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A NOTE ON L^p-BOUNDED POINT EVALUATIONS FOR POLYNOMIALS

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ABSTRACT. We construct a compact nowhere dense subset K of the closed unit disk $\bar{\mathbb{D}}$ in the complex plane $\mathbb{C}$ such that $R(K) = C(K)$ and bounded point evaluations for $P^t(dA|_K)$, $1 \le t < \infty$, is the open unit disk $\mathbb{D}$. In fact, there exists $C = C(t) > 0$ such that

$$\int_{\mathbb{D}} |p|^t dA \le C \int_K |p|^t dA,$$

for $1 \le t < \infty$ and all polynomials p.

1. INTRODUCTION

Let μ be a finite positive compactly supported Borel measure in the complex plane $\mathbb{C}$. For each $1 \le t < \infty$, let $P^t(\mu)$ be the closure of the complex analytic polynomials in $L^t(\mu)$. A point $\lambda_0 \in \mathbb{C}$ is called a bounded point evaluation (bpe) if there exists a constant $C > 0$ such that

$$|p(\lambda_0)| \le C\|p\|_{L^t(\mu)}$$

for all polynomials p. If the above inequality holds for all λ in an open disk $D(\lambda_0, r)$, then λ_0 is called an analytic bounded point evaluation (abpe). For a compact subset K of the complex plane $\mathbb{C}$, $C(K)$ will denote the space of continuous functions on K and $R(K)$ denotes uniform closure in $C(K)$ of the rational functions with poles outside K. Let μ_K denote the area measure dA restricted to the compact subset K.

For a compact subset $E \subset \mathbb{C}$, we define the analytic capacity of E by

$$\gamma(E) = \sup|f'(\infty)|,$$

where the sup is taken over those functions f analytic in $\mathbb{C}_\infty \setminus E$ for which $|f(z)| \le 1$ for all $z \in \mathbb{C}_\infty \setminus E$, and

$$f'(\infty) = \lim_{z\to\infty} z[f(z) - f(\infty)].$$

The analytic capacity of a general $E_1 \subset \mathbb{C}$ is defined to be

$$\gamma(E_1) = \sup\{\gamma(E) : E \subset E_1,\ E \text{ compact}\}.$$

Tolsa, [7] proved the semiadditivity of analytic capacity. That is, there exists an absolute constant $A_T > 0$ such that

$$\gamma(\bigcup_{n=1}^{\infty} E_n) \le A_T \sum_{n=1}^{\infty} \gamma(E_n).$$

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Brennan and Militzer [2] established a connection between uniform rational approximation $R(K)$ and approximation in the mean by polynomials $P^t(d\mu_K),\ 1 \le t < \infty$. The paper proved the following theorem (see Theorem 4.1 and Corollary 4.3 in [2]).

Theorem 1 (Brennan and Militzer)**.** *Let K be a compact subset of $\mathbb{C}$ with empty interior. If $R(K) \neq C(K)$, then there exists at least one point λ_0 that yields a bpe for every $P^t(d\mu_K),\ 1 \le t < \infty$. Moreover, if $\lambda_0 \in K$ is not a peak point for $R(K)$, then λ_0 yields a bpe for $P^t(d\mu_K),\ 1 \le t < \infty$.*

Aleman, Richter, and Sundberg [1] proved the following theorem (see Lemma B in their paper).

Theorem 2 (Aleman, Richter, and Sundberg)**.** *There are absolute constants $\epsilon_1 > 0$ and $C_1 < \infty$ with the following property. Let $E \subset clos\mathbb{D}$ with $\gamma(E) < \epsilon_1$. Then*

$$|p(0)| \le C_1 \int_{clos\mathbb{D}\setminus E} |p| \frac{dA}{\pi}$$

for all polynomials p.

It is not difficult to prove the following corollary from Theorem 2.

Corollary 1. *Let $\epsilon_1 > 0$ be as in Theorem 2. Let $K \subset \mathbb{C}$ be a compact subset and $\lambda_0 \in K$. If there exists $r > 0$ such that*

$$\frac{\gamma(D(\lambda_0, r) \setminus K)}{r} < \epsilon_1,$$

then λ_0 is a bpe for $P^t(d\mu_K),\ 1 \le t < \infty$.

Proof. Let $E = \frac{1}{r}(D(\lambda_0, r) \setminus K - \lambda_0)$; then E is an open subset of $\mathbb{D}$. Using the elementary properties of analytic capacity (see p. 196 of [3]), we see

$$\gamma(E) = \frac{\gamma(D(\lambda_0, r) \setminus K)}{r} < \epsilon_1.$$

Therefore, it follows from Theorem 2 that, for $q(w) = p(\frac{1}{r}(w - \lambda_0))$,

$$\begin{aligned} |q(\lambda_0)| =& |p(0)| \\ \le& C_1 \int_{clos\mathbb{D}\setminus E} |p(z)| \frac{dA(z)}{\pi} \\ \le& \frac{C_1}{r^2} \int_{D(\lambda_0,r)\cap K} |p(\frac{1}{r}(w-\lambda_0))| \frac{dA(w)}{\pi} \\ \le& \frac{C_1}{\pi r^2} \int |q(w)| d\mu_K(w). \end{aligned}$$

The proof is completed.

Corollary 1 is a generalization of Theorem 1. In fact, if $\lambda_0 \in K$ is not a peak point for $R(K)$, then by Melnikov's Theorem (see [3], p. 205),

$$\sum_{i=1}^{\infty} \frac{\gamma(\{\lambda : a^{i+1} \le |\lambda - \lambda_0| < a^i\} \setminus K)}{a^i} < \infty.$$

It follows from the semiadditivity of analytic capacity discussed above that

$$\frac{\gamma(D(\lambda_0, a^n) \setminus K)}{a^n} \le A_T \sum_{i=n}^{\infty} \frac{\gamma(\{\lambda : a^{i+1} \le |\lambda - \lambda_0| < a^i\} \setminus K)}{a^i} \to 0.$$

This implies, from Corollary 1, that λ_0 is a bpe for $P^t(d\mu_K)$.

Corollary 1 seems to suggest that there might exist bpes for $P^t(d\mu_K)$ with $R(K) = C(K)$. Our main theorem in the paper proves this affirmatively.

Main Theorem. *There exists a compact subset K of the closed unit disk $\bar{\mathbb{D}}$ such that $R(K) = C(K)$ and*

$$\int_{\mathbb{D}} |p|^t dA \le C \int_K |p|^t dA,$$

for $1 \le t < \infty$, $C = C(t) > 0$, and all polynomials p. Therefore, $\mathbb{D}$ is the set of bounded point evaluations for $P^t(d\mu_K)$.

2. Proof of the Main Theorem

We fix a compact subset K_0 of the closed unit square such that $R(K_0) = C(K_0)$ and $Area(K_0) = a,\ 0 < a < 1$. In fact, we can construct a planar Cantor set K_0 as follows. Given a sequence $\{\lambda_n\}$ with $0 < \lambda_n < \frac{1}{2}$, let $Q_0 = [0,1] \times [0,1]$. At the first step we take four closed squares inside Q_0, with side length λ_1, with sides parallel to the coordinate axes, and so that each square contains a vertex of Q_0. At the second step we apply the preceding procedure to each of the four squares obtained in the first step, but now using the proportion factor λ_2. In this way, we get 16 squares of side length $\sigma_2 = \lambda_1\lambda_2$. Proceeding inductively, at each step we obtain 4^n squares $Q_j^n,\ j = 1, 2, \dots, 4^n$, with side length $\sigma_n = \lambda_1\lambda_2 \dots \lambda_n$. Now let

$$L_n = \bigcup_{j=1}^{4^n} Q_j^n,\ K_0 = \bigcap_{n=1}^{\infty} L_n,$$

and

$$\lambda_n = \frac{1}{2} a^{\frac{1}{2^{n+1}}}.$$

Then

$$Area(K_0) = \lim_{n\to\infty} 4^n \sigma_n^2 = a,$$

and

$$\limsup_{r\to 0} \frac{\gamma(D(\lambda_0, r) \setminus K_0)}{r} > 0$$

for each $\lambda_0 \in K_0$. Therefore, it follows from Melnikov's Theorem that each point in K_0 is a peak point for $R(K_0)$ and $R(K_0) = C(K_0)$. Let $K(u) = uK_0,\ 0 \le u \le 1$.

Now we use the Thomson [5] coloring scheme argument to construct a closed square. Let us start with $R_0 = [-\frac{r}{2}, \frac{r}{2}] \times [-\frac{r}{2}, \frac{r}{2}]$. Divide R_0 into 16 equal squares and let $S_0 = \{R_j^0,\ j = 1, 2, \dots, 12\}$ be the collection of small squares that have at least one common side with R_0. We construct a square R_1 with the same center and parallel to R_0. The length of R_1 is

$$r_1 = r + 2 \times 2^2 \times \frac{1}{2^2} r + 2 \times \frac{1}{2^3} r = \frac{13}{4} r.$$

We divide R_1 into $26 \times 26 = 676$ small squares with length $\frac{1}{8}r$. Let $S_1 = \{R_j^1,\ j = 1, 2, \dots, 26\}$ be the collection of small squares that have at least one common side with R_1. We continue to construct $R_{n+1},\ r_{n+1},\ S_{n+1}$ as follows. The square R_{n+1} is constructed with the same center and parallel to R_n. The length of R_{n+1} is

$$r_{n+1} = r_n + 2 \times (n+2)^2 \times \frac{1}{2^{n+2}} r + 2 \times \frac{1}{2^{n+3}} r.$$

We divide R_{n+1} into $N_{n+1} \times N_{n+1},\ N_{n+1} = \frac{2^{n+3} r_{n+1}}{r}$ small squares with length $\frac{1}{2^{n+3}} r$. Let $S_{n+1} = \{R_j^{n+1},\ j = 1, 2, \ldots, N_{n+1}\}$ be the collection of small squares that have at least one common side with R_{n+1}. Let R be the limit of R_n. Then the length of the square is $\lim_{n\to\infty} r_n = 8r$. For each small square R_j^i, we use a compact subset K_j^i to replace it, where

$$K_j^i = \frac{r}{2^{j+3}}(K_0 - \text{center of } K_0) + \text{center of } R_j^i.$$

Let

$$K_r = \bigcup_{i,j} K_j^i \cup \partial R.$$

Clearly, every point in K_r is a peak point for $R(K_r)$.

Lemma 1. *There is an absolute constant $C > 0$ such that*

$$|p(0)| \le \frac{C}{r^2} \int |p| d\mu_{K_r}.$$

Proof. We modify the proofs in Section 4 of [5]. For $R \in \{K_{ij}\}$, define $\tau_w = \frac{1}{Area(R)} dA|_R$ for $w \in R$. In our case, $m = 1$ and $\Phi = \chi_R$. The function h in their Lemma 4.5 can be chosen as

$$\|h\| \le \frac{1}{Area(R)} \le \frac{2^{2n}}{ar^2}.$$

The lemma follows from Lemma 4.6 in [5].

Proof of the Main Theorem. Let $\{z_k\}$ be a sampling sequence for the Bergman space $L_a^t(\mathbb{D})$ with

$$\rho(z_i, z_j) = \left| \frac{z_i - z_j}{1 - \bar{z}_i z_j} \right| > 2\delta > 0,$$

for $i \ne j$ (see Seip [4]). Then there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |p|^t dA \le C \sum_{k=1}^{\infty} |p(z_k)|^t (1 - |z_k|^2)^2,$$

for all polynomials p. Let

$$SD(z_k, \delta) = \{z : \rho(z, z_k) < \delta\}.$$

Then the center of the disk is $\frac{1-\delta^2}{1-\delta^2|z_k|^2} z_k$, the radius is $\frac{1-|z_k|^2}{1-\delta^2|z_k|^2}\delta$, and

$$SD(z_i, \delta) \cap SD(z_j, \delta) = \emptyset,$$

for $i \ne j$. It is easy to show that

$$D(z_k, r_k) = \{z : |z - z_k| < r_k\} \subset SD(z_k, \delta),$$

where $r_k = \frac{\delta}{1+\delta}(1 - |z_k|^2)$. Let

$$J_k = \frac{1}{8}(K_{r_k} + z_k) \subset D(z_k, r_k);$$

then $J_i \cap J_j = \emptyset$ for $i \neq j$. Then

$$K = \bigcup_{k=1}^{\infty} J_k \cup \partial\mathbb{D}$$

is a compact subset of the closed unit disk, and by the construction we see that $R(K) = C(K)$. It follows from Lemma 1 that

$$\int_{\mathbb{D}} |p|^t dA \leq C \sum_{k=1}^{\infty} \frac{(1-|z_k|^2)^2}{r_k^{2t}} \left(\int_{K_{r_k}} |p| dA \right)^t \leq C \int_K |p|^t dA.$$

This completes the proof.

3. Remarks

Thomson [6] showed that the sets of bounded point evaluations for $P^t(\mu)$ may vary with the exponent t. In our case, it would be interesting to see if K in the Main Theorem can be constructed so that the sets of bounded point evaluations will vary with the exponents t.

It is not difficult to construct an example that satisfies the condition of Corollary 1 but not Theorem 1.

Example. Let S be the Swiss cheese constructed as the following. Let D_k be a sequence of disjoint open disks in the unit disk, $k = 1, 2, 3, ...$, having radii r_k in such a way that

$$S = clos(\mathbb{D}) \setminus \bigcup_{k=1}^{\infty} D_k$$

has no interior, $0 \in S$,

$$\sum_{k=1}^{\infty} r_k < \epsilon_2,$$

and there exists a subsequence

$$\{D_{k_j}\} \subset \{\frac{1}{2^{j+1}} \leq |z| < \frac{1}{2^j}\}, \; r_{k_j} > \frac{\epsilon_2}{2^{j+2}},$$

for $j = 1, 2, 3, \ldots$ and $\epsilon_2 = \frac{\epsilon_1}{2A_T}$. Then $0 \in S$ satisfies the conditions in Corollary 1 and is a peak point for $R(S)$. Therefore, $0 \in S$ does not satisfy the conditions of Theorem 1.

Proof. By the semiadditivity of analytic capacity,

$$\gamma(\bigcup_{k=1}^{\infty} D_k) \leq A_T \sum_{k=1}^{\infty} r_k < \epsilon_1.$$

It follows from Corollary 1 that 0 is a bpe for $P^t(\mu_S)$. However,

$$\sum_{i=1}^{\infty} 2^i \gamma(\{\lambda : \frac{1}{2^{i+1}} \leq |\lambda - 0| < \frac{1}{2^i}\} \setminus S) = \infty$$

implies, by Melnikov's Theorem (see [3], p. 205), that 0 is a peak point for $R(S)$.

For K constructed in our Main Theorem and every $\lambda_0 \in K$, one can prove that

$$\limsup_{r \to 0} \frac{\gamma(D(\lambda_0, r) \setminus K)}{r} > 0.$$

Therefore, the compact set K does not satisfy the conditions in Theorem 1 for each point $\lambda_0 \in K$. Corollary 1 may not apply to K. This leads to the following question.

Problem. For a compact subset K, is there a sufficient condition for $\lambda_0 \in K$ that covers both Corollary 1 and the Main Theorem so that λ_0 is a bpe for $P^t(\mu|_K),\ 1 \le t < \infty$?

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