then* $S_2 \approx S_1 \Leftrightarrow \exists \eta \in \mathbb{A}$ *and* $\rho \in T$ *so that* $S_2 = M_{\rho\psi\circ\eta} U_{\eta_{-1}\circ\phi\circ\eta}$.

Proof. $S_2 \approx S_1 \Leftrightarrow \exists \eta \in \mathbb{A}$ so that $U_{\eta_{-1}} S_1 U_\eta = S_2$. But

$$\begin{aligned} U_\eta S_1 U_{\eta-1} &= U_\eta M_\psi U_\phi U_{\eta_{-1}} = M_{\psi\circ\eta} U_\eta(\rho_1 U_{\eta_{-1}\circ\phi}) = M_{\psi\circ\eta}\rho_1(U_\eta U_{\eta_{-1}\circ\phi}) \\ &= M_{\psi\circ\eta}\rho_1\rho_2 U_{\eta_{-1}\circ\phi\circ\eta} = \rho M_{\psi\circ\eta} U_{\eta_{-1}\circ\phi\circ\eta}. \end{aligned}$$

Here ρ_1 and ρ_2 are the unimodular constants that arise in Lemma 1 (a), and $\rho = \rho_1\rho_2$. □

With $e(z) = z$ note that if $\psi \in \mathbb{A}$, then $M_\psi = M_\psi U_e$ has codimension one.

Corollary 1. *If* $\psi \in \mathbb{A}$, *then* $S \approx M_\psi$ *if and only if* $S \in \mathbb{I}_1(H^p)$, *and* $S = M_{\widetilde{\psi}}$ *for some* $\widetilde{\psi} \in \mathbb{A}$.

Proof. $S \approx M_\psi \Leftrightarrow \exists \eta \in \mathbb{A}$ so that

$$S = U_\eta M_\psi U_{\eta_{-1}} = M_{\psi\circ\eta} U_\eta U_{\eta_{-1}} = M_{\psi\circ\eta}.$$

Finally, note that $\{\psi \circ \eta : \ \eta \in \mathbb{A}\} = \mathbb{A}$.

We remark that the above result shows that $S \approx M_z \Leftrightarrow S = M_\psi$ for some $\psi \in \mathbb{A}$. We now generalize the last corollary. Fix $\phi \in \mathbb{A}$ and consider when $M_\psi U_\phi \approx M_{\widetilde{\psi}} U_\phi$. Corollary 1 settles the question if $\phi = e$. □

So suppose $\eta \in \mathbb{A}$ and that $U_\eta(M_\psi U_\phi)U_{\eta_{-1}} = M_{\widetilde{\psi}}U_\phi$. The left side simplifies to

$$M_{\psi\circ\eta}U_\eta U_\phi U_{\eta_{-1}} = \rho M_{\psi\circ\eta}U_{\eta_{-1}\circ\phi\circ\eta},$$

and with the notation $\widetilde{\psi} = \rho\psi\circ\eta$ and $\phi = \eta_{-1}\circ\phi\circ\eta$, we have our equality

$$U_\eta(M_\psi U_\phi)U_{\eta_{-1}} = M_{\widetilde{\psi}}U_\phi.$$

It follows that $\phi\circ\eta = \eta\circ\phi$, so ϕ and η commute. Thus we have

Corollary 2. *Let ϕ, ψ and $\widetilde{\psi}$ be in $\mathbb{A}$. Then $M_\psi U_\phi \approx M_{\widetilde{\psi}}U_\phi \Leftrightarrow \widetilde{\psi} = \rho\psi\circ\eta$ for some $\eta \in \mathbb{A}$ with η commuting with ϕ and some $\rho \in T$ satisfying*

$$U_\eta U_\phi U_{\eta_{-1}} = \rho U_{\eta_{-1}\circ\phi\circ\eta}.$$

Remark. We will discuss in Section 7 the classification of the automorphisms commuting with a fixed $\phi \in \mathbb{A}$.

Recall that $\forall\ \psi \in \mathbb{A}$, M_ψ is a Crownover shift. That is,

$$\bigcap_{n=1}^{\infty}(M_\psi)^n H^p = \bigcap_{n=1}^{\infty}(M_{(\psi)^n})H^p = (0).$$

Given an $S = M_\psi U_\phi \in \mathbb{I}_1(H^p)$, when is S Crownover? Now

$$S^2 = (M_\psi U_\phi)(M_\psi U_\phi) = M_\psi M_{\psi\circ\phi}U_\phi U_\phi = \rho M_\psi M_{\psi\circ\phi}U_{\phi_2},$$

for some $\rho \in T$. Iterating, we have

$$S^n = (M_\psi U_\phi)^n = \rho M_\psi M_{\psi\circ\phi}\cdots M_{\psi\circ\phi_{n-1}}U_{\phi_n},$$

where $\rho \in T$ depends on n.

Now U_{ϕ_n} is onto, so $S^n H^p = B_n H^p$, where B_n is the Blaschke product $\prod_{k=0}^{n-1} b_k$, where

$$b_k = \psi\circ\phi_k.$$

Note that b_k is merely the kth term of the sequence $(\psi\circ\phi_k)$ and does not represent the kth iterate of b.

It follows that $\bigcap_{n=1}^\infty S^n H^p = \bigcap_{n=1}^\infty B_n H^p$. If this intersection contains an $f \neq 0$, then each b_k is a factor of f, so by Proposition 1 there is a Blaschke product of the form $B = \prod_{k=0}^\infty \lambda_k b_k$ such that $\bigcap_{n=1}^\infty S^n H^p = BH^p$. Thus the zeros of $(b_k)_{k=0}^\infty$ form a Blaschke sequence. The above discussion shows that

Theorem 2. *$M_\psi U_\phi$ is Crownover $\Leftrightarrow$ the sequence of zeros of $(\psi\circ\phi_k)_{k=0}^\infty$ is not a Blaschke sequence.*

We will elaborate on this result in the next section using the fixed point structure of ϕ.

At this time we maintain the terminology as above, assuming that $S = M_\psi U_\phi$ and that $B = \prod_{k=0}^\infty \lambda_k b_k$ is an infinite Blaschke product with $\lambda_0 = 1$. Note that BH^p is an invariant subspace for S. We will show that $S|_{BH^p} \in \mathbb{I}(BH^P)$. First note that

$$M_\psi C_\phi B = M_\psi C_\phi \prod_0^\infty \lambda_k b_k = \psi\prod_0^\infty \lambda_k b_{k+1} = B.$$

So if $g \in H^p$, then

$$SBg = M_\psi U_\phi Bg = BU_\phi g,$$

and $S|_{BH^p}$ is onto BH^p.

Lastly, we note that $S|_{BH^p}$ is isometrically equivalent to U_ϕ. In fact, let

$$V : H^p \to BH^p$$

be the isometry defined by

$$Vg = Bg, \ \ g \in H^p.$$

Then

$$g \in H^p \Rightarrow (S|_{BH^p})V_g = S(Bg) = BU_\phi g,$$

so that $S|_{BH^p} \approx U_\phi$.

Remark. We now consider the case as above but with $p = 2$. The Wold decomposition for the isometry S (see [9, Th.1.1]) is easy to exhibit. Namely, $H^2 = BH^2 \oplus (BH^2)^\perp$ is a direct sum of invariant subspaces of S. The operator $S|_{BH^2}$ is unitary and is in fact unitarily equivalent to U_ϕ, while $S|_{(BH^2)^\perp}$ is a unilateral shift. If $\psi(z) = \frac{\mu(z-b)}{1-\bar{b}z}$, then the span of the kernel $K_b(z) = \frac{1}{1-\bar{b}z}$ is a wandering subspace for the shift.

5. The Crownover property

Each $\phi \in \mathbb{A}, \phi \neq e$, can be classified as elliptic, hyperbolic, or parabolic according to its fixed points in $\overline{D}$. See [1] or [8] for more detail.

Definition 5. $\phi \in \mathbb{A}$ is elliptic if ϕ has a unique fixed point, say a, in D. Let $\mathbb{E}(a) = \{\psi \in \mathbb{A} : \ \psi(a) = a, \ \ \psi \neq e\}$ denote the set of all elliptic automorphisms of D that fix a.

Choose $\eta \in \mathbb{A}$ with $\eta(a) = 0$ and note that $\eta \circ \mathbb{E}(a) \circ \eta_{-1}$ is the set of nontrivial rotations of D. Thus each of the two sets $\{\mathbb{E}(a) : a \in D\}$ is conjugate to one another.

Definition 6. ϕ is parabolic if it has only one fixed point, say w, in $\mathbb{T}$. In this case $\phi'(w) = 1$. Of course w is also the unique fixed point of ϕ_{-1}. w is attractive for ϕ (and for ϕ_{-1}). That is, for all $c \in D$, $\phi_n(c) \to w$ and $\phi_{-n}(c) \to w$.

Definition 7. $\phi \in \mathbb{A}$, $\phi \neq e$, is called hyperbolic if ϕ has two distinct fixed points, say w_1 and w_2, on T. In this case one of the fixed points, say w_1, is the attractive fixed point for ϕ. Also w_2 is the attractive fixed point for ϕ_{-1}. Further, $\phi'(w_1) < 1$ and $\phi'(w_2) > 1$.

Let

$$\mathbb{H}(w_1 \, , \, w_2) = \{\phi \in \mathbb{A} \, , \phi \neq e : \phi(w_1) = w_1 \, , \, \phi(w_2) = w_2\}$$

be the collection of hyperbolic automorphisms that fix w_1 and w_2. If w_1', w_2' is another pair of distinct points on T and $\eta \in \mathbb{A}$ is chosen so that $\eta(w_1) = w_1', \eta(w_2) = w_2'$, then $\mathbb{H}(w_1', w_2') = \eta \circ \mathbb{H}(w_1, w_2) \circ \eta_{-1}$. Thus all of these sets are conjugate to one another.

As an example, take $w_1 = -1$, $w_2 = 1$. Then one can show $\mathbb{H}(-1, 1) = \{\psi_r : -1 < r < 0 \text{ or } 0 < r < 1\}$, where $\psi_r(z) = \frac{z-r}{1-rz}$.

Proposition 3. *If ϕ is elliptic and $\psi \in \mathbb{A}$, then $M_\psi U_\phi$ is Crownover.*

Proof. Since ϕ is conjugate to a rotation, it is routine to check that the zeros of $\psi \circ \phi_n$ lie on a circle in D and hence cannot be a Blaschke sequence. Now apply Theorem 2. □

Proposition 4. *If $\psi \in \mathbb{A}$ and ϕ is hyperbolic, then $M_\psi U_\phi$ is not Crownover.*

Proof. It is easy to check that if ϕ is hyperbolic and $c \in D$, then $\sum(1-|\phi_n(c)|) < \infty$. (See [8, p. 85, #6].) Suppose $\psi \circ \phi_n(a_n) = 0 \;\forall n \geq 0$. Then $\phi_n(a_n) = \psi_{-1}(0)$ and $a_n = \phi_{-n} \circ \psi_{-1}(0)$. But ϕ_{-1} is also hyperbolic, so $\sum(1 - |\phi_{-n}(\psi_{-1}(0))|) = \sum(1 - |a_n|) < \infty$ and (a_n) is a Blaschke sequence. □

The inequality occurring in the first line of the above proof is easy to obtain since hyperbolic automorphisms have strongly attractive fixed points on the unit circle. However, we will need the inequality for parabolic automorphisms as well, and we establish it in Lemma 5.

Our goal is to show that if $\phi, \psi \in \mathbb{A}$ with ϕ parabolic, then $M_\psi U_\phi$ is not Crownover. That is, the zeros of $(\psi \circ \phi_n)$ form a Blaschke sequence, just as in the case that ϕ is hyperbolic.

Definition 8. For $w \in T$, let $\mathbb{P}(w)$ be the collection of parabolic automorphisms that fix w.

It is easy to see that, as in the elliptic and hyperbolic cases, the sets $\mathbb{P}(w)$, $w \in T$, are conjugate to each other. So we first consider $\mathbb{P}(1)$.

A computation will show that

$$\phi(z) = \frac{\lambda(z-a)}{1-\overline{a}z} \in \mathbb{P}(1) \Leftrightarrow \phi(1) = 1 = \phi'(1).$$

In this case, solving for a and λ, we see that

$$|a - 1/2| = 1/2,\; a \neq 0, 1,$$

and that $\lambda = \frac{1-\overline{a}}{1-a}$. So $a - 1/2 = (c/2)$, where $c \in T, c \neq \pm 1$, and thus $\lambda = \frac{1-\overline{c}}{1-c} = \frac{-1}{c}$. Using these equalities we can write ϕ in the form

$$\phi(z) = \frac{1+c-2z}{2c-(1+c)z},$$

which we write as $\phi_c(z)$.

So we have

$$\mathbb{P}(1) = \{\phi_c(z) = \frac{1+c-2z}{2c-(1+c)z} : \; c \in T\,, c \neq \pm 1\}\,.$$

The functions $\phi_\imath$ and $\phi_{-\imath}$ play a special role in what follows.

Observe that if $\phi \in \mathbb{P}(1)$ and if $\psi \in \mathbb{A}$ with $\psi(1) = 1$, then

$$\psi \circ \phi \circ \psi_{-1} \in \mathbb{P}(1).$$

Here ψ could be hyperbolic. Our approach is to conjugate $\phi_\imath$ (or $\phi_{-\imath}$) by automorphisms $\psi_r \in \mathbb{H}(-1,1)$, discussed after Definition 7. We note that the inverse of ψ_r is ψ_{-r}.

Proposition 5. *Let $\phi_c\,(z) = \frac{1+c-2z}{2c-(1+c)z}$ where $c \in T$, $c \neq \pm i$. If $\Im(c) > 0$, then $\exists\, r \in (-1,1)$, $r \neq 0$ and $\psi_r \in \mathbb{H}(-1,1)$ so that $\phi_c = \psi_r \circ \phi_i \circ \psi_{-r}$, while if $\Im(c) < 0, \exists$ another $r \in (-1,1)$, $r \neq 0$ so that $\phi_c = \psi_r \circ \phi_{-i} \circ \psi_{-r}$,*

Proof. Let $r \in (-1,1) : r \neq 0$. Then $\psi_r \circ \phi_\imath \circ \psi_{-r} \in \mathbb{P}(1)$, so if z_r is the zero of $\psi_r \circ \phi_i \circ \psi_{-r}$, then $|z_r - 1/2| = 1/2$. Letting $c_r = 2z_r - 1$, it suffices to show that $\{c_r \,:\, -1 < r < 1\}$ is the upper half semicircle of T.

A careful computation shows that

$$\psi_r \circ \phi_i \circ \psi_{-r}(z) = \frac{(1-r)^2 - ((1-r)^2 - i(1-r^2))z}{(1-r)^2 + i((1-r^2) - (1-r)^2)z},$$

so that

$$z_r = \frac{(1-r)^2 + i(1-r^2)}{2(1+r^2)}.$$

Thus

$$c_r = 2z_r - 1 = \frac{-2r + i(1-r^2)}{1+r^2}.$$

One checks that as r goes from -1 to 1, c_r traces out the required semicircle. A similar computation for $\phi_{-\imath}$ yields a c_r that traces out the lower semicircle of T. □

Theorem 3. $\mathbb{C}(\phi_i) \cup \mathbb{C}(\phi_{-\imath}) = \mathbb{P}$, *the collection of all parabolic automorphisms.*

Proof. $\mathbb{P} = \bigcup_{w \in T} \mathbb{P}(w)$. Given $w \in T$, choose $\eta \in \mathbb{A}$ so that $\eta(1) = w$. Then we have

$$\eta \circ \mathbb{P}(1) \circ \eta_{-1} = \mathbb{P}(w).$$

Thus each $\psi \in \mathbb{P}(w)$ is conjugate to some $\phi \in \mathbb{P}(1)$, and our previous result shows that ϕ is conjugate to ϕ_i or to ϕ_{-i}. Thus ψ is conjugate to ϕ_i or to ϕ_{-i}. □

We will now examine the automorphism $\phi_i(z)$ and its inverse $\phi_i = \phi_{-i}$.

Lemma 3. *The zeroes of the iterates of* $\phi_\imath$ *(and those of* $\phi_{-\imath}$*) form a Blaschke sequence.*

Proof. Multiplying each coefficient of ϕ_i by $(1-i)/2$ shows that

$$\phi_i(z) = \frac{1-(1-i)z}{1+i-z}.$$

An easy computation shows that

$$\phi_i \circ \phi_i(z) = \frac{2-(2-i)z}{(2+i)-2z},$$

and by induction we see that the nth iterate is given by

$$(\phi_i)_n\,(z) = \frac{n-(n-i)z}{n+i-nz}.$$

Thus $a_n = n/(n-i)$ is the zero of $(\phi_i)_n$, so $|a_n|^2 = n^2/(n^2+1)$. So $\sum(1-|a_n|^2) < \infty$, and (a_n) is a Blaschke sequence. Essentially the same argument shows that the zeroes of $(\phi_{-i})_n$ also form a Blaschke sequence. □

We will use the following in the proof below.

Lemma 4. *Suppose that* (d_n) *is a Blaschke sequence and that* $\eta \in \mathbb{A}$. *Then* $\eta(d_n)$ *is a Blaschke sequence.*

Proof. Suppose $\eta(z) = \lambda\frac{z-a}{1-\overline{a}z}$. Then

$$\begin{aligned} 1 - |\eta(d_n)|^2 &= ((|1-\overline{a}d_n|^2) - (|d_n - a|^2))/|1-\overline{a}d_n|^2 \\ &= ((1-|d_n|^2) + |a|^2(|d_n|^2 - 1))/(|1-\overline{a}d_n|^2) \\ &\le 2(1-|d_n|^2)/(1-|a|), \end{aligned}$$

and the result follows. □

Lemma 5. *Suppose that $\phi \in \mathbb{A}$ with $\phi_n(a_n) = 0, \forall n \Rightarrow (a_n)$ is a Blaschke sequence. Then*

(i) *If $\psi \in \mathbb{A}$ and $\psi \circ \phi_n(b_n) = 0$ $\forall n$, then (b_n) is a Blaschke sequence.*

(ii) *If $\widetilde{\phi} \in \mathbb{C}(\phi)$ and if $(\widetilde{\phi})_n(c_n) = 0$ $\forall n$, then (c_n) is a Blaschke sequence.*

Proof. For (i) assume (a_n) is a Blaschke sequence and consider $\psi \circ \phi_n(b_n) = 0$, with $\phi_n(z) = \frac{\lambda_n(z-a_n)}{1-\overline{a_n}z}$. So $b_n = \phi_{-n} \circ \psi_{-1}(0)$. Then, using essentially the same argument as in Lemma 4,

$$1 - |b_n|^2 = \frac{|1 + \overline{\lambda_n a_n}\alpha|^2 - |\alpha + \lambda_n a_n|^2}{|2 + \overline{\lambda_n a_n}\alpha|^2} \leq \frac{2(1 - |a_n|^2)}{1 - |\alpha|},$$

so (b_n) is a Blaschke sequence.

For part (ii) by our assumption $\widetilde{\phi} = \eta \circ \phi \circ \eta_{-1}$ and so assuming $(\widetilde{\phi})_n(c_n) = 0$, we have $(\widetilde{\phi})_n \circ \eta(d_n) = 0$, where $\eta(d_n) = c_n$. Thus $\eta \circ \phi_n(d_n) = 0$. By part (i) the sequence (d_n) is a Blaschke sequence. Applying Lemma 4 we find that $(c_n) = (\eta(d_n))$ is a Blaschke sequence. □

Theorem 4. *If $\phi \in \mathbb{A}$ is parabolic and $\psi \in \mathbb{A}$, then $M_\psi U_\phi$ is not Crownover.*

Proof. Suppose ϕ is parabolic. Then $\phi \in \mathbb{C}(\phi_\imath)$ or $\mathbb{C}(\phi_{-\imath})$ so by Lemma 3 and Lemma 4 (ii), $\phi_n(c_n) = 0 \;\; \forall n \Rightarrow \{c_n\}$ is a Blaschke sequence. Now by Lemma 4 (i), $\Psi \circ \phi_n(b_n) = 0 \;\; \forall n \Rightarrow \{b_n\}$ is a Blaschke sequence. The result now follows from Theorem 2. □

6. Isometries of codimension greater than one

Recall that if S is an isometry on H^p $(p \neq 2)$ of codimension $d < \infty$, then $S = M_\Psi U_\phi$ for some $\phi \in \mathbb{A}$ and Ψ is a d-fold Blaschke product. Our key results for codimension one isometries carry over easily to this setting. Thus

(i) if $\widetilde{S} \in B(H^p)$, then

$$\widetilde{S} \approx S \Leftrightarrow \widetilde{S} = \rho M_{\Psi \circ \eta} U_{\eta_{-1} \circ \phi \circ \eta}$$

for some $\eta \in \mathbb{A}$ and $\rho \in T$ which is determined by ϕ and ρ.

(ii) $\bigcap_n^\infty S^n H^p = (0) \Leftrightarrow \phi$ is elliptic.

For (ii) , one observes that the zeros of $(\Psi \circ \phi_n, \, n \geq 0)$ can be written as a union of d Blaschke sequences.

We now consider isometries S of infinite codimension. These can arise in two ways. S could have the form $M_F C_\Phi$ where Φ is inner and $\Phi \notin \mathbb{A}$. The other possibility is that $S = M_\Psi U_\phi$ where $\phi \in \mathbb{A}$ and Ψ is inner and not a finite Blaschke product. We focus on this latter case.

Note that $\bigcap_n^\infty S^n H^p = (0)$ if ϕ is elliptic.

Proposition 6. *Suppose $\phi \in \mathbb{A}$ and ϕ is parabolic or hyperbolic. Depending on this choice of Blaschke product Ψ, the isometry*

$$S = M_\Psi U_\phi$$

can satisfy either $\bigcap_1^\infty S^n H^p = (0)$ or $\bigcap_1^\infty S^n H^p \neq (0)$.

Proof. The following proof will utilize the results of Theorems 2 and 4. Suppose that $\phi \in \mathbb{A}$ is parabolic and thus by Proposition 1 we choose $\{\lambda_n\}$ in T such that $\Psi = \prod_1^\infty \lambda_n \phi_{-n}$ is a convergent Blaschke product.

Note that

$$\Psi \circ \phi = \prod_1^\infty \lambda_n \phi_{-n+1} = e\,\Psi.$$

Iterating this step, we see that $\Psi \circ \phi_n$ has Ψ as a factor $\forall n > 0$. Thus $\prod_1^\infty \Psi \circ \phi_n$ cannot be a convergent Blaschke product. Hence $\bigcap_1^\infty S^n H^p = \{0\}$ for $S = M_\Psi U_\phi$.

Now suppose that $\phi_n(a_n) = 0\,\forall n > 0$, so that we must have $\sum_1^\infty (1 - |a_n|) = R < \infty$. Choose $1 < n_1 < n_2 < \dots$, such that $\forall k \geq 1$,

$$\sum_{n=n_k}^\infty (1 - |a_n|) < R/(2^k)$$

and let $\Psi = \prod_{k=1}^\infty \phi_{n_k}$. Since

$$\sum_{k=1}^\infty \sum_{n=n_k}^\infty (1 - |a_n|) < \sum_{k=1}^\infty R/(2^k) < \infty,$$

we see that the zeroes of $\prod_1^\infty \Psi \circ \psi_n$ form a Blaschke sequence. Thus $\prod_1^\infty \Psi \circ \phi_n$ is convergent, and $\bigcap_1^\infty S^n H^p \neq \{0\}$ for $S = M_\Psi U_\phi$. □

7. Commuting automorphisms

In this section we elaborate on the conclusion in Corollary 2 by describing the automorphisms of D that commute with a fixed automorphism. These results are undoubtedly known and we outline proofs using the results from the last section.

Definition 9. For $\phi \in \mathbb{A}$, let

$$Com(\phi) = \{\psi \in \mathbb{A} :\ \psi \circ \phi = \phi \circ \psi\}.$$

Clearly $Com(\phi)$ is a subgroup of $\mathbb{A}$, and $Com(\,e\,) = \mathbb{A}$.

Proposition 7. *Let $\phi \in \mathbb{A}, \phi \neq e$.*

(i) *If $\phi \in \mathbb{E}(a)$, then $Com(\phi) = \mathbb{E}(a) \cup \{e\}$.*
(ii) *If $\phi \in \mathbb{P}(w)$, then $Com(\phi) = \mathbb{P}(w) \cup \{e\}$.*
(iii) *If $\phi \in \mathbb{H}(w_1,\ w_2\,)$, then $Com(\phi) = \mathbb{H}(w_1, w_2\,) \cup \{e\}$.*
(iv) *In each of these cases, $Com(\phi)$ is abelian.*

Proof. Observe that $Com(\eta \circ \phi \circ \eta_{-1}) = \eta \circ Com(\phi) \circ \eta_{-1}$. So it will suffice to consider $a = 0\,, w = 1$, $w_1 = -1$, and $w_2 = 1$.

For (i), if $\phi \in \mathbb{E}(0) = \{\eta_\lambda(z) = \lambda z : \lambda \in T, \lambda \neq 1\}$, then $\mathbb{E}(0) \subset Com(\phi)$. But if $\psi \in Com(\phi)$, then $\psi(0) = \psi(\phi(0)) = \phi(\psi(0))$, so $\psi(0) = 0$. Thus $\psi \in \mathbb{E}(0)$ or $\psi = e$.

For (ii), if

$$\phi \in \mathbb{P}(1) = \{\phi_c = \frac{1 + c - 2z}{1 + i - z} : c \in T\ ,\ c \neq \pm 1\},$$

then a computation shows that $\mathbb{P}(1)$ is an abelian set. The same argument as in (i) shows that if $\psi \in Com(\phi)$, then $\psi = e$ or ψ has 1 as its unique fixed point. So $Com(\phi) = \mathbb{P}(1) \cup (\{e\})$.

For (iii), suppose that

$$\phi \in \mathbb{H}(-1\ ,\ 1) = \{\psi_r(z) = \frac{z - r}{1 + rz} : -1 < r < 0 \text{ or } 0 < r < 1\}.$$

Observe that $\mathbb{H}(-1\,,\,1)$ is an abelian set. Let $\psi \in Com(\phi)$. It follows easily that

$$\{\psi(1)\,,\psi(-1)\,\} = \{\pm 1\}.$$

If $\psi(1) = 1$ and $\psi(-1) = -1$, then $\psi \in \mathbb{H}(-1\,,\,1)$ as desired. So suppose $\psi(-1) = 1$ and $\psi(1) = -1$. Then $-\psi \in \mathbb{H}(-1\,,\,1)$ so $\psi(z) = \frac{r_1 - z}{1 - r_1 z}$ for some r_1. A computation will show that ψ does not commute with automorphisms in $\mathbb{H}(-1\,,\,1)$. Thus $Com(\phi) = \mathbb{H}(-1\,,\,1) \cup \{e\}$. □

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