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GENERALIZED RECIPROCAL IDENTITIES

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Abstract. Included in Ramanujan's Notebooks are two reciprocal identities. The first identity connects the Rogers-Ramanujan continued fraction with an eta quotient. The second identity is a level thirteen analogue. These are special cases of a more general class of relations between eta quotients and modular functions defined by product generalizations of the Rogers-Ramanujan continued fraction. Each identity is shown to be a relation between generators for a certain congruence subgroup. The degree, form, and symmetry of the identities is determined from behavior at cusps of the congruence subgroup whose field of functions the parameters generate. The reciprocal identities encode information about fundamental units and class numbers for real quadratic fields.

1. Introduction

In his second notebook [16], Ramanujan derived the symmetric reciprocal identity [2, p. 267]

$$\frac{1}{R^5} - 11 - R^5 = \frac{1}{q}\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}, \qquad R(q) = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}, \tag{1.1}$$

where $(a;q)_\infty = \prod_{k=0}^\infty (1-aq^k)$. Ramanujan also [2, 16] gave a level thirteen analogue of (1.1)

$$\frac{1}{\mathcal{R}} - 3 - \mathcal{R} = \frac{1}{q}\frac{(q;q)_\infty^2}{(q^{13};q^{13})_\infty^2}, \qquad \mathcal{R} = \mathcal{R}(q) = q\prod_{n=1}^\infty (1-q^n)^{\left(\frac{n}{13}\right)}, \tag{1.2}$$

where $\left(\frac{n}{p}\right)$ is the Legendre symbol. The product $R(q)$ of (1.1) is equal to the Rogers-Ramanujan continued fraction [17]

$$R(q) = q^{1/5}\prod_{n=1}^\infty (1-q^n)^{\left(\frac{n}{5}\right)} = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}}. \tag{1.3}$$

Identities (1.1) and (1.2) are representative of relations between generators for the field of rational functions invariant on a subgroup of the Hecke congruence group of level Δ, where Δ is the discriminant of a real quadratic number field. At higher levels we will show that similar identities exist with corresponding symmetric form.

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For example, at level $\Delta = 17$ we obtain

$$r + \frac{1}{r} - 2\sqrt{\frac{4}{r} - 4r - 15} = \frac{1}{q^2}\frac{(q;q)_\infty^3}{(q^{17};q^{17})_\infty^3}, \qquad r = q^2 \prod_{n=1}^{16} (q^n;q^{17})_\infty^{\left(\frac{n}{17}\right)}. \tag{1.4}$$

More generally, for prime $\Delta = p \equiv 1 \bmod 4$ and χ the real primitive quadratic character modulo p, there exists a quasi-symmetric polynomial relation $\Phi_p(R_p, S_p) = 0$ between the η-quotient

(1.5)

$$S_p = \frac{\eta^{b_p}(p\tau)}{\eta^{b_p}(\tau)} = q^{a_p}\frac{(q^p;q^p)_\infty^{b_p}}{(q;q)_\infty^{b_p}}, \qquad \frac{p-1}{24} = \frac{a_p}{b_p}, \qquad \gcd(a_p,b_p) = 1, \qquad q = e^{2\pi i \tau}$$

and a product analogue of the Rogers-Ramanujan continued fraction

$$R_p = q^{\ell_p}\prod_{n=1}^{\infty}(1-q^n)^{\chi(n)}, \qquad \ell_p = -\frac{1}{2}L\left(-1,\chi\right). \tag{1.6}$$

Similar identities hold for composite levels. Listed below is a rational relation for $\Delta = 8$. The level 8 analogue of the function R is the Ramanujan-Gölnitz-Gordon continued fraction, and the corresponding identity is equivalent to a known reciprocal relation [4, Theorem 2.1, (iii)]

$$r^2 + \frac{1}{r^2} - 6 = \frac{(q;q)_\infty^4 (q^4;q^4)_\infty^2}{q(q^2;q^2)_\infty^2 (q^8;q^8)_\infty^4}, \qquad r = q^{1/2}\frac{(q;q^8)_\infty (q^7;q^8)_\infty}{(q^3;q^8)_\infty (q^5;q^8)_\infty}. \tag{1.7}$$

We describe an infinite class of identities for composite Δ, where the corresponding S_Δ is defined up to certain conditions. Among the identities for composite Δ, we list those for $\Delta = 12$ and 21:

(1.8)

$$r + \frac{1}{r} - 4 = \frac{1}{q}\frac{(q;q)_\infty^3 (q^4;q^4)_\infty (q^6;q^6)_\infty^2}{(q^2,q^2)_\infty^2 (q^3;q^3)_\infty (q^{12};q^{12})_\infty^3}, \qquad r = q\frac{(q;q^{12})_\infty (q^{11};q^{12})_\infty}{(q^5;q^{12})_\infty (q^7;q^{12})_\infty},$$

$$r^2 + \frac{1}{r^2} + \frac{29r}{2} + \frac{29}{2r} - 22 + \left(\frac{7r}{2} - \frac{7}{2r}\right)\sqrt{4r + \frac{4}{r} + 1} = \frac{1}{q^4}\frac{(q;q)_\infty^5 (q^7;q^7)_\infty}{(q^3;q^3)_\infty (q^{21};q^{21})_\infty^5},$$
$$r = q^2\prod_{n=1}^{\infty}(1-q^n)^{\left(\frac{21}{n}\right)}.$$

In both the prime and composite cases, the more general identities described here retain the form of (1.1), (1.2), and (1.7). We therefore term such identities reciprocal relations. We explain the existence of each class of identities and classify them according to the groups on which the functions are modular. The uniform symmetry of the identities follows from transformation formulas for the η-quotient S_Δ and the product R_Δ under action by the Hecke congruence subgroup of level Δ,

$$\Gamma_0(\Delta) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \bmod \Delta\}.$$

The identities themselves are a consequence of the fact that R_Δ and S_Δ generate the field $A_0(\Gamma_0^2(\Delta))$ of invariant functions for the index two subgroup corresponding to the real primitive character χ,

$$\Gamma_0^2(p) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \bmod p \text{ and } \chi(d) = 1\}.$$

The purpose of this paper is to establish these facts, determine the degree of the relations with respect to each parameter, and demonstrate that the coefficients encode information about the fundamental unit and class number for the real quadratic field $\mathbb{Q}(\sqrt{\Delta})$. In cases where S_Δ has a pole only at ∞, the values S_Δ are closely related to the McKay-Thompson series for the Monster group. In particular, the right sides of identities of (1.7) and (1.8) appear in Table 3 of [6].

The functions $R_\Delta(\tau)$ were first studied in detail by T. Horie and N. Kanou [9], where several instances of relations were derived between R_Δ and the Conway-Norton Hauptmoduln [6] for $\Gamma_\Delta(p)$ extended by the Atkin-Lehner involution. The relations discussed in the present work generalize relations beyond those from [9, Theorem 2.2]. We first consider cases for prime Δ with the η-quotient defined by (1.5). Identity (1.4) is equivalent to a quadratic in $s = S_7$ with coefficients in $r = R_7$,

$$0 = r^2 - 2\left(r^2+1\right)rs + \left(r^2+8r-1\right)^2 s^2. \tag{1.9}$$

For $\Delta = 29$, the relation $\Phi_\Delta(r,s) = 0$ is given, for $r = R_{29}$ and $s = S_{29}$, by

$$\begin{aligned} 0 = r^7 &+ \left(25r^4 - 92r^3 + 25r^2 + 92r + 25\right) r^5 s \\ &+ \left(4r^2 - 9r - 4\right)\left(34r^6 - 621r^5 + 389r^4 - 917r^3 - 389r^2 - 621r - 34\right) r^3 s^2 \\ &+ \left(r^2 + 5r - 1\right)^7 s^3. \end{aligned} \tag{1.10}$$

For $\Delta = 37$, we have the quintic

$$\begin{aligned} 0 = r^3 - 5r^3 s - r^2\left(13r^2 - 63r - 13\right)s^2 + r^2\left(51r^2 - 91r - 51\right)s^3 \\ + r\left(3r^2 - 38r - 3\right)\left(11r^2 - 16r - 11\right)s^4 + \left(r^2 + 12r - 1\right)^3 s^5. \end{aligned} \tag{1.11}$$

At level $\Delta = 41$, the identity $\Phi_\Delta(r,s) = 0$ takes the shape

$$0 = r^5 - r^4\left(58r^2 + 5r - 58\right)s^2 - 1777r^4\left(r^2+1\right)s^3 + \cdots + \left(r^2 + 64r - 1\right)^5 s^8. \tag{1.12}$$

For each fundamental discriminant Δ, an explicit formula for ℓ_Δ in agreement with (1.6) may be derived from a formula [1, §12.6] for Dirichlet L-values in terms of the Hurwitz zeta function

$$\ell_\Delta = -\frac{1}{2}L(\chi,-1) = \sum_{n=1}^{\frac{\Delta-1}{2}} \frac{n(n-\Delta)}{2\Delta}\chi(n), \tag{1.13}$$

where $\chi(n) = (\frac{\Delta}{n})$ is the real even character to the modulus Δ. Here an upper limit of $\frac{\Delta-1}{2}$ on a sum means that the summation index ranges over integers strictly less than $\frac{\Delta}{2}$. The values a_p, b_p, and ℓ_p from (1.6)–(1.9) for primes $p \equiv 1 \pmod 4$, $p \leq 157$ are listed below:

p	5	13	17	29	37	41	53	61	73	89	97	101	109	113	137	149	157
ℓ_p	1/5	1	2	3	5	8	7	11	22	26	34	19	27	36	48	35	43
a_p	1	1	2	7	3	5	13	5	3	11	4	25	9	14	17	37	13
b_p	6	2	3	6	2	3	6	2	1	3	1	6	2	3	3	6	2

From this data and identities (1.9)–(1.12) we make the following preliminary observations about $\Phi_p(r,s)$:

Observation 1.1. Let $p \equiv 1 \bmod 4$ be a prime fundamental discriminant.

(1) The identity $\Phi_p(r,s)$ is a polynomial of degree ℓ_p in $s = S_p$ and of degree $2a_p$ in $r = R_p$.
(2) Coefficients of successive powers of s are symmetric up to sign about the middle terms.
(3) As polynomials in s, and up to multiplication by a constant, $\Phi_p(r,s)$ has constant term 1 and leading term $s^{\ell_p}(r^2 + j_p r - 1)^{a_p}$, where $j_p = x$ is the first coordinate $(x,y) \in \mathbb{Z}^2$ of the least positive integer solution (x,y) to the Pell equation $x^2 - Dy^2 = -4$.

The third observation is related to the fact (see [3, p. 294]) that the fundamental unit $\epsilon_\Delta = (U + T\sqrt{\Delta})/2$ for $\mathbb{Q}(\sqrt{\Delta})$ is determined by the pair (U,T) corresponding to the least positive integer T satisfying $T^2 - \Delta U^2 = \pm 4$. The observation is correct when $\Delta = p$ is prime and the class number $h_p = 1$. More generally we show that the coefficient j_p in the leading term $s^{\ell_p}(r^2 + j_p r - 1)^{a_p}$ of Φ_p is given by $j_p = \epsilon_p^{h_p} - \epsilon_p^{-h_p}$. The arithmetic content results from a confluence of formulas in the expression for a product of Klein forms and a special case of the Class Number Formula [8, p. 228]

$$\epsilon_p^{h_p} = \prod_{k=1}^{(p-1)/2} \sin(k\pi/p)^{-\left(\frac{k}{p}\right)}, \tag{1.14}$$

where (k/p) denotes the Kronecker symbol. A precise statement of each of these observations is given in the next section along with proofs relying on the modular properties of $R_p(\tau)$ and $S_p(\tau)$. In Section 3, we prove corresponding claims for more generally defined parameters R_Δ and S_Δ of composite moduli Δ. Section 4 concludes with a construction of η-quotients S_Δ for composite Δ.

2. Properties of the reciprocal identities for prime levels

In this section, common features of the relations $\Phi_p(r,s) = 0$ are derived for prime fundamental discriminants $\Delta = p$. The main result of [9] is that ℓ_p is an integer for $p > 5$, so the function R_p belongs to the function field of $\Gamma_0^2(p)$ for these p. An amusing consequence of the analysis here is that ℓ_p is in fact even if $p \equiv 1 \bmod 8$. The first step in analyzing the function R is to note that ℓ_p is positive. Since this fact holds for any Δ, we record this fact here.

Proposition 2.1. *For any fundamental discriminant $\Delta > 1$, ℓ_Δ is positive and we have*

$$\frac{\Delta^{3/2}}{60} < \ell_\Delta < \frac{\Delta^{3/2}}{24}.$$

Proof. The functional equation for Dirichlet L-functions [7, Chapter 9] gives

$$\left(\frac{\pi}{\Delta}\right)^{(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) L(1-s,\chi) = \frac{\Delta^{1/2}}{\tau(\chi)} \left(\frac{\pi}{\Delta}\right)^{s/2} \Gamma\left(\frac{s}{2}\right) L(s,\chi),$$

where (see [5, §2.2.5])

$$\tau(\chi) = \sum_{n=1}^{\Delta} \chi(n) e^{2\pi i n/\Delta} = \Delta^{1/2} \tag{2.1}$$

is the Gauss sum attached to χ. Therefore,

$$\ell_\Delta = -\frac{1}{2}L(-1,\chi) = \frac{\Delta^{3/2}L(2,\chi)}{4\pi^2},$$

and the bound follows from

$$\frac{\pi^2}{15} = \prod_{p\text{ prime}} (1+p^{-2})^{-1} < L(2,\chi) < \prod_{p\text{ prime}} (1-p^{-2})^{-1} = \frac{\pi^2}{6}. \qquad \square$$

The identities are motivated by showing that R_p and S_p generate the field of invariant functions of $\Gamma_0^2(p)$ for $p > 5$. By calculating the order of each function at the inequivalent cusps of $\Gamma_0^2(p)$, the degree of each relation is derived. Observation 1.1 is proved in the case that the class number $h_p = 1$. We start with basic facts for the group $\Gamma_0^2(p)$ and its inequivalent cusps. Denoting two entries of a matrix by stars indicates that the entries should be completed in an arbitrary (but fixed) manner so the resulting matrix is in $\mathrm{SL}_2(\mathbb{Z})$.

Lemma 2.2. *Let* $5 < p \equiv 1 \bmod 4$ *be a prime and let* g *be a fixed integer with* $\chi(g) = -1$. *Then*

(1) $\Gamma_0(p) = \Gamma_0^2(p) \cup \left(\begin{smallmatrix} g & * \\ p & * \end{smallmatrix}\right)\Gamma_0^2(p) = \Gamma_0^2(p) \cup \left(\begin{smallmatrix} * & * \\ p & g \end{smallmatrix}\right)\Gamma_0^2(p)$,

(2) $\overline{\mathbb{H}}/\Gamma_0^2(p)$ *has four inequivalent cusps* $\frac{1}{0}$, $\frac{g}{p}$, $\frac{0}{1}$, $\frac{1}{g}$ *with widths* 1, 1, p, *and* p, *respectively.*

Proof. These properties follow directly from the fact that $\Gamma_0^2(p)$ is a subgroup of $\Gamma_0(p)$ of index 2, and the inequivalent cusps of $\Gamma_0(p)$ are $\frac{0}{1}$ and $\frac{1}{0}$ of widths 1 and p, respectively. $\square$

For the order of a non-constant function f at a cusp $\frac{a}{c} \in \overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, we say $\mathrm{ord}_{\frac{a}{c}}(f(z)) = n$ if

$$f\left(\frac{a\tau + *}{c\tau + *}\right) \propto q^n(1 + \text{ higher powers of } q).$$

The invariant order of f with respect to any finite-index subgroup of the modular group, Γ, at the cusp $\frac{a}{c}$ is then defined as $\mathrm{ord}_{\frac{a}{c}}(f,\Gamma) = m \cdot \mathrm{ord}_{\frac{a}{c}}(f)$, where m, known as the width of the cusp $\frac{a}{c}$, is the smallest positive integer such that $\pm\left(\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right)^{-1} \in \Gamma$. We then denote by $\mathrm{ord}_\Gamma(f)$ the number of poles modulo Γ of f counted according to multiplicity. All of the functions considered here will be holomorphic in $\mathbb{H}$, so we are only concerned with their orders at points in $\overline{\mathbb{Q}}$.

Proposition 2.3. *Let* p, g *be as in Lemma* 2.2, *and define* a_p, ℓ_p, R_p, *and* S_p *by* (1.5)–(1.6). *Then*

$$\begin{array}{llll} \mathrm{ord}_{\frac{1}{0}} S_p = a_p, & \mathrm{ord}_{\frac{g}{p}} S_p = a_p, & \mathrm{ord}_{\frac{0}{1}} S_p = -a_p, & \mathrm{ord}_{\frac{1}{g}} S_p = -a_p, \\ \mathrm{ord}_{\frac{1}{0}} R_p = \ell_p, & \mathrm{ord}_{\frac{g}{p}} R_p = -\ell_p, & \mathrm{ord}_{\frac{0}{1}} R_p = 0, & \mathrm{ord}_{\frac{1}{g}} R_p = 0. \end{array}$$

Proof. To calculate the expansion for R_p at the cusps of $\Gamma_0^2(p)$, we begin with a representation for R_p as a product of Klein forms. We have [9, Theorem 6.2]

$$R_p(\tau) = \prod_{j=1}^{\frac{p-1}{2}} \mathfrak{k}_{\frac{j}{p},\frac{0}{p}}(p\tau)^{\chi(j)},$$

where χ is the quadratic character modulo p and $\mathfrak{k}_{a,b}$ is the Klein form defined by

$$\mathfrak{k}_{a,b}(\tau) = e^{i\pi(a-1)b} q^{\frac{1}{2}(a-1)a} \frac{(e^{-2\pi i b} q^{1-a};q)_\infty (e^{2\pi i b} q^a;q)_\infty}{(q;q)_\infty^2}.$$

From the transformation formula for Klein forms [12, Chapter 2, §1], we have
(2.2)

$$\begin{aligned}
R_p(\tfrac{-1}{p\tau}) &= \prod_{j=1}^{\frac{p-1}{2}} \mathfrak{k}_{\frac{j}{p},\frac{0}{p}}(\tfrac{-1}{\tau})^{\chi(j)} \\
&= \prod_{j=1}^{\frac{p-1}{2}} \mathfrak{k}_{\frac{0}{p},\frac{j}{p}}(\tau)^{\chi(j)} \\
&= \prod_{j=1}^{\frac{p-1}{2}} \Big(e^{-\frac{\pi i j}{p}} (e^{\frac{2\pi i j}{p}};q)_\infty (e^{-\frac{2\pi i j}{p}} q;q)_\infty\Big)^{\chi(j)} \\
&= \prod_{j=1}^{\frac{p-1}{2}} e^{-\frac{\pi i j}{p}\chi(j)} (1-e^{\frac{2\pi i j}{p}})^{\chi(j)} (1-e^{\frac{-2\pi i j}{p}} q + \cdots)^{\chi(j)} (1-e^{\frac{2\pi i j}{p}} q+\cdots)^{\chi(j)} \\
&= \prod_{j=1}^{\frac{p-1}{2}} (e^{-\frac{\pi i j}{p}} - e^{\frac{\pi i j}{p}})^{\chi(j)} (1-\chi(j)(e^{\frac{-2\pi i j}{p}} + e^{\frac{2\pi i j}{p}})q + \cdots) \\
&= \epsilon_p^{-h_p}(1-\sqrt{p}q + \tfrac{1}{2}(p - (\tfrac{2}{p})\sqrt{p} - 2\sqrt{p})q^2 + \cdots).
\end{aligned}$$

The expansions of R_p and S_p at the cusps $\frac{1}{0}$, $\frac{g}{p}$, $\frac{0}{1}$, $\frac{1}{g}$ are, respectively,
(2.3)

$$\begin{aligned}
R_p(\tau) &= q^{\ell_p}(1+\cdots), & S_p(\tau) &= q^{+a_p}(1+\cdots), \\
R_p\left(\frac{g\tau+*}{p\tau+*}\right) &= -q^{-\ell_p}(1+\cdots), & S_p\left(\frac{g\tau+*}{p\tau+*}\right) &= q^{+a_p}((-1)^{b_p}+\cdots), \\
R_p\left(\frac{-1}{\tau}\right) &= \epsilon_p^{-h_p}(1-\sqrt{p}q^{1/p}+\cdots), & S_p\left(\frac{-1}{\tau}\right) &= q^{-a_p/p}(1+\cdots), \\
R_p\left(\frac{\tau+*}{g\tau+*}\right) &= -\epsilon_p^{h_p}(1+\sqrt{p}q^{1/p}+\cdots), & S_p\left(\frac{\tau+*}{g\tau+*}\right) &= q^{-a_p/p}((-1)^{b_p}+\cdots).
\end{aligned}$$

Here we used (2.2), and the fact that R_p and S_p are invariant under $\Gamma_0^2(p)$, and the modular transformations [9] for R_p and S_p under $\Gamma_0(p)$. In particular, for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p) \setminus \Gamma_0^2(p)$,

$$S_p\left(\frac{a\tau+b}{c\tau+d}\right) = (-1)^{b_p} S_p(\tau), \qquad R_p\left(\frac{a\tau+b}{c\tau+d}\right) = -R_p(\tau)^{-1}. \tag{2.4}$$

The second two identities of (2.3) follow from the first, and the fourth identities from the third. □

We next show that S_p and R_p generate the field of functions for $\Gamma_0^2(p)$. To prove the theorem, we apply an argument similar to that in [10]. This relies on the fundamental fact [18, Proposition 2.11]

$$[A_0(\Gamma) : \mathbb{C}(f)] = \mathrm{ord}_\Gamma(f),$$

for any function $f \in A_0(\Gamma)$.

Theorem 2.4. *The field $A_0(\Gamma_0^2(p))$ of functions invariant under $\Gamma_0^2(p)$ is generated by R_p and S_p. Moreover, with respect to $\Gamma_0^2(p)$, the total orders of R_p and S_p are ℓ_p and $2a_p$, respectively.*

Proof. Choose any integer m such that $ma_p \geq \ell_p$. From the q-series expansions, it is clear that S_p has a pole of order a_p at $\frac{0}{1}$ and $\frac{1}{g}$, while $(\epsilon_p^{h_p} R_p - 1)S_p^m$ has a pole of order $ma_p - 1$ at $\frac{0}{1}$ and a pole of order ma_p at $\frac{1}{n}$. In total,

$$\operatorname{ord}_{\Gamma_0^2(p)} S_p = 2a_p, \qquad \operatorname{ord}_{\Gamma_0^2(p)} (\epsilon_p^{h_p} R_p - 1)S_p^m = 2ma_p - 1.$$

Let L be the subfield of $A_0(\Gamma_0^2(p))$ generated by S_p and $(\epsilon_p^{h_p} R_p - 1)S_p^m$. The degree $[A_0(\Gamma_0^2(p)) : L]$ is a divisor of $[A_0(\Gamma_0^2(p)) : \mathbb{C}(f)]$ for $f = S_p$ and $f = (\epsilon_p^{h_p} R_p - 1)S_p^m$. Since these two indices are relatively prime, it follows that $[A_0(\Gamma_0^2(p)) : \mathbb{C}(x)] = 1$, and $L = A_0(\Gamma_0^2(p))$. Since S_p and $(\epsilon_p^{h_p} R_p - 1)S_p^m$ generate the function field of $\Gamma_0^2(p)$, this means that S_p and R_p are also generators. □

Proposition 2.5. *For prime $5 < p \equiv 1 \bmod 4$, let ϵ_p and h_p be the fundamental unit and class number of $\mathbb{Q}(\sqrt{p})$, respectively. Let $\Phi_p(r, s)$ be the irreducible polynomial with $\Phi_p(R_p(q), S_p(q)) = 0$. Then*

(1) $\deg_r \Phi_p(r, s) = 2a_p$.
(2) $\deg_s \Phi_p(r, s) = \ell_p$.
(3) $\Phi_p(r, s) = (-1)_p^a r^{2a_p} \Phi_p(-1/r, (-1)^{b_p} s)$.
(4) *Observation 1.1 is correct for $h_p = 1$:*

$$r^{-a_p}\Phi_p(r, s) \propto (r - r^{-1} + \epsilon_p^{h_p} - \epsilon_p^{-h_p})^{a_p} s^{\ell_p} + \textit{lower powers of } s,$$

$$r^{-a_p}\Phi_p(r, s) \propto 1 + \textit{higher powers of } s.$$

Proof. The assertions $\deg_r \Phi_p(r, s) = 2a_p$ and $\deg_s \Phi_p(r, s) = \ell_p$ follow immediately from Theorem 2.4. Next, for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p) \setminus \Gamma_0^2(p)$, we have (2.4), and so $\Phi_p(r, s) \propto r^{2a_p}\Phi_p(-1/r, (-1)^{b_p} s)$. In order to deduce that this constant is $(-1)_p^a$, we use the second result of (4). For (4), write

$$\Phi_p(r, s) = f_{\ell_p}(r)s^{\ell_p} + f_{\ell_p - 1}(r)s^{\ell_p - 1} + \cdots + f_1(r)s + f_0(r),$$

where the $f_i(r)$ are polynomials in r that transform properly under $(r, s) \mapsto (-1/r, (-1)^{b_p} s)$. Note that near the cusps $\frac{0}{1}$ and $\frac{1}{g}$, all of the $f_i(R_p(\tau))$ remain finite while $S_p(\tau)^i$ has a pole of order ia_p. This forces $f_{\ell_p}(r)$ to have a zero of order at least a_p in order to cancel the pole of order $\ell_p a_p$ introduced by the factor $S_p(\tau)^{\ell_p}$, since the remaining terms, $f_{\ell_p - 1}(r)s^{\ell_p - 1} + \cdots + f_0(r)$, contribute a pole of order at most $(\ell_p - 1)a_p$. Since $\frac{0}{1}$ is a zero of $f_{\ell_p}(r)$ of order at least a_p, it must contain $(r - \epsilon_p^{-h_p})^{a_p}$ as a factor by (2.3). Similarly, $f_{\ell_p}(r)$ must contain $(r + \epsilon_p^{h_p})^{a_p}$ as a factor in order to have a zero of order at least a_p at $\frac{1}{g}$. Since the degree of $f_{\ell_p}(r)$ is not more than $2a_p$, we must have

$$f_{\ell_p}(r) \propto (r - \epsilon_p^{-h_p})^{a_p}(r + \epsilon_p^{h_p})^{a_p}.$$

We can also deduce that $f_0(r) \propto r^{a_p}$. Since $\Phi_p(r, s)$ is irreducible, $f_0(r)$ can have zeros only at the two locations of the zeros of S_p, namely $\frac{1}{0}$ and $\frac{g}{p}$. Since R_p has a pole at $\frac{g}{p}$, the only possible zero of $f_0(r)$ is at $\frac{1}{0}$, and so $f_0(r)$ is proportional to a power of r. Since $\Phi_p(r, s) \propto r^{2a_p}\Phi_p(-1/r, (-1)^{b_p} s)$, this power must be a_p, and we have $f_0(r) \propto r^{a_p}$. □

Corollary 2.6. *If $p \equiv 1 \bmod 8$, then ℓ_p is even.*

Proof. The product $b_p \ell_p$ is even since $r^{-a_p}\Phi_p(r,s)$ is invariant under $(r,s) \mapsto (-1/r, (-1)^{b_p} s)$. □

3. Construction of reciprocal identities for general fundamental discriminants

Let Δ be a fundamental discriminant of a real quadratic field, that is, $\Delta > 1$ and

$$\Delta \equiv 1 \bmod 4 \text{ and } \Delta \text{ is square free}$$
$$\text{or } \Delta \equiv 8, 12 \bmod 16 \text{ and } \Delta/4 \text{ is square free.}$$

We can still define our function R_Δ as

$$R_\Delta(\tau) = q^{\ell_\Delta} \prod_{n=1}^{\infty} (1-q^n)^{\chi(n)}, \quad \ell_\Delta = \sum_{n=1}^{\frac{\Delta-1}{2}} \frac{n(n-\Delta)}{2\Delta} \chi(n),$$

where $\chi(n) = (\frac{\Delta}{n})$ is the real even character to the modulus Δ. As before, we let the fundamental unit of $\mathbb{Q}(\sqrt{\Delta})$ be ϵ_Δ and h_Δ its class number. If we set $\chi'(n)$ to be $\chi(n)$ if Δ is prime and $\chi'(n)$ to be the principal character modulo Δ when Δ is not prime, then the results of [9] give the transformation formula

$$R_\Delta\left(\frac{a\tau+b}{c\tau+d}\right) = \chi'(d) R_\Delta(\tau)^{\chi(d)}$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\Delta)$ and $\Delta > 8$.

We next characterize four important classes of cusps that are analogous to the four classes of cusps of $\overline{\mathbb{H}}/\Gamma_0^2(p)$ in the case of prime discriminants.

Proposition 3.1. *Let $\Gamma(\beta)$ denote the Γ-orbit of the cusp $\beta \in \overline{\mathbb{Q}}$. If g denotes any integer with $\chi(g) = -1$, then*

$$\Gamma_0^2(\Delta)(\tfrac{1}{\Delta}) = \{\tfrac{a}{c} \mid \chi(a) = +1 \text{ and } \Delta | c\},$$
$$\Gamma_0^2(\Delta)(\tfrac{g}{\Delta}) = \{\tfrac{a}{c} \mid \chi(a) = -1 \text{ and } \Delta | c\},$$
$$\Gamma_0^2(\Delta)(\tfrac{1}{1}) = \{\tfrac{a}{c} \mid \chi(c) = +1\},$$
$$\Gamma_0^2(\Delta)(\tfrac{1}{g}) = \{\tfrac{a}{c} \mid \chi(c) = -1\}.$$

The analogue of the η-quotient S_p is slightly more difficult to describe. We will choose any η-quotient $S(\tau)$ that has the following properties:

(1) S is in $A_0(\Gamma_0^2(\Delta))$.
(2) S has poles only at the cusps $\frac{1}{1}$ (and $\frac{1}{g}$) of $\overline{\mathbb{H}}/\Gamma_0^2(\Delta)$.
(3) S has zeros at the cusps $\frac{1}{\Delta}$ (and $\frac{g}{\Delta}$) of $\overline{\mathbb{H}}/\Gamma_0^2(\Delta)$.

Such an η-quotient always exists, and we give an explicit construction in Section 4. The order of the poles of S at $\frac{1}{1}$ and $\frac{1}{g}$ are the same, and we set a_Δ to be this common value. This is the analogue of a_p in the case of prime discriminants. Strengthening condition (3) to allow zeros *only* at $\frac{1}{\Delta}$ and $\frac{g}{\Delta}$ gives the strictest analogue of S_p, but such a condition might give a large order a_Δ and is unecessary in Proposition 3.4 below.

We need the following lemma on Dirichlet characters which follows from [13, Theorem 9.4].

Lemma 3.2. *Suppose that χ is a primitive Dirichlet character to the modulus q and d is a proper divisor of q. If $f : \mathbb{Z} \to \mathbb{C}$ has a period d, then*

$$\sum_{i=1}^{q} \chi(i) f(i) = 0.$$

A consequence of Lemma 3.2 is that the function R has the remarkable property of having only one pole and only one zero on $\overline{\mathbb{H}}/\Gamma_0^2(\Delta)$.

Proposition 3.3. *R has a zero of order ℓ_Δ at $\frac{1}{\Delta}$ and a pole of order ℓ_Δ at $\frac{g}{\Delta}$. R takes finite non-zero values at every other point in $\overline{\mathbb{Q}}/\Gamma_0^2(\Delta)$. That is,*

$$\operatorname{ord}_{\frac{a}{c}}(R, \Gamma_0^2(\Delta)) = \begin{cases} \chi(a)\ell_\Delta, & \Delta \mid c, \\ 0, & \Delta \nmid c. \end{cases} \tag{3.1}$$

Proof. Recall the representation

$$\begin{aligned} R(\tau) &= \prod_{i=1}^{\frac{\Delta-1}{2}} \mathfrak{k}_{\frac{i}{\Delta}, \frac{0}{\Delta}}(\Delta\tau)^{\chi(i)} \\ &\propto \prod_{i=1}^{\frac{\Delta-1}{2}} \prod_{j=1}^{\Delta} \mathfrak{k}_{\frac{i}{\Delta}, \frac{j}{\Delta}}(\tau)^{\chi(i)}. \end{aligned}$$

From the fact that the Klein form satisfies [12, Chapter 2, §1]

$$\mathfrak{k}_{u,v}\left(\frac{a\tau + *}{c\tau + *}\right) \propto q^{\binom{\operatorname{frac}(au+cv)}{2}}(1 + \text{positive powers of } q),$$

where $\operatorname{frac}(x)$ is the fractional part of x with $0 \le \operatorname{frac}(x) < 1$, we obtain

$$\operatorname{ord}_{\frac{a}{c}}(R) = \sum_{i=1}^{\frac{\Delta-1}{2}} \sum_{j=1}^{\Delta} \chi(i) \binom{\operatorname{frac}\left(\frac{ai+cj}{\Delta}\right)}{2},$$

and we need to show that this is zero whenever $\Delta \nmid c$, so assume first that $\Delta \nmid c$. Set

$$f_{a,c}(i) = \sum_{j=1}^{\Delta} \binom{\operatorname{frac}\left(\frac{ai+cj}{\Delta}\right)}{2}.$$

Two easy observations on f are that it has c as a period and that it is even, i.e.

$$f_{a,c}(i + c) = f_{a,c}(i), \qquad f_{a,c}(-i) = f_{a,c}(i).$$

The function $f_{a,c}(i)$ also has Δ as a period, so $f_{a,c}(i)$ has a period $\gcd(\Delta, c)$. Since f is even, we may extend the sum to the whole range Δ as

$$\operatorname{ord}_{\frac{a}{c}}(R) = \tfrac{1}{2} \sum_{i=1}^{\Delta} \chi(i) f_{a,c}(i).$$

Applying Lemma 3.2 with $d = \gcd(\Delta, c)$ immediately gives $\operatorname{ord}_{\frac{a}{c}}(R) = 0$ for $\Delta \nmid c$.

If $\Delta \mid c$, then, since now we have $\gcd(a, \Delta) = 1$, the order of R is given by

$$\begin{aligned}\operatorname{ord}_{\frac{a}{c}}(R) &= \tfrac{1}{2}\sum_{i=1}^{\Delta}\chi(i)\sum_{j=1}^{\Delta}\binom{\operatorname{frac}\left(\frac{ai+cj}{\Delta}\right)}{2}\\ &= \tfrac{1}{2}\sum_{i=1}^{\Delta}\chi(i)\Delta\binom{\operatorname{frac}\left(\frac{ai}{\Delta}\right)}{2}\\ &= \tfrac{1}{2}\sum_{k=1}^{\Delta}\chi(ak)\Delta\binom{\operatorname{frac}\left(\frac{k}{\Delta}\right)}{2} \quad (\text{set } k = ai)\\ &= \tfrac{1}{2}\chi(a)\sum_{k=1}^{\Delta}\chi(k)\Delta\tfrac{1}{2}\tfrac{k}{\Delta}(\tfrac{k}{\Delta}-1)\\ &= \chi(a)\ell_{\Delta}.\end{aligned}$$

Since the cusps with denominator divisible by Δ have width 1 with respect to $\Gamma_0^2(\Delta)$, this is also the invariant order of the R at these cusps. □

Proposition 3.4. *Let $\Delta > 8$ be a fundamental discriminant. If S is an η-quotient satisfying the conditions* (1)-(3) *above, then the function field of $\Gamma_0^2(\Delta)$ is generated by R_Δ and S.*

Proof. We know from Proposition 3.3 that $R = R_\Delta$ takes a non-zero finite value at the cusp $\frac{0}{1}$. The q-series expansion at this point may be derived analogously to the expansion of $R_p(-1/p\tau)$ and we have

$$R\left(\tfrac{-1}{\Delta\tau}\right) = \prod_{j=1}^{\frac{\Delta-1}{2}}\left(e^{-\frac{\pi i j}{\Delta}} - e^{\frac{\pi i j}{\Delta}}\right)^{\chi(j)}(1 - \sqrt{\Delta}q + \cdots), \tag{3.2}$$

where the evaluation of the Gauss sum in (2.1) is used to obtain the coefficient of q, although for our purposes we only need that this coefficient is non-zero. By the class number formula, we have

$$R(0) = \epsilon_\Delta^{-h_\Delta}.$$

Since $R(\frac{1}{1}) = R(0)$, and $R(\frac{1}{g}) = \chi'(g)R(0)^{-1}$, we have

$$R(\tfrac{1}{1}) \neq R(\tfrac{1}{g}). \tag{3.3}$$

Choose an integer m that is large enough so that $(R(\tau)-R(0))S(\tau)^m$ has poles only at $\frac{1}{1}$ and $\frac{1}{g}$. This is possible because S was supposed to have a zero at the location of the pole of R and poles only at $\frac{1}{1}$ and $\frac{1}{g}$. The function $(R(\tau)-R(0))S(\tau)^m$ has a pole of order $\alpha m - 1$ at $\frac{1}{1}$ by (3.2) and a pole of order αm at $\frac{1}{g}$ by (3.3). Therefore, S and $(R - R(0))S^m$ have total orders 2α and $2\alpha m - 1$, which are relatively prime, and so the proof is complete. □

4. Constructing the η-quotient for composite Δ

For any Δ, define the valuation matrix M_Δ by

$$M_\Delta = \left(\frac{\Delta\gcd(l,c)^2}{24l\gcd(\Delta,c^2)}\right)_{c|\Delta,l|\Delta}.$$

The divisors c and l are taken in increasing order. For example,

$$M_8 = \frac{1}{24}\begin{pmatrix} 8 & 4 & 2 & 1 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 1 & 2 & 4 & 8 \end{pmatrix}.$$

Recall that every cusp is equivalent under $\Gamma_0(\Delta)$ to one of the form $\frac{a}{c}$ where $c|\Delta$, and that ∞ is equivalent to any cusp with denominator divisible by Δ, while 0 is equivalent to any cusp whose denominator is relatively prime to Δ. If $f(\tau) = \prod_{l|\Delta} \eta(l\tau)^{v_l}$ and v is the vector of the v_l, then the order of f at any cusp with denominator c ($c|\Delta$) is given by the corresponding entry of $M_\Delta.v$.

Proposition 4.1. *Let $f(\tau) = \prod_{l|\Delta} \eta(l\tau)^{v_l}$ and let v be the vector of the v_l. Set $m = \prod_{l|\Delta} l^{v_l}$. If $\sum_{l|\Delta} v_l = 0$ and the first and last entries of $M_\Delta.v$ are integers, then the remaining entries are integers as well, and $f \in A_0(\Gamma_0(\Delta), \left(\frac{m}{\cdot}\right))$.*

Proof. The order of vanishing of f at the cusp $\frac{a}{c}$ with respect to $\Gamma_0(\Delta)$ is given by

$$\operatorname{ord}_{\frac{a}{c}}(f, \Gamma_0(\Delta)) = \sum_{l|\Delta} \frac{\Delta \gcd(l,c)^2}{24l \gcd(\Delta, c^2)} v_l$$

[15, Theorem 1.65]. Hence the corresponding entry of $M_\Delta.v$ is the order of f at any cusp with denominator c. The second assertion is due to Newman [14]. □

The matrix M_Δ is invertible [11], so we can prescribe the locations of the zeros and poles of f among the equivalent classes of cusps as we wish. We are constrained only by the fact that the number of zeros and poles must be equal when counted according to multiplicity and the fact that the entries of both v and $M_\Delta.v$ must be integers.

As an example, take $\Delta = 21$. We want f to have a pole only at $\frac{1}{1}$ and to have a zero at $\frac{1}{21}$. Thus we could take the vector x of orders corresponding to the cusps with denominators 1, 3, 7, and 21 to be $(-1, 0, 0, 1)$. As

$$M_{21}^{-1}.(-1,0,0,1) = \tfrac{1}{4}(-5, 1, -1, 5),$$

we must rescale x to be $(-4, 0, 0, 4)$ in order to obtain integer orders. The value of α is then 4 and our S is

$$S = \eta(\tau)^{-5}\eta(3\tau)^1\eta(7\tau)^{-1}\eta(21\tau)^5.$$

We then find the relation Φ_{21} is simply

$$r^{-4}\Phi_{21}(r,s) = 1 + \left(-2r^2 - \frac{2}{r^2} - 29r - \frac{29}{r} + 44\right)s + \left(r + \frac{1}{r} - 5\right)^4 s^2,$$

which has all of the properties observed for prime discriminants including the constant 5 corresponding to the fundamental unit $\frac{1}{2}(5+\sqrt{21})$ of $\mathbb{Q}(\sqrt{21})$.

In the $\Delta = 12$ case, the η-quotient $S = \eta(\tau)^{-3}\eta(2\tau)^2\eta(3\tau)^1\eta(4\tau)^{-1}\eta(6\tau)^{-2}\eta(12\tau)^3$ attains $a_{12} = 1$, and we have the relation

$$r^{-1}\phi_{12}(r,s) = 1 - \left(r + \frac{1}{r} - 4\right)s.$$

The constant 4 in the leading term is consistent with the fundamental unit

$$2 + \sqrt{3} = \tfrac{1}{2}(4 + 2\sqrt{3}).$$

As a last example corresponding to a case where Observation 1.1 turns out not to hold because the class number is 2, take $\Delta = 65$. If we prescribe that S has zeros only at $\frac{1}{65}$, then the η-quotient of lowest order arises from

$$M_{65}^{-1}(-21,0,0,21) = (-8,1,-1,8),$$

with the corresponding η-quotient having $a_{65} = 21$. However, if we allow S to have a zero at $\frac{1}{5}$, then we can obtain an η-quotient with $a_{65} = 7$ by

$$M_{65}^{-1}(-7,4,0,3) = (-3,2,0,1).$$

Note that the corresponding η-quotient $S = \eta(\tau)^{-3}\eta(5\tau)^2\eta(65\tau)$ has multiplier system $(\frac{5^2 65}{\cdot}) = (\frac{65}{\cdot})$ with respect to $\Gamma_0(65)$, so it is invariant under $\Gamma_0^2(65)$. The corresponding relation is

$$r^{-7}\Phi_{64}(r,s) = -\left(\sqrt{r}+\frac{1}{\sqrt{r}}\right)^8 + 13\left(\sqrt{r}+\frac{1}{\sqrt{r}}\right)^5\left(7r^{3/2}-\frac{7}{r^{3/2}}+85\sqrt{r}-\frac{85}{\sqrt{r}}\right)s$$
$$+\cdots+\left(r+\frac{1}{r}-258\right)^7 s^{16}.$$

The constant 258 appearing in the leading terms corresponds to the second power of the fundamental unit of $\mathbb{Q}(\sqrt{65})$,

$$(8+\sqrt{65})^2 = \tfrac{1}{2}(258+32\sqrt{65}).$$

In such situations, where S has a zero other than ∞, the coefficient of s^0 is no longer a constant.

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