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A NOTE ON NON-ORDINARY PRIMES

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ABSTRACT. Suppose that O_L is the ring of integers of a number field L, and suppose that

$$f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k \cap O_L[[q]]$$

(note: $q := e^{2\pi iz}$) is a normalized Hecke eigenform for $\mathrm{SL}_2(\mathbb{Z})$. We say that f is non-ordinary at a prime p if there is a prime ideal $\mathfrak{p} \subset O_L$ above p for which

$$a_f(p) \equiv 0 \pmod{\mathfrak{p}}.$$

For any finite set of primes S, we prove that there are normalized Hecke eigenforms which are non-ordinary for each $p \in S$. The proof is elementary and follows from a generalization of work of Choie, Kohnen and the third author.

1. Introduction and statement of results

If $k \geq 4$ is even, then let M_k (resp. S_k) denote the finite dimensional $\mathbb{C}$-vector space of weight k holomorphic modular forms (resp. cusp forms) on $\mathrm{SL}_2(\mathbb{Z})$. Furthermore, let $M_k^!$ denote the infinite dimensional space of weakly holomorphic modular forms of weight k with respect to $\mathrm{SL}_2(\mathbb{Z})$. Recall that a meromorphic modular form is weakly holomorphic if its poles (if any) are supported at cusps. We shall identify a modular form on $\mathrm{SL}_2(\mathbb{Z})$ by its Fourier expansion at infinity

$$f(z) = \sum_{n \gg -\infty} a_f(n)q^n,$$

where $q := e^{2\pi iz}$.

Suppose that O_L is the ring of integers of a number field L, and suppose that

$$f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k \cap O_L[[q]]$$

is a normalized Hecke eigenform for $\mathrm{SL}_2(\mathbb{Z})$. We say that f is non-ordinary at a prime p if there is a prime ideal $\mathfrak{p} \subset O_L$ above p for which

$$a_f(p) \equiv 0 \pmod{\mathfrak{p}}.$$

Very little is known about the distribution of non-ordinary primes. We recall the following well-known open problem (see Gouvêa's expository article [2]).

Problem. Are there infinitely many non-ordinary primes for a generic normalized Hecke eigenform $f(z)$?

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We do not solve this problem here. It remains open. However, we establish the following related result.

Theorem 1.1. *If S is a finite set of primes, then there are infinitely many normalized Hecke eigenforms for* $\mathrm{SL}_2(\mathbb{Z})$ *which are non-ordinary for each* $p \in S$.

Remark. The proof of Theorem 1.1 relies on a general theorem about the Fourier coefficients of weakly holomorphic modular forms modulo p (see Theorem 2.5). For normalized Hecke eigenforms, this general result incorporates classical results of Hatada [3] (in the case where $p = 2$ and 3) and Hida [4–6] (for primes $p \geq 5$) on non-ordinary primes.

Remark. The proof of Theorem 1.1 is constructive. Suppose that $S = \{p_1, p_2, \ldots, p_m\}$ is a finite set of primes. Suppose that $k \geq 12$ is an even integer. If for each $p \in S$ there is a choice of $t \in A = \{4, 6, 8, 10, 14\}$ for which $(p-1)|(k-t)$, then every prime in S is non-ordinary for every normalized Hecke eigenform $f \in S_k$. The earlier work of Choie, Kohnen and the third author [1] is eclipsed by this result thanks to the flexibility in the choice of t above.

In Section 2 we recall certain facts about modular forms and we prove Theorem 2.5. The proof is elementary. In Section 3 we obtain Theorem 1.1 as a simple consequence when $p \geq 5$, combining with the known result on $p = 2, 3$, and in Section 4 we offer some numerical examples.

2. Preliminaries

2.1. **Nuts and bolts.** As usual, let $\Delta(z) \in S_{12}$ be the cusp form

$$\Delta(z) := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + \ldots, \tag{2.1}$$

and, for even $k \geq 4$, let $E_k(z) \in M_k$ be the normalized Eisenstein series

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \left(\sum_{1 \leq d | n} d^{k-1} \right) q^n, \tag{2.2}$$

where the rational numbers B_k are the usual Bernoulli numbers given by the generating function

$$\sum_{k=0}^{\infty} B_k \cdot \frac{t^k}{k!} = \frac{t}{e^t - 1} = 1 - \frac{1}{2}t + \frac{1}{12}t^2 - \ldots.$$

For convenience, we let $E_0(z) := 1$. Finally, we let $j(z)$ be the usual modular function

$$j(z) := \frac{E_4(z)^3}{\Delta(z)} = q^{-1} + 744 + 196884q + \ldots. \tag{2.3}$$

Finally, for convenience, if $k \in 2\mathbb{Z}$, then throughout we define $\delta(k) \in \{0, 4, 6, 8, 10, 14\}$ so that

$$\delta(k) \equiv k \pmod{12}. \tag{2.4}$$

In the proof, we need the following propositions.

Proposition 2.1. *A normalized Hecke eigenform is non-ordinary at p if there is an $m \geq 1$ such that* $a_f(p^m) \equiv 0 \pmod{p}$.

Proof. This follows from the fact that $T_p f(z) = a_f(p)f(z)$ for every prime p when $f(z)$ is a normalized Hecke eigenform of weight k. Here T_p is the p-th Hecke operator. In particular, on prime power exponents, we have

$$a_f(p)a_f(p^m) = a_f(p^{m+1}) + p^{k-1}a_f(p^{m-1}) \equiv a_f(p^{m+1}) \pmod{p}$$

for every non-negative integer n. By induction, we find that

$$a_f(p^m) \equiv a_f(p)^m \pmod{p}.$$

This proves the proposition. □

The following well-known propositions play a central role in the proof of Theorem 2.5.

Proposition 2.2. *If $p \geq 5$ is prime, then as a q-series, $E_{p-1}(z) \equiv 1 \pmod{p}$.*

Proof. This can be found on page 38 of [7]. □

Proposition 2.3. *If $f(z) = \sum_{n \gg -\infty} a_f(n)q^n \in M_2^!$, then $a_f(0) = 0$.*

Proof. By a simple generalization of Lemma 2.34 of [7], it is known that every weakly holomorphic modular form $h(z)$ of weight 2 may be represented as $P(j(z))E_{14}(z)\Delta(z)^{-1}$, where $P(x)$ is a polynomial of x. Dropping the dependence on z for convenience, we have the following well-known identities:

$$-\frac{1}{2\pi i}\frac{d}{dz}j = \frac{E_{14}}{\Delta},$$

$$j^w \frac{d}{dz}j = \frac{1}{w+1}\frac{d}{dz}j^{w+1},$$

where $w \in \mathbb{Z}_{\geq 0}$. Therefore, it follows that h is the derivative of a polynomial in j, and so its constant term in the Fourier expansion is zero. □

Remark. For more standard facts about modular forms the reader may see [7].

2.2. **Our main technical result.** In 2005 Choie, Kohnen and the third author proved the following (see Corollary 1.3 of [1]). This result recovered earlier aforementioned results of Hatada and Hida.

Theorem 2.4. *Let p be a prime, and suppose that $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k$ is a normalized Hecke eigenform. Let L_f be the number field generated by the coefficients of $f(z)$, and let $\mathfrak{p} \in O_{L_f}$ be any prime ideal above p.*

(1) *If $p = 2, 3$, then*
$$a_f(p) \equiv 0 \pmod{\mathfrak{p}}.$$
(2) *If $p \geq 5$, $\delta(k) \in \{4, 6, 8, 10, 14\}$ and $k \equiv \delta(k) \pmod{p-1}$, then*
$$a_f(p) \equiv 0 \pmod{\mathfrak{p}}.$$

Here we strengthen this result for primes $p \geq 5$ by extending it to all k without any condition on $\delta(k)$.

Theorem 2.5. *Let $p \geq 5$ be prime, and suppose that $f(z) = \sum_{n \gg -\infty}^{\infty} a_f(n)q^n \in M_k^! \cap O_L[[q]]$, where $k \in 2\mathbb{Z}$ and O_L is the ring of algebraic integers of a number field L.*

(1) *Suppose that* $a \geq 0$ *and* $m \in A = \{4, 6, 8, 10, 14\}$ *are integers for which*

$$k - 2 \leq (m-2)p^a.$$

If $\mathrm{ord}_\infty(f) > -p^a$ *and* $(p-1)|(k-m)$, *then for any integer* $b \geq a$, *we have*

$$a_f(p^b) \equiv -\frac{2m}{B_m} a_f(0) \pmod{p}.$$

(2) *Suppose that* $k \leq 2$, $r, s \in \mathbb{Z}_{\geq 0}$ *and* $t, u \in \mathbb{Z}_{>0}$ *are integers for which*

$$2 - k = r(p-1) + sp^t,$$

where $s \neq 2$. *If* $\mathrm{ord}_\infty(f) > -p^u, u \leq t$, *then for any integer* v *such that* $u \leq v \leq t$, *we have*

$$a_f(p^v) \equiv a_f(0) \equiv 0 \pmod{p}.$$

Proof. The proofs in both cases begin with the construction of suitable weakly holomorphic modular forms of weight $2-k$. The product of such forms with f have weight 2, and so Proposition 2.3 implies that their constant terms vanish.

For case (1), first note that $(k-2) - (m-2)p^b \equiv k - m \pmod{p-1}$. As we have $(p-1)|(k-m)$ and $k - 2 \leq (m-2)p^b$, we may find a non-negative integer c such that

$$2 - k = c(p-1) - (m-2)p^b.$$

Let g_m be the function

$$g_m := j\frac{E_6^{(1+i^m)/2}}{E_4^{(m+1+3i^m)/4}} = \begin{cases} j\frac{E_6}{E_4^2} & \text{for} \quad m = 4 \\ j\frac{1}{E_4} & \text{for} \quad m = 6 \\ j\frac{E_6}{E_4^3} & \text{for} \quad m = 8 \\ j\frac{1}{E_4^2} & \text{for} \quad m = 10 \\ j\frac{1}{E_4^3} & \text{for} \quad m = 14 \end{cases} \in M_{2-m}^!.$$

Then we have

$$g_m^{p^b} E_{p-1}^c \in M_{2-k}^!.$$

That is to say, the constant term of $g_m^{p^b} E_{p-1}^c f$ is zero. From Proposition 2.2 we know that

$$E_{p-1} \equiv 1 \pmod{p}.$$

Then we have that the constant term of $g_m^{p^b} f$ is zero modulo p. By using Fermat's little theorem to compute the multinomials, we get

$$\begin{aligned} g_m^{p^b} f &= (q^{-1} + 744 + O(q))^{p^b} (1 - 504q + O(q^2))^{\frac{p^b(1+i^m)}{2}} \\ &\quad (1 + (-240)q + O(q^2))^{\frac{p^b(m+1+3i^m)}{4}} f \\ &\equiv (q^{-p^b} + 744 + O(q^{p^b}))(1 - 252(1+i^m)q^{p^b} + O(q^{2p^b})) \\ &\quad (1 - 60(m+1+3i^m)q^{p^b} + O(q^{2p^b})) \textstyle\sum_{n \gg -\infty}^{\infty} a_f(n)q^n \\ &\equiv (q^{-p^b} + 432 - 60m - 432i^m + O(q^{p^b})) \textstyle\sum_{n \gg -\infty}^{\infty} a_f(n)q^n \pmod{p}. \end{aligned}$$

We already know that $\mathrm{ord}_\infty(f) > -p^a \geq -p^b$, so we know that the constant term $c_{m,p}$ of $g_m^{p^b} f$ must satisfy the congruence

$$c_{m,p} \equiv a_f(p^b) + (432 - 60m - 432i^m)a_f(0) \pmod{p}.$$

As $c_{m,p}$ is known to be zero modulo p and for $m \in A$,

$$\frac{2m}{B_m} = 432 - 60m - 432i^m,$$

we get the conclusion.

For case (2), as we have $2 - k = r(p-1) + sp^t$ and $sp^{t-u} \neq 2$, we can find $c_1, c_2 \in \mathbb{Z}_{\geq 0}$ such that $4c_1 + 6c_2 = sp^{t-u}$. Then we have

$$(E_4^{c_1} E_6^{c_2})^{p^u} E_{p-1}^r f \in M_2^!.$$

Hence we have that the constant term of $(E_4^{c_1} E_6^{c_2})^{p^u} E_{p-1}^r f$ is zero. As

$$(E_4^{c_1} E_6^{c_2})^{p^u} E_{p-1}^r f \equiv (1 + O(q^{p^u}))f \pmod{p}$$

and $\mathrm{ord}_\infty(f) > -p^u$, we know $a_f(0) \equiv 0 \pmod{p}$. To prove the case of $a_f(p^v)$ for $u \leq v \leq t$, we may find $c_1', c_2' \in \mathbb{Z}_{\geq 0}$ such that $4c_1' + 6c_2' = sp^{t-v}$. Then we have

$$j^{p^v} (E_4^{c_1'} E_6^{c_2'})^{p^v} E_{p-1}^r f \in M_2^!.$$

Hence the constant term of $j^{p^v} (E_4^{c_1'} E_6^{c_2'})^{p^v} E_{p-1}^r f$ is zero. As

$$(jE_4^{c_1'} E_6^{c_2'})^{p^v} E_{p-1}^r f \equiv (q^{-p^v} + 744 + 240c_1' - 504c_2' + O(q^{p^v}))f \pmod{p}$$

and $\mathrm{ord}_\infty(f) > -p^u \geq -p^v$, we get

$$a_f(p^v) + (744 + 240c_1' - 504c_2')a_f(0) \equiv 0 \pmod{p}.$$

Knowing that $a_f(0) \equiv 0 \pmod{p}$, we get the conclusion. $\square$

3. Proof of Theorem 1.1

By Theorem 2.4, $p = 2$ and 3 are non-ordinary for every normalized Hecke eigenform on $\mathrm{SL}_2(\mathbb{Z})$. Therefore, we may assume that S consists only of primes $p \geq 5$.

For the given finite set of primes S, let $k_S(j,m) := j \prod_{p \in S}(p-1) + m$, where j is an arbitrary non-negative integer, $m \in A$. For each j and m let $b_S(j,m)$ be any integer for which

$$k_S(j,m) - 2 < (m-2)p^{b_S(j,m)}$$

for all $p \in S$. Let $f = \sum_{n=1}^{\infty} a_f(n)q^n$ be any Hecke eigenform of weight $k_S(j,m)$. By Theorem 2.5 (1), since $a_f(0) = 0$, we have

$$a_f(p^{b_S(j,m)}) \equiv 0 \pmod{p}$$

for all $p \in S$. Applying Proposition 2.1, we know that f is non-ordinary for each $p \in S$. As j can be chosen freely, we get the conclusion.

4. Examples

Example. Let $S = \{2, 3, 5, 7, 11, 13, 17, 19\}$. In the following table we list some of the weights k for which Hecke eigenforms are non-ordinary at each prime p.

p	$12 \leq k \leq 42$ such that all Hecke eigenforms S_k are non-ordinary at p															
2	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42
3	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42
5	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42
7	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42
11		14	16	18	20		24	26	28	30		34	36	38	40	
13		14	16	18	20	22		26	28	30	32	34		38	40	42
17		14			20	22	24	26		30			36	38	40	42
19		14				22	24	26	28		32				40	42

In particular, we consider the case $k = 26$ and check its non-ordinariness. We have the following q-expansion of the normalized weight 26 Hecke eigenform $f_{26} = \Delta E_6 E_4^2$:

$$
\begin{aligned}
f_{26}(z) = q &- 48q^2 - 195804q^3 - 33552128q^4 - 741989850q^5 \\
&+ 9398592q^6 + 39080597192q^7 \\
&+ 3221114880q^8 - 808949403027q^9 + 35615512800q^{10} + 8419515299052q^{11} \\
&+ 6569640870912q^{12} - 81651045335314q^{13} - 1875868665216q^{14} \\
&+ 145284580589400q^{15} + 1125667983917056q^{16} - 2519900028948078q^{17} \\
&+ 38829571345296q^{18} - 6082056370308940q^{19} + O(q^{20}).
\end{aligned}
$$

We can easily check that $a_{f_{26}}(p) \equiv 0 \pmod{p}$ for each $p \in S$. Of course we can also choose weights k of the form $k = 26 + 720j$, for every $j \in \mathbb{N}$. Note that $720 = [5-1, 7-1, 11-1, 13-1, 17-1, 19-1]$.

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