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A NOTE ON INNER QUASIDIAGONAL C*-ALGEBRAS

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ABSTRACT. In this paper, we give two new characterizations of separable inner quasidiagonal C*-algebras. Based on these characterizations, we show that a unital full free product of two inner quasidiagonal C*-algebras is itself inner quasidiagonal. As an application, we show that a unital full free product of two inner quasidiagonal C*-algebras with amalgamation over a full matrix algebra is inner quasidiagonal. Meanwhile, we conclude that a unital full free product of two AF algebras with amalgamation over a finite-dimensional C*-algebra is inner quasidiagonal if there are faithful tracial states on each of these two AF algebras such that the restrictions of these states to the common subalgebra coincide.

1. INTRODUCTION

Quasidiagonal (QD) C*-algebras have now been studied for more than 30 years. Voiculescu [18] gives a characterization of quasidiagonal C*-algebras as follows:

Definition 1. A C*-algebra $\mathcal{A}$ is quasidiagonal if, for every $x_1, \cdots, x_n \in \mathcal{A}$ and $\varepsilon > 0$, there is a representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, and a finite-rank projection $P \in \mathcal{B}(\mathcal{H})$ such that $\|P\pi(x_i) - \pi(x_i)P\| < \varepsilon, \|P\pi(x_i)P\| > \|x_i\| - \varepsilon$ for $1 \leq i \leq n$.

Voiculescu showed that $\mathcal{A}$ is QD if and only if $\pi(\mathcal{A})$ is a quasidiagonal set of operators for a faithful essential representation π of $\mathcal{A}$. In [3], it was shown that all separable QD C*-algebras are Blackadar and Kirchberg's MF algebras. It is well known that the reduced free group C*-algebra $C_r^*(F_2)$ is not QD. Haagerup and Thorbjørnsen showed that $C_r^*(F_2)$ is MF [11]. It follows that the family of all separable QD C*-algebras is strictly contained in the set of MF C*-algebras.

The concept of MF algebras was first introduced by Blackadar and Kirchberg in [3]. Many properties of MF algebras were discussed in [3]. In the same article, Blackadar and Kirchberg study NF algebras and strong NF algebras as well. A separable C*-algebra is a strong NF algebra if it can be written as a generalized inductive limit of a sequential inductive system of finite-dimensional C*-algebras in which the connecting maps are complete order embedding and asymptotically multiplicative in the sense of [3]. An NF algebra is a C*-algebra which can be written as the generalized inductive limit of such a system, where the connecting

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maps are only required to be completely positive contractions. It was shown that a separable C*-algebra is an NF algebra if and only if it is nuclear and quasidiagonal. Is the class of NF algebra distinct from the class of strong NF algebras? To solve this question, Blackadar and Kirchberg introduced the concept of inner quasidiagonality by slightly modifying Voiculescu's characterization of quasidiagonal C*-algebras:

Definition 2 ([4])**.** A C*-algebra $\mathcal{A}$ is inner quasidiagonal if, for every $x_1, \cdots, x_n \in \mathcal{A}$ and $\varepsilon > 0$, there is a representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, and a finite-rank projection $P \in \pi(\mathcal{A})''$ such that $\|P\pi(x_i) - \pi(x_i)P\| < \varepsilon, \|P\pi(x_i)P\| > \|x_i\| - \varepsilon$ for $1 \le i \le n$.

It was shown that a separable C*-algebra is a strong NF algebra if and only if it is nuclear and inner quasidiagonal [4]. Blackadar and Kirchberg also gave examples of separable nuclear C*-algebras which are quasidiagonal but not inner quasidiagonal, hence of NF algebras which are not strong NF. Therefore, the preceding question has been solved.

In this note, we are interested in the question of whether the unital full free products of inner QD C*-algebras are inner QD again. Note that every RFD C*-algebra is inner QD [4]. It was shown that a unital full free product of two RFD C*-algebras is RFD [17]. A similar result holds for unital QD C*-algebras [2]. Based on these results and the relationship between RFD C*-algebras, inner QD C*-algebras and QD C*-algebras, it is natural to ask whether the same things happen with inner QD C*-algebras. In this note we show that a unital full free product of two unital inner QD C*-algebras is inner again. As an application, we consider the unital full free products of two inner QD C*-algebras with amalgamation over finite-dimensional C*-algebras.

All C*-algebras in this note are unital and separable. Let us briefly outline the paper. In Section 2, we fix some notation and give two new characterizations of inner QD C*-algebras. Section 3 deals with results on the unital full free products of two unital inner QD C*-algebras. We first consider unital full free products of unital inner QD C*-algebras. As an application, we show that a unital full free product of two inner quasidiagonal C*-algebras with amalgamation over a full matrix algebra is inner quasidiagonal. Meanwhile, we conclude that a unital full free product of two AF algebras with amalgamation over a finite-dimensional C*-algebra is inner quasidiagonal if there are faithful tracial states on each of these two AF algebras such that the restrictions of these states to the common subalgebra coincide.

2. Inner quasidiagonal C*-algebras

We denote the set of all bounded operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$.

Suppose $\{\mathcal{M}_{k_n}(\mathbb{C})\}_{n=1}^{\infty}$ is a sequence of complex matrix algebras. We introduce the C*-direct product $\prod_{m=1}^{\infty} \mathcal{M}_{k_m}(\mathbb{C})$ of $\{\mathcal{M}_{k_n}(\mathbb{C})\}_{n=1}^{\infty}$ as follows:

$$\prod_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C}) = \{(Y_n)_{n=1}^{\infty} \mid \forall\, n \geq 1,\ Y_n \in \mathcal{M}_{k_n}(C) \text{ and } \|(Y_n)_{n=1}^{\infty}\| = \sup_{n\geq 1} \|Y_n\| < \infty\}.$$

Furthermore, we can introduce a norm-closed two sided ideal in $\prod_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C})$ as follows:

$$\sum_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C}) = \left\{ (Y_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C}) : \lim_{n\to\infty} \|Y_n\| = 0 \right\}.$$

Let π be the quotient map from $\prod_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C})$ to $\prod_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C})/\sum_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C})$. Then

$$\prod_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C})/\sum_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C})$$

is a unital C*-algebra. If we denote $\pi\left((Y_n)_{n=1}^{\infty}\right)$ by $[(Y_n)_n]$, then

$$\|[(Y_n)_n]\| = \limsup_{n\to\infty} \|Y_n\| \leq \sup_n \|Y_n\| = \|(Y_n)_n\|$$

for $(Y_n)_n \in \prod_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C})$.

Recall that a C*-algebra is said to be residually finite-dimensional (RFD) if it has a separating family of finite-dimensional representations. If a separable C*-algebra $\mathcal{A}$ can be embedded into $\prod_k \mathcal{M}_{n_k}(\mathbb{C}) / \sum_k \mathcal{M}_{n_k}(\mathbb{C})$ for a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$, then $\mathcal{A}$ is called an MF algebra. Many properties of MF algebras were discussed in [3]. Note that the family of all RFD C*-algebras is strictly contained in the family of all inner QD C*-algebras, and all QD C*-algebras are MF C*-algebras.

Continuing the study of generalized inductive limits of finite-dimensional C*-algebras, Blackadar and Kirchberg define a refined notion of quasidiagonality for C*-algebras, called inner quasidiagonality. A cleaner alternative definition of inner quasidiagonality can be given using the socle of the bidual.

Definition 3. If $\mathcal{B}$ is a C*-algebra, then a projection $p \in \mathcal{B}$ is in the socle if $p\mathcal{B}p$ is finite-dimensional. Denote the set of the socle in $\mathcal{B}$ by *socle* $(\mathcal{B})$

Theorem 1 ([8], Definition 3.1, Chapter 11). *A separable C*-algebra $\mathcal{A}$ is inner QD if there are projections $p_n \in \mathcal{A}^{**}$ such that*

(1) $\|[p_n, a]\| \longrightarrow 0$ *for all* $a \in \mathcal{A} \subseteq \mathcal{A}^{**}$,
(2) $\|a\| = \lim \|p_n a p_n\|$ *for all* $a \in \mathcal{A}$ *and*
(3) $p_n \in$ *socle* $(\mathcal{A}^{**})$ *for every* n.

Theorem 2 ([4], Proposition 3.7). *Let $\mathcal{A}$ be a separable C*-algebra. Then $\mathcal{A}$ is inner QD if and only if there is a sequence of irreducible representation $\{\pi_n\}$ of $\mathcal{A}$ on Hilbert space $\mathcal{H}_n$, and finite-rank projection $p_n \in \mathcal{B}(\mathcal{H}_n)$, such that $\|[p_n, \pi_n(x)]\| \longrightarrow 0$ and $\limsup \|p_n \pi_n(x) p_n\| = \|x\|$ for all $x \in \mathcal{A}$.*

The principal shortcoming of the definition of inner QD C*-algebra is that it is often difficult to determine directly whether a C*-algebra is inner QD. The following result for the separable case is much easier to check.

Theorem 3 ([5]). *A separable C*-algebra is inner QD if and only if it has a separating family of quasidiagonal irreducible representations.*

Let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H})$ be the canonical mapping onto the Calkin algebra and $\mathcal{A}$ be a unital C*-algebra. Suppose $\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a unital completely positive map; then we say that φ is a representation modulo the compacts if $\pi \circ \varphi : \mathcal{A} \to \mathcal{Q}(\mathcal{H})$ is a *-homomorphism. If $\pi \circ \varphi$ is injective, then we say that φ is a faithful representation modulo the compacts.

For an MF C*-algebra, we are able to embed it into $\prod_k \mathcal{M}_{n_k}(\mathbb{C}) / \sum_k \mathcal{M}_{n_k}(\mathbb{C})$ for a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$. For an RFD C*-algebra, we can embed it into $\prod_k \mathcal{M}_{n_k}(\mathbb{C})$. Meanwhile, for a QD C*-algebra, we cannot only embed it into

$\prod\limits_{k}\mathcal{M}_{n_k}(\mathbb{C})/\sum_k \mathcal{M}_{n_k}(\mathbb{C})$, but also lift this embedding to a faithful representation into $\prod\mathcal{M}_{k_m}(\mathbb{C})$ modulo the compacts [13]. Is there a similar characterization for the inner QD C*-algebras? We will answer this question in the following theorem.

The following lemma is a well-known result about a completely positive map. We use c.p. to abbreviate "completely positive", u.c.p. for "unital completely positive" and c.c.p. for "contractive completely positive".

Lemma 1 (Stinespring)**.** *Let $\mathcal{A}$ be a unital C*-algebra and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ be a c.p. map. Then, there exist a Hilbert space $\widehat{\mathcal{H}}$, a *-representation $\pi_\varphi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ and an operator $V : \mathcal{H} \longrightarrow \widehat{\mathcal{H}}$ such that*

$$\varphi(a) = V^* \pi_\varphi(a) V$$

for every $a \in \mathcal{A}$. In particular, $\|\varphi\| = \|V^*V\| = \|\varphi(1)\|$.

We call the triplet $\left(\pi_\varphi, \widehat{\mathcal{H}}, V\right)$ in the preceding lemma a Stinespring dilation of φ. When φ is unital, $V^*V = \varphi(I) = I$, and hence V is an isometry. So in this case we may assume that V is a projection P and $\varphi(a) = P\pi_\varphi(a)|_{\mathcal{H}}$. In general there could be many different Stinespring dilations, but we may always assume that a dilation $\left(\pi_\varphi, \widehat{\mathcal{H}}, V\right)$ is minimal in the sense that $\pi_\varphi(\mathcal{A}) V\mathcal{H}$ is dense in $\widehat{\mathcal{H}}$. We know that, under this minimality condition, a Stinespring dilation is unique up to unitary equivalence. It is well known that for $\left(\pi_\varphi, \widehat{\mathcal{H}}, V\right)$ minimal Stinespring dilation of $\varphi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$, there exists a *-homomorphism $\rho : \varphi(\mathcal{A})' \longrightarrow \pi_\varphi(\mathcal{A})' \subseteq \mathcal{B}\left(\widehat{\mathcal{H}}\right)$ such that $\varphi(a)x = V^*\pi_\varphi(a)\rho(x)V$ for every $a \in \mathcal{A}$ and $x \in \varphi(\mathcal{A})'$. It follows that the commutant $\varphi(\mathcal{A})' \subseteq \mathcal{B}(\mathcal{H})$ also lifts to $\mathcal{B}\left(\widehat{\mathcal{H}}\right)$.

Lemma 2. *Let $\mathcal{A}$ be a unital C*-algebra and $\varphi : \mathcal{A} \longrightarrow \mathcal{M}_n(\mathbb{C})$ be a surjective u.c.p. map. Suppose $\left(\pi_\varphi, \widehat{\mathcal{H}}, P\right)$ is a minimal Stinespring dilation of φ where P is a projection in $\mathcal{B}\left(\widehat{\mathcal{H}}\right)$. Then the *-homomorphism $\rho : \varphi(\mathcal{A})' \longrightarrow \pi_\varphi(\mathcal{A})' \subseteq \mathcal{B}\left(\widehat{\mathcal{H}}\right)$ is unital, i.e.,*

$$\rho(\mathbb{C}I) = \mathbb{C}I \subseteq \pi_\varphi(\mathcal{A})'.$$

Proof. Since φ is surjective, $\varphi(\mathcal{A})' = \mathbb{C}I$. Note ρ is a *-homomorphism. It is easy to check that

$$(I-P)\rho(\alpha I)P = P\rho(\alpha I)(I-P) = 0$$

and $(I-P)\rho(I)(I-P)$ is a projection. We know $\left(\pi_\varphi, \widehat{\mathcal{H}}, P\right)$ is a minimal Stinespring dilation; then $\pi_\varphi(\mathcal{A})'$ has no proper projection larger than P. It implies $(I-P)\rho(I)(I-P) = 0$, i.e., $\rho(I) = I$. Hence $\rho(\mathbb{C}I) = \mathbb{C}I \subseteq \pi_\varphi(\mathcal{A})'$. □

Note that $\Pi\mathcal{M}_{k_m}(\mathbb{C})$ can be viewed as a C*-subalgebra of $\mathcal{B}\left(\bigoplus_{m=1}^\infty \mathbb{C}^{k_m}\right)$. If $\mathcal{K}\left(\bigoplus_{m=1}^\infty \mathbb{C}^{k_m}\right)$ stands for the set of all compact operators on $\bigoplus_{m=1}^\infty \mathbb{C}^{k_m}$, then it is not hard to see that $\mathcal{K}\left(\bigoplus_{m=1}^\infty \mathbb{C}^{k_m}\right) \cap \Pi\mathcal{M}_{k_m}(\mathbb{C}) = \sum \mathcal{M}_{k_m}(\mathbb{C})$. Let P_n be the projection from $\bigoplus_{m=1}^\infty \mathbb{C}^{k_m}$ onto $\mathbb{C}^{k_n}$ for some $n \in \mathbb{N}$; then for any u.c.p. mapping $\Phi : \mathcal{A} \longrightarrow \Pi\mathcal{M}_{k_m}(\mathbb{C})$, the mapping $\varphi_n : \mathcal{A} \longrightarrow P_n\Pi\mathcal{M}_{k_m}(\mathbb{C})P_n$ with $\varphi_n(A) = P_n\Phi(A)P_n$ is itself u.c.p. Now, we are ready to give a new characterization of inner QD C*-algebras.

Theorem 4. *Let $\mathcal{A}$ be a unital C*-algebra. Then $\mathcal{A}$ is inner QD if and only if there is a faithful representation modulo compacts $\Phi : \mathcal{A} \longrightarrow \Pi\mathcal{M}_{k_n}(\mathbb{C})$ for a sequence $\{k_n\}$ of integers such that the u.c.p. map $\varphi_n : \mathcal{A} \longrightarrow \mathcal{M}_{k_n}(\mathbb{C})$ which naturally comes from Φ is surjective for every n and the *-homomorphism*

$$\rho : \varphi_n(\mathcal{A})' \longrightarrow \pi_{\varphi_n}(\mathcal{A})' \subseteq \mathcal{B}\left(\widehat{\mathcal{H}}_n\right)$$

is surjective where $\left(\pi_{\varphi_n}, \widehat{\mathcal{H}}_n, p_n\right)$ is a minimal Stinespring dilation of φ_n.

Proof. ($\Longrightarrow$) Suppose $\mathcal{A}$ is inner QD. Then, by applying Theorem 2, we can find sequences of irreducible representations $\{\pi_n\}$ and finite-rank projections $\{p_n\}$ where $p_n \in \pi_n(\mathcal{A})''$ such that $\Phi : \mathcal{A} \longrightarrow \Pi p_n\pi_n(\mathcal{A})p_n$ is a faithful representation modulo compacts. Meanwhile, we have $p_n\pi_n(\mathcal{A})p_n \cong \mathcal{M}_{k_n}(\mathbb{C})$ for some integer k_n and $(\pi_n(\mathcal{A}))' = \mathbb{C}I$ since π_n is irreducible. Define

$$\varphi_n : \mathcal{A} \longrightarrow p_n\pi_n(\mathcal{A})p_n \cong \mathcal{M}_{k_n}(\mathbb{C})$$

by $\varphi_n(a) = p_n\pi_n(a)p_n$. Then φ_n is u.c.p. and surjective for every n. Note $(\pi_n, \mathcal{H}_n, p_n)$ is a minimal Stinespring dilation of φ_n since π_n is irreducible. Therefore the *-homomorphism

$$\rho : \varphi_n(\mathcal{A})' \longrightarrow \pi_n(\mathcal{A})' \subseteq \mathcal{B}(\mathcal{H}_n)$$

is surjective by Lemma 2 and the fact that $\pi(\mathcal{A})' = \mathbb{C}I$.

($\Longleftarrow$) Suppose there is a faithful representation modulo compacts $\Phi : \mathcal{A} \longrightarrow \Pi\mathcal{M}_{k_n}(\mathbb{C})$ for a sequence $\{k_n\}$ such that the u.c.p. maps $\varphi_n : \mathcal{A} \longrightarrow \mathcal{M}_{k_n}(\mathbb{C})$ which naturally comes from Φ is surjective and the *-homomorphism

$$\rho : \varphi_n(\mathcal{A})' \longrightarrow \pi_{\varphi_n}(\mathcal{A})' \subseteq \mathcal{B}\left(\widehat{\mathcal{H}}_n\right)$$

is surjective where $\left(\pi_{\varphi_n}, \widehat{\mathcal{H}}_n, p_n\right)$ is a minimal Stinespring dilation of φ_n. Then

$$\rho\left(\varphi_n(\mathcal{A})'\right) = \rho(\mathbb{C}I) = \mathbb{C}I$$

by Lemma 2. It implies that $\pi_{\varphi_n}(\mathcal{A})' = \mathbb{C}I$ since ρ is surjective. Hence π_{φ_n} is irreducible and $p_n \in \pi_{\varphi_n}(\mathcal{A})''$. So, for these irreducible representations $\{\pi_{\varphi_n}\}$ of $\mathcal{A}$ on Hilbert space $\widehat{\mathcal{H}}_n$ and finite-rank projection $p_n \in \pi_{\varphi_n}(\mathcal{A})''$, we have $\|[p_n, \pi_{\varphi_n}(x)]\| \longrightarrow 0$ and $\limsup \|p_n\pi_{\varphi_n}(x)p_n\| = \|x\|$ for all $x \in \mathcal{A}$. It implies that $\mathcal{A}$ is inner QD by Theorem 2. □

Suppose $\mathcal{A}$ is a unital C*-algebra and $p \in socle(\mathcal{A}^{**})$. Define

$$\mathcal{A}_p = \{a \in \mathcal{A} : [a, p] = 0\}.$$

Then it is easy to see that $\mathcal{A}_p$ is a C*-subalgebra of $\mathcal{A}$.

Lemma 3 ([4], Corollary 3.5)**.** *Let $p \in socle(\mathcal{A}^{**})$. Then $d(a, \mathcal{A}_p) = \|[a, p]\|$for all $a \in \mathcal{A}$.*

Lemma 4 ([4], Proposition 3.4)**.** *Let $\mathcal{A}$ be a C*-algebra, and $p \in socle(\mathcal{A}^{**})$. Then*

(1) $p\mathcal{A}_p = p\mathcal{A}_p p = p\mathcal{A}^{**}p = p\mathcal{A}p$.
(2) *The weak closure of $\mathcal{A}_p$ in $\mathcal{A}^{**}$ (i.e. $\mathcal{A}_p^{**}$) is $p\mathcal{A}^{**}p + (1-p)\mathcal{A}^{**}(1-p)$.*

Lemma 5. *Let $\mathcal{A}$ be a C*-algebra, $p_1, \cdots, p_k \in socle(\mathcal{A}^{**})$ with $p_1 \leq \cdots \leq p_k$. Then*

$$p_k\left(\bigcap_{i=0}^{k}\mathcal{A}_{p_i}\right) = p_k\left(\bigcap_{i=0}^{k}\mathcal{A}_{p_i}^{**}\right)$$
$$= p_0\mathcal{A}p_0 + (p_1 - p_0)\,\mathcal{A}\,(p_1 - p_0) + \cdots + (p_k - p_{k-1})\,\mathcal{A}\,(p_k - p_{k-1}).$$

Proof. We only prove the case when $k = 2$. Since $p_1 \leq p_2 \in socle(\mathcal{A}^{**})$, we have

$$p_2\left(\mathcal{A}_{p_1}^{**} \cap \mathcal{A}_{p_2}^{**}\right) = p_1\mathcal{A}p_1 + (p_2 - p_1)\,\mathcal{A}\,(p_2 - p_1)$$

by Lemma 4. So it is obvious that

$$p_2\left(\mathcal{A}_{p_1} \cap \mathcal{A}_{p_2}\right) \subseteq p_1\mathcal{A}p_1 + (p_2 - p_1)\,\mathcal{A}\,(p_2 - p_1).$$

Meanwhile,

$$p_2\mathcal{A}_{p_2} = p_2\mathcal{A}p_2 = p_2\mathcal{A}^{**}p_2$$
$$\supseteq p_1\mathcal{A}p_1 + (p_2 - p_1)\,\mathcal{A}\,(p_2 - p_1)$$

by Lemma 4 and the fact that $p_1 \leq p_2 \in socle(\mathcal{A}^{**})$. Then for every $b_1 \in p_1\mathcal{A}p_1$ and $b_2 \in (p_2 - p_1)\,\mathcal{A}\,(p_2 - p_1)$, there is $a \in \mathcal{A}_{p_2}$ such that $p_2a = b_1 + b_2$. Hence $a \in \mathcal{A}_{p_1} \cap \mathcal{A}_{p_2}$. It implies that

$$p_2\left(\mathcal{A}_{p_1} \cap \mathcal{A}_{p_2}\right) \supseteq p_1\mathcal{A}p_1 + (p_2 - p_1)\,\mathcal{A}\,(p_2 - p_1).$$

This completes the proof. □

Lemma 6. *Let $\mathcal{A}$ be a C*-algebra, $\{p_n\}$ be a sequence of projections in $socle(\mathcal{A}^{**})$ with $p_0 \leq p_1 \leq \cdots$ and $p_i \xrightarrow{s.o.t.} I$ (strong operator topology). Then*

$$\bigcap_{i=0}^{\infty}\mathcal{A}_{p_i} = p_0\mathcal{A}p_0 + \sum_{i=1}^{\infty}\left[(p_i - p_{i-1})\,\mathcal{A}\,(p_i - p_{i-1})\right] \subseteq \mathcal{A}.$$

Proof. By Lemma 5 and the fact that $p_1 \leq p_2 \leq \cdots$ with $p_i \in socle(\mathcal{A}^{**})$, we have

$$p_k\left(\bigcap_{i=0}^{\infty}\mathcal{A}_{p_i}\right) = p_k\left(\bigcap_{i=0}^{k}\mathcal{A}_{p_i}\right) = p_k\left(\bigcap_{i=0}^{k}\mathcal{A}_{p_i}^{**}\right)$$
$$= p_0\mathcal{A}p_0 + (p_1 - p_0)\,\mathcal{A}\,(p_1 - p_0) + \cdots + (p_k - p_{k-1})\,\mathcal{A}\,(p_k - p_{k-1})$$

for every k. Therefore

$$\bigcap_{i=0}^{\infty}\mathcal{A}_{p_i} = p_0\left(\bigcap_{i=0}^{\infty}\mathcal{A}_{p_i}\right) + \sum_{i=1}^{\infty}(p_i - p_{i-1})\left(\bigcap_{k=0}^{\infty}\mathcal{A}_{p_k}\right)$$
$$= p_0\left(\bigcap_{i=0}^{\infty}\mathcal{A}_{p_i}\right) + \sum_{i=1}^{\infty}(p_i - p_{i-1})\left(\bigcap_{k=0}^{i}\mathcal{A}_{p_k}\right)$$
$$= p_0\mathcal{A}p_0 + \sum_{i=1}^{\infty}\left[(p_i - p_{i-1})\,\mathcal{A}\,(p_i - p_{i-1})\right].$$

□

Remark 1. Suppose $\mathcal{A}$ is a unital inner QD C*-algebra; then there is a sequence $\{p_n\}$ of projections in $socle(\mathcal{A}^{**})$ such that $\|[p_n, a]\| \longrightarrow 0$ for all $a \in \mathcal{A} \subseteq \mathcal{A}^{**}$ and $\|a\| = \lim\|p_nap_n\|$ for all $a \in \mathcal{A}$ by Theorem 1. Therefore we can define a sequence of u.c.p. maps $\varphi_n : \mathcal{A} \longrightarrow p_n\mathcal{A}^{**}p_n$ by compression. It is obvious that

$\mathcal{A}_{p_n} = \mathcal{M}_{\varphi_n}$ where $\mathcal{M}_{\varphi_n}$ is the multiplicative domain of φ_n and $\|a\| = \lim \|\varphi_n(a)\|$, $d(a, \mathcal{M}_{\varphi_n}) \longrightarrow 0$ for all $a \in \mathcal{A}$ by Lemma 3. Actually, this is a sufficient condition for a given C*-algebra to be an inner QD C*-algebra.

Theorem 5 ([8], Corollary 3.7, Chapter 11)**.** *$\mathcal{A}$ is inner QD if and only if there is a sequence of c.c.p. maps $\varphi_n : \mathcal{A} \longrightarrow \mathcal{M}_{k_n}(\mathbb{C})$ such that $\|a\| = \lim \|\varphi_n(a)\|$ and $d(a, \mathcal{M}_{\varphi_n}) \longrightarrow 0$ for all $a \in \mathcal{A}$, where $\mathcal{M}_{\varphi_n}$ is the multiplicative domain of φ_n.*

Now, we are ready to give another characterization of unital inner QD C*-algebras.

Theorem 6. *Let $\mathcal{A}$ be a unital separable C*-algebra. Then $\mathcal{A}$ is inner QD if and only if there is a sequence of unital RFD C*-subalgebras $\{\mathcal{A}_n\}_{n=1}^{\infty}$ of $\mathcal{A}$ such that $\bigcup_{n=1}^{\infty} \mathcal{A}_n$ is norm dense in $\mathcal{A}$.*

Proof. ($\Longrightarrow$) Suppose $\mathcal{F} \subseteq \mathcal{A}$ is a finite subset and $\varepsilon > 0$. Let

$$\{1\} \cup \mathcal{F} \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$$

be the sequence of finite subsets of $\mathcal{A}$ such that $\overline{\bigcup_i \mathcal{F}_i} = \mathcal{A}$. Then, from Remark 1 and Theorem 5, we can find

$$P_0 \le P_1 \le P_2 \le \cdots$$
$$\text{with } P_i \in socle(\mathcal{A}^{**}) \quad \text{and} \quad P_i \overset{s.o.t.}{\longrightarrow} P \in \mathcal{A}^{**} \ (i \longrightarrow \infty)$$

such that

$$\begin{aligned} d(a, \mathcal{A}_{P_i}) &= \|[a, P_i]\| \\ &= \|(1 - P_i) a P_i + P_i a (1 - P_i)\| < \frac{\varepsilon}{2 \cdot 2^{i+1}} \end{aligned}$$

and $\|P_i a P_i\| > \|a\| - \frac{\varepsilon}{2^{i+1}}$ for every $a \in \mathcal{F}_i$ ($i \in \mathbb{N}$). Since $P_i \overset{s.o.t.}{\longrightarrow} P \in \mathcal{A}^{**}$ (as $i \longrightarrow \infty$) and $P \ge P_i$, we have

$$\|PaP\| \ge \|P_i a P_i\| \ge \|a\| - \frac{\varepsilon}{2^{i+1}} \quad \text{for} \quad \forall a \in \bigcup_i \mathcal{F}_i \quad \text{and} \quad i.$$

It implies that $\|PaP\| = \|a\|$ for $\forall a \in \overline{\bigcup_i \mathcal{F}_i} = \mathcal{A}$, therefore $P = I$. Now let

$$\mathcal{A}_\varepsilon = \bigcap_{i=0}^{\infty} \mathcal{A}_{P_i} = P_0 \mathcal{A} P_0 + (P_1 - P_0) \mathcal{A} (P_1 - P_0) + \cdots$$

by Lemma 6. So, for any $a \in \mathcal{F}$, let

$$x = P_0 a P_0 + (P_1 - P_0) a (P_1 - P_0) + \cdots \in \mathcal{A}_\varepsilon.$$

We have

$$\begin{aligned} d(a, \mathcal{A}_\varepsilon) &\le \|a - x\| \\ &= ||P_0 a (P_1 - P_0) + P_1 a (P_2 - P_1) + \cdots \\ &+ (P_1 - P_0) a P_0 + (P_2 - P_1) a P_1 + \cdots || \\ &\le \|P_0 a (1 - P_0)\| \, \|P_1 - P_0\| + \cdots + \|P_1 - P_0\| \, \|(1 - P_0) a P_0\| \\ &< \sum_{i=0}^{\infty} \frac{\varepsilon}{2 \cdot 2^{i+1}} + \sum_{i=0}^{\infty} \frac{\varepsilon}{2 \cdot 2^{i+1}} = \varepsilon. \end{aligned}$$

Note that $\mathcal{A}_\varepsilon$ is an RFD C*-subalgebra of $\mathcal{A}$, hence we can find a sequence of unital RFD C*-subalgebras $\{\mathcal{A}_n\}_{n=1}^{\infty}$ of $\mathcal{A}$ such that $\bigcup_{n=1}^{\infty}\mathcal{A}_n$ is norm dense in $\mathcal{A}$.

($\Longleftarrow$) Suppose $\{\mathcal{A}_n\}_{n=1}^{\infty}$ is a sequence of unital RFD C*-subalgebras in $\mathcal{A}$ such that $\overline{\bigcup_{n=1}^{\infty}\mathcal{A}_n}^{\|\cdot\|}=\mathcal{A}$, $\mathcal{F}\subseteq\mathcal{A}$ is a finite subset and $\varepsilon>0$. Then there is an RFD C*-subalgebra $\mathcal{A}_n$ and $b_a\in\mathcal{A}_n$ such that $\|a-b_a\|<\frac{\varepsilon}{3}$ for every $a\in\mathcal{F}$. It follows that

$$\|a\|-\frac{\varepsilon}{3}\leq\|b_a\|.$$

Since $\mathcal{A}_n$ is RFD, we can find a projection P such that $\Phi_P:\mathcal{A}_n\longrightarrow P\mathcal{A}_nP\subseteq\mathcal{M}_t(\mathbb{C})$ is a *-homomorphism for some $t\in\mathbb{C}$ and $\|\Phi_P(b_a)\|\geq\|b_a\|-\frac{\varepsilon}{3}$. Then there is a u.c.p. extension $\widetilde{\Phi_P}:\mathcal{A}\longrightarrow\mathcal{M}_t(\mathbb{C})$ such that $\mathcal{A}_n\subseteq\mathcal{M}_{\widetilde{\Phi_P}}$ and

$$\left\|\widetilde{\Phi_P}(b_a)\right\|\geq\|b_a\|-\frac{\varepsilon}{3}$$

where $\mathcal{M}_{\widetilde{\Phi_P}}$ is the multiplicative domain of $\widetilde{\Phi_P}$. It follows that

$$\left\|\widetilde{\Phi_P}(b_a)\right\|=\left\|\widetilde{\Phi_P}(b_a-a)+\widetilde{\Phi_P}(a)\right\|\leq\frac{\varepsilon}{3}+\left\|\widetilde{\Phi_P}(a)\right\|\text{ for every }a\in\mathcal{F}.$$

So from the above inequalities, we have

$$\begin{aligned}\left\|\widetilde{\Phi_P}(a)\right\|&\geq\left\|\widetilde{\Phi_P}(b_a)\right\|-\frac{\varepsilon}{3}\\&\geq\|b_a\|-\frac{2\varepsilon}{3}\geq\|a\|-\varepsilon\text{ for every }a\in\mathcal{F}.\end{aligned}$$

By the preceding discussion, for a finite subset $\mathcal{F}$ and $\varepsilon>0$, there is a u.c.p. map $\widetilde{\Phi_P}:\mathcal{A}\longrightarrow\mathcal{M}_t(\mathbb{C})$ for some $t\in\mathbb{N}$ such that $d\left(a,\mathcal{M}_{\widetilde{\Phi_P}}\right)<\varepsilon$ and $\left\|\widetilde{\Phi_P}(a)\right\|\geq\|a\|-\varepsilon$ for every $a\in\mathcal{F}$. So by Theorem 5, $\mathcal{A}$ is inner QD. □

Remark 2. After slightly modifying the proof of Theorem 6, we can see that a unital separable C*-algebra $\mathcal{A}$ is inner QD if and only if there is an increasing sequence $\{\mathcal{A}_n\}$ of unital RFD C*-subalgebras such that $\bigcup_{n=1}^{\infty}\mathcal{A}_n$ is norm dense in $\mathcal{A}$.

3. Unital full free products of two inner QD algebras

In this section we consider the question as to whether a unital full free product of inner QD C*-algebras is inner QD again. To show the main result in this section, we need the following lemma.

Lemma 7 (Theorem 3.2, [17]). *Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital C*-algebras. Then the unital full free product $\mathcal{A}=\mathcal{A}_1*_{\mathbb{C}}\mathcal{A}_2$ is RFD if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ are both RFD.*

Now, we are ready to give the main result of this section.

Theorem 7. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be unital inner quasidiagonal C*-algebras. Then $\mathcal{A}_1*_{\mathbb{C}}\mathcal{A}_2$ is inner QD.*

Proof. Let τ be a fixed state on $\mathcal{A}_1*_{\mathbb{C}}\mathcal{A}_2$ and $\mathcal{F}$ be an arbitrary finite subset of $\mathcal{A}_1*_{\mathbb{C}}\mathcal{A}_2$. Let

$$\mathcal{A}_j^0=\{a\in\mathcal{A}_j:\tau(a)=0\},j=1,2.$$

Without loss of generality, we may assume that every $b \in \mathcal{F}$ can be decomposed into a finite sum with respect to τ, that is,

$$b = \alpha_0 I + \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} a_{i_1} a_{i_2} \cdots a_{i_n} \quad \alpha_0 \in \mathbb{C},\, a_{i_j} \in \mathcal{A}^0_{i_j}, i_1 \neq i_2 \neq \cdots \neq i_n$$

where $\mathcal{A}^0_{i_j} = \mathcal{A}^0_1$ or $\mathcal{A}^0_2$. Denote by $\mathcal{F}^j_0$, $j = 1, 2$, the set of such elements of $\mathcal{A}^0_j$ which appear in the decomposition of elements from $\mathcal{F}$. Then we can find an RFD C*-subalgebra $\mathcal{A}^j_\varepsilon$ of $\mathcal{A}_j$ for $j = 1, 2$ such that

$$d(a, \mathcal{A}^j_\varepsilon) < \varepsilon \quad \text{for} \quad \forall a \in \mathcal{F}^j_0, j = 1, 2. \tag{1}$$

Since $\mathcal{F}$ is a finite set, we may assume without loss of generality that $\mathcal{F} = \{b\}$. We slightly change our notation for convenience. We may assume without loss of generality that b can be decomposed into the form

$$\alpha I + \sum_{i=1}^{l} a^i_{1,1} a^i_{2,1} a^i_{1,2} a^i_{2,2} a^i_{1,3} \cdots a^i_{1,n_i}.$$

The subscript on $a_{1,j}$ means that it is the jth element in the expression

$$a^i_{1,1} a^i_{2,1} a^i_{1,2} a^i_{2,2} a^i_{1,3} \cdots a^i_{1,n_i}$$

lying in $\mathcal{F}^1_0$ and the subscript on $a_{2,j}$ means that it is the jth element in the expression

$$a^i_{1,1} a^i_{2,1} a^i_{1,2} a^i_{2,2} a^i_{1,3} \cdots a^i_{1,n_i}$$

lying in $\mathcal{F}^2_0$. Therefore

$$\left\{a^i_{1,1}, a^i_{1,2}, \cdots, a^i_{1,n_i}\right\} \subseteq \mathcal{F}^1_0$$

and

$$\left\{a^i_{2,1}, a^i_{2,2}, \cdots, a^i_{2,n}\right\} \subseteq \mathcal{F}^2_0.$$

Then, for any $a^i_{j,k}$, we can find $\widetilde{a^i_{j,k}} \in \mathcal{A}^j_\varepsilon$ such that $\left\| a^i_{j,k} - \widetilde{a^i_{j,k}} \right\| < \varepsilon$ by (1). Note that

$$\left|\tau\left(a^i_{j,k} - \widetilde{a^i_{j,k}}\right)\right| = \left|\tau\left(a^i_{j,k}\right) - \tau\left(\widetilde{a^i_{j,k}}\right)\right| = \left|\tau\left(\widetilde{a^i_{j,k}}\right)\right| < \varepsilon;$$

then

$$\widetilde{a^i_{j,k}} = \tau\left(\widetilde{a^i_{j,k}}\right) I + \left(\widetilde{a^i_{j,k}} - \tau\left(\widetilde{a^i_{j,k}}\right) I\right) \quad \text{and} \quad \left\|\left(\widetilde{a^i_{j,k}} - \tau\left(\widetilde{a^i_{j,k}}\right) I\right) - a^i_{j,k}\right\| < 2\varepsilon.$$

Therefore, without loss of generality, we may assume that $\widetilde{a^i_{1,k}} \in \left(\mathcal{A}^1_\varepsilon\right)^0$ with $\left\| a^i_{1,k} - \widetilde{a^i_{1,k}} \right\| < 2\varepsilon$ where $k = 1, \cdots, n_i, i = 1, \cdots, l$. Similarly, we assume $\widetilde{a^i_{2,k}} \in \left(\mathcal{A}^2_\varepsilon\right)^0$ with $\left\| a^i_{2,k} - \widetilde{a^i_{2,k}} \right\| < 2\varepsilon$ where $k = 1, \cdots, n_i, i = 1, \cdots, l$. Let $\widetilde{b} = \alpha I + \sum_{i=1}^{l} \widetilde{a^i_{1,1}}\, \widetilde{a^i_{2,1}}\, \widetilde{a^i_{1,2}} \cdots \widetilde{a^i_{1,n_i}} \in \mathcal{A}^1_\varepsilon *_{\mathbb{C}} \mathcal{A}^2_\varepsilon$. There is an integer $M_b > 0$ such that

$$\begin{aligned} &\left\| b - \widetilde{b} \right\| \\ &= \left\| \alpha I + \sum_{i=1}^{l} a^i_{1,1} a^i_{2,1} a^i_{1,2} a^i_{2,2} a^i_{1,3} \cdots a^i_{1,n_i} - \left(\alpha I + \sum_{i=1}^{l} \widetilde{a^i_{1,1}}\, \widetilde{a^i_{2,1}}\, \widetilde{a^i_{1,2}} \cdots \widetilde{a^i_{1,n_i}}\right) \right\| \\ &\leq M_b \varepsilon. \end{aligned}$$

For the other decomposition forms, we can use a similar discussion. Now $\mathcal{A}^1_\varepsilon *_{\mathbb{C}} \mathcal{A}^2_\varepsilon$ is an RFD C*-algebra by Lemma 7; then by Theorem 6, $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$ is inner QD. □

Since every strong NF algebra is inner, then from [3], every AF algebra and AH algebra are inner. Hence we have the following two corollaries.

Corollary 1. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are both AF algebras; then $\mathcal{A} *_{\mathbb{C}} \mathcal{B}$ is an inner QD algebra.*

Corollary 2. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are both AH algebras; then $\mathcal{A} *_{\mathbb{C}} \mathcal{B}$ is an inner QD algebra.*

What will happen when the above amalgamation is over some other C*-algebras instead of $\mathbb{C}I$? In [14], one example was given to show that a full amalgamated free product of two QD algebras may fail to be MF again, even for a unital full free product of two full matrix algebras with amalgamation over a two dimensional C*-algebra which is *-isomorphic to $\mathbb{C} \oplus \mathbb{C}$. Therefore, a unital full amalgamated free product of two unital inner QD C*-algebras may fail to be inner QD again. But we can give affirmative answers for some specific cases.

The following result can be found in [16] or [14].

Lemma 8. *Suppose that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ are unital C*-algebras. Then*

$$(\mathcal{A} \otimes_{\max} \mathcal{D}) \underset{\mathcal{D}}{*} (\mathcal{B} \otimes_{\max} \mathcal{D}) \cong \left(\mathcal{A} \underset{\mathbb{C}}{*} \mathcal{B} \right) \otimes_{\max} \mathcal{D}.$$

Lemma 9 ([4])**.** *Let $\mathcal{A}$ be a C*-algebra. Then, for any k, $\mathcal{A}$ is inner QD if and only if $\mathcal{M}_k(\mathcal{A}) = \mathcal{A} \otimes \mathcal{M}_k(\mathbb{C})$ is inner QD.*

Proposition 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras. If $\mathcal{D}$ can be embedded as a unital C*-subalgebra of $\mathcal{A}$ and $\mathcal{B}$ respectively, and $\mathcal{D}$ is *-isomorphic to a full matrix algebra $\mathcal{M}_n(\mathbb{C})$ for some integer n, then the unital full amalgamated free product $\mathcal{A} \underset{\mathcal{D}}{*} \mathcal{B}$ is inner QD if $\mathcal{A}$ and $\mathcal{B}$ are both inner QD.*

Proof. Since $\mathcal{D}$ is *-isomorphic to a full matrix algebra, from Lemma 6.6.3 in [12], it follows that $\mathcal{A} \cong \mathcal{A}' \otimes \mathcal{D}$ and $\mathcal{B} \cong \mathcal{B}' \otimes \mathcal{D}$. Then $\mathcal{A}'$ and $\mathcal{B}'$ are inner QD by Lemma 9. So the desired conclusion follows from Theorem 7, Lemma 8 and Lemma 9. ☐

Next, we consider the case in which the free products are amalgamated over some finite-dimensional C*-algebras.

Lemma 10 ([5])**.** *An arbitrary inductive limit (with injective connecting maps) of inner quasidiagonal C*-algebras is inner quasidiagonal.*

Lemma 11 ([16], Theorem 4.2)**.** *Assume that we have embeddings of C*-algebras $\mathcal{C} \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{C} \subseteq \mathcal{B}_1 \subseteq \mathcal{B}_2$; then the natural morphism $\sigma : \mathcal{A}_1 *_{\mathcal{C}} \mathcal{B}_1 \longrightarrow \mathcal{A}_2 *_{\mathcal{C}} \mathcal{B}_2$ is injective.*

Lemma 12 ([16], Corollary 4.13)**.** *If $(\mathcal{A}_n)$ and $(\mathcal{B}_n)$ are increasing sequences of C*-algebras, all of which contain a common C*-subalgebra $\mathcal{C}$, then there is a natural isomorphism*

$$\varinjlim (\mathcal{A}_n *_{\mathcal{C}} \mathcal{B}_n) = \varinjlim \mathcal{A}_n *_{\mathcal{C}} \varinjlim \mathcal{B}_n$$

where $\varinjlim$ denotes the ordinary direct limit.

The following lemma is a well-known property of AF algebras. We can find it in [9].

Lemma 13. *A C*-algebra $\mathcal{A}$ is AF if and only if it is separable and*

(∗) *for all $\varepsilon > 0$ and $A_1, \cdots, A_n$ in $\mathcal{A}$, there exists a finite-dimensional C*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that dist $(A_i, \mathcal{B}) < \varepsilon$ for $1 \leq i \leq n$.*

Moreover, if $\mathcal{A}_1$ is a finite-dimensional subalgebra of $\mathcal{A}$, then we may choose $\mathcal{B}$ so that it contains $\mathcal{A}_1$.

Lemma 14 ([1], Theorem 4.2)**.** *Consider unital inclusions of C*-algebras $\mathcal{A} \supseteq \mathcal{C} \subseteq \mathcal{B}$ with $\mathcal{A}$ and $\mathcal{B}$ finite-dimensional. Let $\mathcal{A} *_{\mathcal{C}} \mathcal{B}$ be the corresponding full amalgamated free product. Then $\mathcal{A} *_{\mathcal{D}} \mathcal{B}$ is RFD if and only if there are faithful tracial states $\tau_{\mathcal{A}}$ on $\mathcal{A}$ and $\tau_{\mathcal{B}}$ on $\mathcal{B}$ with*

$$\tau_{\mathcal{A}}(x) = \tau_{\mathcal{B}}(x), \qquad \forall\, x \in \mathcal{C}.$$

Corollary 3. *Let $\mathcal{A}$ and $\mathcal{B}$ be AF algebras. Consider unital inclusions of C*-algebras $\mathcal{A} \supseteq \mathcal{C} \subseteq \mathcal{B}$ with $\mathcal{C}$ finite-dimensional. If there are faithful tracial states $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ on $\mathcal{A}$ and $\mathcal{B}$ respectively, such that*

$$\tau_{\mathcal{A}}(x) = \tau_{\mathcal{B}}(x), \qquad \forall\, x \in \mathcal{C},$$

then $\mathcal{A} *_{\mathcal{C}} \mathcal{B}$ is inner QD..

Proof. Since $\mathcal{C}$ is a finite-dimensional C*-subalgebra, then we can find a sequence of finite-dimensional C*-subalgebras $\{\mathcal{A}_n\}_{n=1}^{\infty}$ and $\{\mathcal{B}_n\}_{n=1}^{\infty}$ such that $\mathcal{C} \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ with $\overline{\bigcup \mathcal{A}_n} = \mathcal{A}$ and $\mathcal{C} \subseteq \mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \cdots$ with $\overline{\bigcup \mathcal{B}_n} = \mathcal{B}$ by Lemma 13. Note that $\mathcal{A}_n *_{\mathcal{C}} \mathcal{B}_n$ is RFD by Lemma 14; then $\mathcal{A} *_{\mathcal{C}} \mathcal{B} = \varinjlim (\mathcal{A}_n *_{\mathcal{C}} \mathcal{B}_n)$ is inner by Lemma 12 and Lemma 10. □

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