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PROJECTIONS IN $L^1(G)$: THE UNIMODULAR CASE

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Abstract. We consider the issue of describing all self-adjoint idempotents (projections) in $L^1(G)$ when G is a unimodular locally compact group. The approach is to take advantage of known facts concerning subspaces of the Fourier-Stieltjes and Fourier algebras of G and the topology of the dual space of G. We obtain an explicit description of any projection in $L^1(G)$ which happens to also lie in the coefficient space of a finite direct sum of irreducible representations. This leads to a complete description of all projections in $L^1(G)$ for G belonging to a class of groups that includes $\mathrm{SL}_2(\mathbb{R})$ and all second countable almost connected nilpotent locally compact groups.

1. Introduction

Let G be a unimodular locally compact group, and let $L^1(G)$ denote the Banach $*$-algebra of integrable functions on G. Let $M(G)$ denote the Banach $*$-algebra of bounded regular Borel measures on G, and recall that the measure algebra $M(G)$ contains $L^1(G)$ as a closed ideal. Self-adjoint idempotents in $L^1(G)$ (respectively $M(G)$) are called L^1-projections (respectively projection measures). The study of projections originated with Rudin [20] and Helson [11]. A full characterization of idempotents of the measure algebra of a locally compact abelian group was obtained in [4] through identifying such measures with certain subsets of the dual group. Note that in the abelian case, idempotents of $M(G)$ are automatically projection measures. For nonabelian compact groups, the orthogonality relations for coefficient functions of irreducible representations show that any pure positive definite function, properly scaled, is an L^1-projection. Such a projection is strongly minimal in the sense defined later. Conversely, every strongly minimal L^1-projection is just a pure positive definite coefficient function, and is associated uniquely (up to equivalence) with a particular irreducible representation (namely the unique representation in its "support"). Moreover, every L^1-projection of a compact group is just a finite sum of such strongly minimal projections.

The support of an L^1-projection p of a locally compact group is the collection of (equivalence classes of) all irreducible unitary representations π which satisfy

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$\pi(p) \neq 0$. It turns out that an L^1-projection in a compact or an abelian locally compact group can be understood using its support. For a noncompact unimodular group G, it was shown in [2] and independently in [22] that, similar to the case of a compact group, strongly minimal L^1-projections are singly supported. That is, for a strongly minimal projection $p \in L^1(G)$, there is a unique (up to equivalence of representations) irreducible representation π of G such that $\pi(p) \neq 0$. Moreover, the representation π is an open point in the dual space of G, and p is nothing but a positive definite coefficient function of $\overline{\pi}$.

To our knowledge, the first explicit construction of projections in $L^1(G)$ for a nonunimodular G was carried out by Eymard and Terp [7] for the group of affine transformations of any locally compact field. In [10] and [14], groups of the form $G = A \rtimes H$, with A abelian, were studied. The nature of the action of H on $\widehat{A}$ determines whether or not there are nonzero projections in $L^1(G)$. Groups of the form $A \rtimes H$ are often nonunimodular and, although we know how to construct many examples of projections, we are a long way from a characterization of projections for nonunimodular groups that is comparable to that for compact or abelian groups.

In this paper, we restrict our attention to unimodular groups G with the purpose of building on the results of [2] and [22] and moving closer to a complete description of all projections in $L^1(G)$. In particular, we study projections with finite support in detail and show that, for many groups (precisely the unimodular, second countable, type I groups), the finite support of a projection identifies the smallest coefficient function space which contains the projection. This article provides partial generalizations to some earlier results about projections of certain unimodular groups. For G a connected nilpotent group, all projections in $L^1(G)$ are explicitly described in [13]. Some more headway was made in [15] for [FC$^-$]-groups; that is, groups for which every conjugacy class is relatively compact. Note that nilpotent and [FC$^-$]-groups are unimodular.

This article is organized as follows. We collate the necessary background and tools in Section 2. In Section 3, we prove that every L^1-projection can be represented by an element of the Fourier algebra. We then study projections that lie in certain subspaces of the Fourier algebra, namely coefficient spaces associated with finite sums of irreducible representations. In Theorem 3.4, we show that every such projection is of a rather special form, i.e. it is just a finite sum of coefficient functions, where each summand is a strongly minimal L^1-projection in its own right. This in particular proves that every one of the irreducible representations has to be "integrable". We use the results of [2], together with careful study of coefficient function spaces of irreducible representations, to prove this theorem. (It is worth mentioning that we know of no direct way to answer even a very simple version of this question, namely when the projection is assumed to be just the sum of coefficient functions of two inequivalent irreducible representations.) In Section 4, we study projections through their support, and show that in special cases, the support of the projection identifies its location in the Fourier algebra.

As perhaps the most useful consequence of this study, Corollary 4.4 provides a complete description of all projections in $L^1(G)$ when G is a unimodular, second countable, type I locally compact group with the property that every compact open subset of the dual of G is a finite subset of the reduced dual. This class of groups includes $\mathrm{SL}_2(\mathbb{R})$ and any almost connected nilpotent group.

We finish the paper with an application of projections to $*$-homomorphisms between L^1-algebras of (unimodular) locally compact groups.

2. Notation and background

For the rest of this paper, G is a unimodular locally compact group unless otherwise stated. The space of equivalence classes of irreducible unitary representations of G is denoted by $\widehat{G}$. There is a natural topology on $\widehat{G}$ that is not, in general, Hausdorff (see [16]). Whenever we refer to a representation of G, we mean a weakly continuous unitary representation. The support of a representation π, denoted by $\mathrm{supp}(\pi)$, is the set of all representations in $\widehat{G}$ which are weakly contained in π. For the left regular representation λ of G, the support of λ is called the *reduced dual* of G and is denoted by $\widehat{G}_r$. For detailed accounts of the representation theory of locally compact groups see [5] and [8].

For a representation π on G and $\xi, \eta \in \mathcal{H}_\pi$, we define the corresponding *coefficient function* to be the function $\pi_{\xi,\eta}(x) := \langle \pi(x)\xi, \eta\rangle$, for $x \in G$. Let F_π denote the linear span of $\{\pi_{\xi,\eta} : \xi, \eta \in \mathcal{H}_\pi\}$. Then F_π is a subspace of the space of bounded continuous complex-valued functions on G.

An irreducible representation π is called *integrable* if there exists $\xi \in \mathcal{H}_\pi$, $\xi \neq 0$, such that $\pi_{\xi,\xi} \in L^1(G)$. This is equivalent to the existence of a dense subspace $\mathcal{H}'$ of $\mathcal{H}_\pi$ such that for all $\xi, \eta \in \mathcal{H}'$, the coefficient function $\pi_{\xi,\eta}$ belongs to $L^1(G)$. An irreducible representation π of G is said to be *square-integrable* if there exist nonzero vectors $\xi, \eta \in \mathcal{H}_\pi$ such that $\pi_{\xi,\eta} \in L^2(G)$. Note that every integrable representation is square-integrable but the converse is not true. When π is a square-integrable representation of a unimodular group, every coefficient function of π is square-integrable. See Chapter 14 of [5] for the basic properties of square-integrable and integrable representations of unimodular groups.

Square-integrable representations satisfy orthogonality relations similar to the ones held for coefficient functions of irreducible representations of compact groups. In particular, let $\sigma = \bigoplus_{i=1}^n \pi_i$ for mutually nonequivalent square-integrable representations $(\pi_i)_{i=1}^n$, and let $\xi_i, \xi_i', \eta_i, \eta_i' \in \mathcal{H}_{\pi_i}$ for $i \in 1, \ldots, n$. Then for $\xi = \bigoplus_{i=1}^n \xi_i$, $\xi' = \bigoplus_{i=1}^n \xi_i'$, $\eta = \bigoplus_{i=1}^n \eta_i$, and $\eta' = \bigoplus_{i=1}^n \eta_i'$ we have

$$
\begin{aligned}
(1) \qquad \int_G \langle \xi, \sigma(x)\eta\rangle \overline{\langle \xi', \sigma(x)\eta'\rangle} dx &= \sum_{i,j=1}^n \int_G \langle \xi_i, \pi_i(x)\eta_i\rangle \overline{\langle \xi_j', \pi_j(x)\eta_j'\rangle} dx \\
&= \sum_{i=1}^n \frac{1}{k_i} \langle \xi_i, \xi_i'\rangle \langle \eta_i', \eta_i\rangle,
\end{aligned}
$$

where each positive real quantity k_i is called the *formal dimension* of π_i. For such a representation σ, F_σ has additional structure. With ξ, η, ξ', η' as above,

$$
(2) \qquad \sigma_{\xi,\eta} * \sigma_{\xi',\eta'} = \sum_{i=1}^n \frac{1}{k_i} \langle \xi_i, \eta_i'\rangle \pi_{i,\xi_i',\eta_i} \in F_\sigma,
$$

where $\pi_{i,\xi_i,\eta_i} = \langle \pi_i(\cdot)\xi_i, \eta_i\rangle$. Using (2), we observe that if G is unimodular and σ is a direct sum of finitely many square-integrable representations, then F_σ forms a $*$-algebra, where $*$ is the involution.

For a locally compact group G, let $B(G)$ denote the set of all coefficient functions generated by representations of G. Eymard first introduced $B(G)$ for a general locally compact group in [6]. Clearly, $B(G)$ is an algebra with respect to the pointwise operations. Eymard showed that $B(G)$ is in fact a Banach algebra with the norm defined as follows. For each $u \in B(G)$, $\|u\|_{B(G)} := \inf \|\xi\|\|\eta\|$ where the infimum is taken over all possibilities of representations σ of G and $\xi, \eta \in \mathcal{H}_\sigma$ with $u(x) = \langle \sigma(x)\xi, \eta \rangle$. The Banach algebra $B(G)$ is called the *Fourier-Stieltjes algebra* of the group G. Further, $B(G)$ is invariant with respect to left and right translations by elements of G.

For a representation σ of G, F_σ is a subspace of $B(G)$ which is not necessarily closed. The closure of F_σ with respect to the norm of $B(G)$ is denoted by $A_\sigma(G)$, or A_σ when there is no risk of confusion. These subspaces were defined and studied by Arsac in [1] where it was shown that

$$A_\sigma = \left\{ \sum_{j=1}^{\infty} \sigma_{\xi_j, \eta_j} : \xi_j, \eta_j \in \mathcal{H}_\sigma, \sum_{j=1}^{\infty} \|\xi_j\|\|\eta_i\| < \infty \right\}.$$

In addition, every u in A_σ can be represented as $u = \sum_{i=1}^{\infty} \sigma_{\xi_i, \eta_i}$ in such a way that

$$\|u\|_{B(G)} = \sum_{j=1}^{\infty} \|\xi_j\|\|\eta_j\|.$$

Moreover, the subspace A_σ can be realized as a quotient of the trace class operator algebra of $\mathcal{H}_\sigma$, $\mathcal{T}(\mathcal{H}_\sigma)$, through the map ψ defined as

$$\psi : \mathcal{T}(\mathcal{H}_\sigma) \longrightarrow A_\sigma, \quad \psi(T)(x) = \mathrm{Tr}(T\sigma(x)),$$

for every $T \in \mathcal{T}(\mathcal{H}_\sigma)$ and $x \in G$. In the special case where σ is irreducible, the above map defines an isometry. In particular, we conclude that $\|\sigma_{\xi,\eta}\|_{B(G)} = \|\xi\|\|\eta\|$. In our computations, we will use the following proposition which is merely a weaker version of Corollaire (3.13) of [1].

Proposition 2.1. *Let $\sigma = \bigoplus_{i \in I} \pi_i$, where $\{\pi_i : i \in I\}$ is a collection of nonequivalent irreducible representations of G. Then $A_\sigma = \ell^1\text{-}\bigoplus_{i \in I} A_{\pi_i}$.*

If λ denotes the left regular representation of G, A_λ turns out to be a closed ideal of $B(G)$, and is simply denoted by $A(G)$. The algebra $A(G)$, called the *Fourier algebra* of G, was also introduced by Eymard in [6]. In particular, Eymard proved that each element of $A(G)$ can be written in the form of a coefficient function of λ; that is, $\lambda_{f,g}$ for some $f, g \in L^2(G)$.

An element p in $L^1(G)$ is called a *projection* if $p * p = p = p^*$; that is, if p is a self-adjoint idempotent in $L^1(G)$. Let $\mathcal{P}L^1(G)$ denote the set of projections in $L^1(G)$.

For $p \in \mathcal{P}L^1(G)$, define the *support* of p to be $S(p) := \{\pi \in \widehat{G} : \pi(p) \neq 0\}$. For any $p \in \mathcal{P}L^1(G)$, $S(p)$ is a compact open subset of $\widehat{G}$ (see 3.3.2 and 3.3.7 of [5]), but $S(p)$ is not necessarily closed (see [13, Example 2]). Thus, if $\widehat{G}$ has no nonempty compact open subsets, then $\mathcal{P}L^1(G) = \{0\}$.

The set $\mathcal{P}L^1(G)$ carries a partial order $\leq$ that is $q \leq p$ if $q * p = q$ (or equivalently $p * q = q$) for $p, q \in \mathcal{P}L^1(G)$. A nonzero $p \in \mathcal{P}L^1(G)$ is called a *minimal projection* if for any other $q \in \mathcal{P}L^1(G)$, $q \leq p$ implies that q is either p or 0. A projection p in $L^1(G)$ is called *strongly minimal* if the left ideal $L^1(G) * p$ is a minimal left ideal in $L^1(G)$. Equivalently, for a strongly minimal projection p,

$$p * f * p = \alpha_f p, \tag{3}$$

where $\alpha_f \in \mathbb{C}$, for every $f \in L^1(G)$. It is clear that strong minimality implies minimality of a projection. But the following example shows that the converse is not true.

Example 2.2. Let G be a locally compact abelian group with a nontrivial compact open subgroup K such that G/K is infinite and torsion free ($\mathbb{T} \times \mathbb{Z}$ is such a group). Normalize Haar measure on G so that the measure of K is 1 and let $p = 1_K$, the characteristic function of K. Noting that $\widehat{p}$ is the characteristic function of the connected set $\widehat{G/K}$ in $\widehat{G}$, one sees that p is minimal but not strongly minimal.

Strongly minimal projections were studied in [2] and [22] where they were called "minimal". Since in this article we study two types of minimality for projections, namely minimal and strongly minimal, we use different terminology. It has been shown that, for unimodular groups, there is a one-to-one correspondence between the set of equivalence classes of integrable representations and strongly minimal projections. We will clarify the relation between a strongly minimal projection and the corresponding irreducible representation, for unimodular groups, in Section 3.

3. Main results

Our objective for this paper is to study L^1-projections of unimodular groups. Our motivation is the result of Barnes in [2] which states that every strongly minimal L^1-projection of a unimodular group is a coefficient function of an integrable representation. In particular, strongly minimal projections of $L^1(G)$ lie in some A_π with π integrable (which implies $A_\pi \subseteq A(G)$).

We begin this section with an analogous key observation on idempotents in $L^1(G)$. Note that unimodularity of G guarantees that many of the significant dense left ideals of $L^1(G)$ are (two-sided) ideals. Let us recall that an element p of an algebra $\mathcal{A}$ is called an idempotent if $p^2 = p$. The following proposition was formerly proved in [17, Theorem 8] in a more general setting. We present the proof here to be self-contained.

Proposition 3.1. *Let G be a unimodular locally compact group and let p be an idempotent in $L^1(G)$. Then, $p \in A(G) \cap L^r(G)$ for every $1 < r < \infty$.*

Proof. Suppose p is a nonzero idempotent in $L^1(G)$. Let $\mathcal{J}$ be any one of the ideals $A(G) \cap L^1(G)$ or $L^1(G) \cap L^r(G)$ for some $1 \leq r < \infty$. Since $\mathcal{J}$ is dense in $L^1(G)$, there is some $u \in \mathcal{J}$ such that $\|u - p\|_1 < \|p\|_1^{-1}$. Define $b := \sum_{n=1}^{\infty} (p - u * p)^{*n} \in L^1(G)$, where $*n$ denotes the n-fold convolution. Note that $b * p = b$. Moreover, $b * (p - u * p) = b - (p - u * p)$. Therefore, $u * p + (b - b * (p - u * p)) = p$. On the other hand, $u * p + (b - b * (p - u * p)) = u * p + b * u * p \in \mathcal{J}$. This implies that $p \in \mathcal{J}$. $\square$

Note that the proof of Proposition 3.1 does not work for nonunimodular locally compact groups, as having two-sided ideals is essential for the proof.

The following gives most of [2, Theorem 1], but from a perspective more suitable to our needs.

Proposition 3.2. *Let G be a unimodular locally compact group and p a projection in $L^1(G)$.*

(i) *Let $\pi = \lambda(\cdot)|_{L^2(G) * \overline{p}}$. Then $p \in A_\pi$.*

(ii) *If p is strongly minimal, then π is irreducible and integrable. Further, $p = \pi_{\overline{p},\overline{p}}$.*

Proof. Since G is unimodular, and $p^* = p$, we have that $\check{p}$ is equal to $\overline{p}$ a.e. where $\check{p}(s) := p(s^{-1})$ and $\overline{p}(s) := \overline{p(s)}$. Proposition 3.1 tells us that $p \in A(G) \cap L^2(G)$. Hence we have

$$p = p * p = p * \check{\overline{p}} = \langle p, \lambda(\cdot)p\rangle = \langle \lambda(\cdot)\overline{p}, \overline{p}\rangle,$$

which gives (i).

Let p be a strongly minimal projection. Let u be the element in $L^\infty(G)$ which is associated with the linear functional $f \mapsto \alpha_f$ defined in (3), i.e.

$$p * f * p = (\int_G uf)\, p \qquad (\text{for } f \in L^1(G)). \tag{4}$$

Notice that $\int_G u(p*f) = \int_G (\check{p} * u) f = \int_G (\overline{p} * u) f$. So for every $f \in L^1(G)$,

$$(\int_G uf)\, p = p * f * p = p * (p * f) * p = (\int_G u(p*f))\, p = (\int_G (\overline{p} * u) f)\, p,$$

which implies that $\overline{p} * u = u$. Likewise $u * \overline{p} = u$. Now if (u_ι) is a net from $C_c(G)$ which is weak* convergent to u, then we have

$$u = \overline{p} * u * \overline{p} = \text{w*-}\lim_\iota \overline{p} * u_\iota * \overline{p} = \lim_\iota \left(\int_G u_\iota \overline{u}\right) \overline{p}.$$

In particular, $\alpha := \lim_\iota \left(\int_G u_\iota \overline{u}\right)$ exists and $u = \alpha\overline{p}$. But $p = p * p * p = \alpha(\int_G p\overline{p})p$, so $\alpha = \|p\|_2^{-2}$.

Now we follow a procedure from [2]. If σ is any representation for which $\sigma(\overline{p}) \neq 0$, find ξ in $\mathcal{H}_\sigma$ such that $\sigma(\overline{p})\xi = \xi$ and $\|\xi\|^2 = \alpha^{-1}$. Interchanging roles of p and $\overline{p}$ in (4), we have for f in $L^1(G)$ that

$$\|\sigma(f)\xi\|^2 = \langle \sigma(\overline{p} * f^* * f * \overline{p})\xi, \xi\rangle = \alpha\|\xi\|^2 \int_G (f^* * f)\, p = \langle f^* * f * \overline{p}, \overline{p}\rangle = \|f * \overline{p}\|_2^2,$$

where the fact that $p = \pi_{\overline{p},\overline{p}}$ was used in the penultimate equality. Hence $U : L^1(G) * \overline{p} \to \mathcal{H}_\sigma$ given by $U(f * \overline{p}) = \sigma(f)\xi$ extends to an isometry from $L^2(G) * \overline{p}$ to $\mathcal{H}_\sigma$ which intertwines π and σ. In particular, with choice of irreducible σ, we see that π is necessarily irreducible as well. Since $p = \pi_{\overline{p},\overline{p}}$ is integrable, π is an integrable representation. □

The following remark gives the converse to (ii), above.

Remark. Let G be a unimodular locally compact group and let π be an integrable irreducible representation of G. Then there is a dense subspace $\mathcal{H}'_\pi$ of $\mathcal{H}_\pi$ consisting of elements ξ for which $\pi_{\xi,\xi}$ is a multiple of a projection p_ξ in $L^1(G)$. Furthermore, by the calculation of [22, Lemma 2.2], each p_ξ is strongly minimal.

The preceding propositions give a description of strongly minimal projections. In what follows, we will study projections in $L^1(G)$ given that they belong to certain subspaces of the Fourier algebra. We begin with the following lemma which says that, similar to the compact case, A_π for a square-integrable representation π, is a Banach $*$-algebra with respect to convolution.

Lemma 3.3. *Let $\sigma = \bigoplus_{i=1}^n \pi_i$ for square-integrable representations π_i of a unimodular locally compact group G. Then $A_\sigma \subseteq L^2(G)$ and $\sqrt{k_\sigma}\|\cdot\|_2 \leq \|\cdot\|_{B(G)}$*

on A_σ where $k_\sigma = \min\{k_{\pi_i} : i = 1, \ldots, n\}$. Furthermore, $(A_\sigma, k_\sigma^{-1}\|\cdot\|_{B(G)})$ is a Banach $*$-algebra when it is equipped with convolution.

Proof. By orthogonality relations stated in (1), for each $\xi = \bigoplus_{i=1}^n \xi_i, \eta = \bigoplus_{i=1}^n \eta_i \in \bigoplus_{i=1}^n \mathcal{H}_{\pi_i}$, $\sigma_{\xi,\eta} \in L^2(G)$,

$$\|\sigma_{\xi,\eta}\|_2^2 = \sum_{i=1}^n \frac{1}{k_{\pi_i}}\|\xi_i\|^2\|\eta_i\|^2 \leq \frac{1}{k_\sigma}\left(\sum_{i=1}^n \|\xi_i\|\|\eta_i\|\right)^2 \leq \frac{1}{k_\sigma}\|\sigma_{\xi,\eta}\|_{B(G)}^2,$$

where we used Proposition 2.1 in the last inequality. Let $u = \sum_{k=1}^\infty \sigma_{\xi_k,\eta_k} \in A_\sigma$ be represented such that $\|u\|_{B(G)} = \sum_{k=1}^\infty \|\xi_k\|\|\eta_k\|$. Then,

$$\|u\|_2 \leq \sum_{k=1}^\infty \|\sigma_{\xi_k,\eta_k}\|_2 \leq \sum_{k=1}^\infty \frac{1}{\sqrt{k_\sigma}}\|\sigma_{\xi_k,\eta_k}\|_{B(G)} \leq \frac{1}{\sqrt{k_\sigma}}\sum_{k=1}^\infty \|\xi_k\|\|\eta_k\| = \frac{1}{\sqrt{k_\sigma}}\|u\|_{B(G)}.$$

Therefore, $A_\sigma \subseteq L^2(G)$. Moreover, since G is unimodular, for every $u, v \in A_\sigma$, their convolution is defined, and $u * v = \langle \lambda(\cdot)\check{v}, \overline{u}\rangle$. Thus,

$$\|u * v\|_{B(G)} \leq \|\overline{u}\|_2\|\check{v}\|_2 \leq k_\sigma^{-1}\|u\|_{B(G)}\|v\|_{B(G)}.$$

This completes the proof. □

The following is a partial generalization of [13, Theorem 3], where conditions on the set $S(p)$ were assumed. We take the perspective of assuming p itself consists of certain types of matrix coefficients.

Theorem 3.4. *Let G be a unimodular locally compact group. Let $\pi_1, \ldots, \pi_n$ be a family of pairwise inequivalent members of $\widehat{G}$ and let $\sigma = \bigoplus_{i=1}^n \pi_i$. If p in $L^1(G) \cap A_\sigma$ is a projection which belongs to no $A_{\sigma'}$ for any proper subrepresentation σ' of σ, then*

(i) *$p = \sum_{i=1}^n p_i$ where each p_i is a projection in $L^1(G) \cap A_{\pi_i}$ and $p_i * p_{i'} = 0$ for $i \neq i'$;*
(ii) *each $p_i = \sum_{j=1}^{r_i} p_{ij}$ where each p_{ij} is strongly minimal and $p_{ij} * p_{ij'} = 0$ if $j \neq j'$;*
(iii) *each π_i is integrable; and*
(iv) *$S(p) = \{\overline{\pi_1}, \ldots, \overline{\pi_n}\}$.*

Proof. First note that $A_\sigma \cap A(G) \neq \{0\}$, since $p \in L^1(G) \cap A(G)$ by Proposition 3.1. Thanks to [1, (3.12)] there is a subrepresentation σ' of σ for which $A_{\sigma'} = A_\sigma \cap A(G)$, but then $p \in A_{\sigma'}$, and our assumptions ensure that $\sigma' = \sigma$. In particular, for each i, $A_{\pi_i} \subseteq A_\sigma \subseteq A(G)$, and hence by [5, 14.3.1], each π_i is square-integrable. Thus, by Lemma 3.3, $(A_\sigma, k_\sigma^{-1}\|\cdot\|_{B(G)})$ is an involutive Banach algebra when equipped with convolution.

Recall that $A_\sigma^* \cong VN_\sigma = \ell^\infty\text{-}\bigoplus_{i=1}^n \mathcal{B}(\mathcal{H}_{\pi_i})$. We have the usual duality $\mathcal{T}(\overline{\mathcal{H}})^* \cong \mathcal{B}(\mathcal{H})$ given by $(\overline{\eta} \otimes \overline{\xi}^*, T) \mapsto \mathrm{Tr}(\xi \otimes \eta^* T) = \langle T\xi, \eta\rangle$, where $\overline{\eta} \otimes \overline{\xi}^*$ is the rank-one operator on $\overline{\mathcal{H}}$ given by $\overline{\zeta} \mapsto \langle \xi, \zeta\rangle\overline{\eta}$. Combining these facts gives us an isometric Banach space isomorphism

$$\phi : A_\sigma \to \mathcal{T} = \ell^1\text{-}\bigoplus_{i=1}^n \mathcal{T}(\overline{\mathcal{H}}_{\pi_i}), \quad \sum_{i=1}^n\sum_{j=1}^\infty \pi_{i,\xi_{ij},\eta_{ij}} \mapsto \sum_{i=1}^n\sum_{j=1}^\infty \overline{\eta}_{ij} \otimes \overline{\xi}_{ij}^*,$$

where $\pi_{i,\xi_{ij},\eta_{ij}} = \langle \pi_i(\cdot)\xi_{ij}, \eta_{ij}\rangle$. Consider the new mapping Φ as follows:

$$\Phi : (A_\sigma, k_\sigma^{-1}\|\cdot\|_{B(G)}) \to \mathcal{T}, \quad \sum_{i=1}^{n}\sum_{j=1}^{\infty} \pi_{i,\xi_{ij},\eta_{ij}} \mapsto \sum_{i=1}^{n} \frac{1}{k_{\pi_i}} \sum_{j=1}^{\infty} \overline{\eta}_{ij} \otimes \overline{\xi}_{ij}^{*},$$

where k_σ is the constant from the preceding lemma. It is straightforward to check that Φ is a continuous bijective algebra homomorphism when the domain is endowed with the convolution product. Indeed, one checks that Φ is a homomorphism on the dense subspace F_σ of finite sums of matrix coefficients of σ, and observes that $\Phi(F_\sigma)$ is the space of finite rank operators, which is dense in $\mathcal{T}$.

Now we let $\mathcal{A} = p * L^1(G) * p$. Since $L^1(G) \cap A_\sigma$ is a left ideal in $L^1(G)$, $\mathcal{A}$ is an involutive convolution algebra with unit p, and $\mathcal{A} \subseteq A_\sigma$. Hence $\Phi(\mathcal{A})$ is a unital involutive subalgebra of $\mathcal{T}$, whence of the algebra of compact operators $\mathcal{K} = c_0\text{-}\bigoplus_{i=1}^{n} \mathcal{K}(\mathcal{H}_{\pi_i})$. So if $P = \Phi(p)$, P is a compact idempotent and thus it is finite dimensional. Moreover, we have that $\Phi(\mathcal{A}) \subset P\mathcal{K}P$ which is a finite dimensional $*$-algebra, hence semisimple and thus, by Wedderburn's theorem, isomorphic to a finite direct sum of full matrix algebras $\bigoplus_{j=1}^{N} M_{r_j}(\mathbb{C})$. Let $\{E_{kl}^j\}_{k,l=1}^{r_j}$ be the matrix units of $M_{r_j}(\mathbb{C})$. Note that for each $j \in 1,\ldots,N$ and $k \in 1,\ldots,r_j$, E_{kk}^j is a strongly minimal projection in $\bigoplus_{i=1}^{N} M_{r_i}(\mathbb{C})$, as $E_{kk}^j(\bigoplus_{i=1}^{N} M_{r_i}(\mathbb{C}))E_{kk}^j = \mathbb{C}E_{kk}^j$. Hence $p_{jk} := \Phi^{-1}(E_{kk}^j)$ is a strongly minimal projection in $\mathcal{A}$ with $p_{jk} \le p$, and subsequently for each $f \in L^1(G)$ we have

$$p_{jk} * f * p_{jk} = p_{jk} * (p * f * p) * p_{jk} \in \mathcal{A}.$$

We now appeal to Proposition 3.2, and observe that p_{jk} must be a coefficient function of an integrable representation. On the other hand $p_{jk} \in A_\sigma$, and therefore p_{jk} is a coefficient function of π_i for some i. In particular, π_i is integrable.

In the remaining, we prove that $N = n$. In what follows, we assume that π_i belongs to $\{\pi_1,\ldots,\pi_n\}$. We show the following two facts:

(a) If $k \ne k'$ and $p_{jk} \in A_{\pi_i}$ then $p_{jk'} \in A_{\pi_i}$.

(b) If $j \ne j'$, then p_{jk} and $p_{j'l}$ do not belong to the same A_{π_i}.

To prove (a), fix j and $k \ne k'$. Towards a contradiction, assume that p_{jk} and $p_{jk'}$ are coefficient functions of representations π_i and $\pi_{i'}$ respectively, with $i \ne i'$. Since $A_{\pi_i} * A_{\pi_{i'}} = \{0\}$, for every $f \in L^1(G)$, we have $p_{jk} * f * p_{jk'} = 0$ as $f * p_{jk'}$ still belongs to $A_{\pi_{i'}}$. But this is a contradiction, as for $f = \Phi^{-1}(E_{kk'}^j)$, we get $p_{jk} * f * p_{jk'} = \Phi^{-1}(E_{kk}^j E_{kk'}^j E_{k'k'}^j) = f \ne 0$.

For (b), recall that for each $f \in L^1(G)$, $p_{jk} * f * p_{j'l} = p_{jk} * (p * f * p) * p_{j'l} = 0$, since Φ is a homomorphism. Now towards a contradiction, suppose that $p_{jk} = \pi_{i,\xi,\xi}$ and $p_{j'l} = \pi_{i,\eta,\eta}$ for some $\xi, \eta \in \mathcal{H}_{\pi_i}$. Since π_i is irreducible, there is some $g \in L^1(G)$ such that $\langle \pi_i(\overline{g})\eta, \xi\rangle \ne 0$. Therefore,

$$p_{jk} * g * p_{j'l} = \pi_{i,\xi,\xi} * \pi_{i,\eta,\pi_i(\overline{g})\eta} = \langle \xi, \pi_i(\overline{g})\eta\rangle \pi_{i,\eta,\xi} \ne 0,$$

which is a contradiction. So with $p_i := \sum_{j=1}^{r_i} p_{ij}$, properties (i) and (ii) hold.

To prove (iv), note that by Lemma 1.1 of [22], the support of a strongly minimal projection is a singleton. In fact, for a nonzero L^1-projection of the form $\pi_{\xi,\xi}$, we have $S(\pi_{\xi,\xi}) = \{\overline{\pi}\}$, since $\langle \overline{\pi}(\pi_{\xi,\xi})\overline{\xi}, \overline{\xi}\rangle = \|\pi_{\xi,\xi}\|_2^2 > 0$. Thus, for every i, $S(p_i) = \{\overline{\pi_i}\}$, since p_i is a finite sum of strongly minimal projections, each of which is a coefficient function of π_i. This fact, together with the orthogonality relations for square-integrable representations, implies that $S(p) = \{\overline{\pi_1},\ldots,\overline{\pi_n}\}$. $\square$

Corollary 3.5. *Let π be an irreducible representation of a unimodular locally compact group G, and let $p \in A_\pi$ be an L^1-projection. Then p is a minimal projection if and only if it is strongly minimal.*

Proof. By Theorem 3.4, a projection p in A_π can be written as a finite sum of strongly minimal projections of the form $\pi_{\xi,\xi}$. Therefore, p is minimal if and only if it is strongly minimal. □

4. Support of projections

Recall that the support of an L^1-projection p, denoted by $S(p)$, is the collection of all (equivalence classes of) irreducible representations π of G such that $\pi(p) \neq 0$. In this section, we show that the support sheds some light on the structure of the projection itself. This is evident in the abelian case, where the Fourier transform of a projection is just the characteristic function of the conjugate of its support. Recall that the support of a projection is always open and compact in the Fell topology of the dual. For compact groups, the support of a projection is the finite set of irreducible representations which are used to construct the projection. We study similar cases (projections with finite support) for general unimodular groups in more detail.

We start this section by a general observation linking the support of a projection and the support of its GNS representation.

Proposition 4.1. *Let G be unimodular, second countable, and type I, and let p be a projection in $L^1(G)$. Then $S(p) \cap \widehat{G}_r$ is dense in* $\mathrm{supp}\overline{\pi}_p$.

Proof. We have the following Plancherel picture of the left and right regular representations (see [8, Section 7.5]). There is a Borel subset B of $\widehat{G}$ which is dense in $\widehat{G}_r$ and a measure μ on $\widehat{G}$ which is carried by B for which we have unitary equivalences

$$\lambda \cong \int_B^{\oplus} I \otimes \overline{\pi}\, d\mu(\pi) \text{ and } \rho \cong \int_B^{\oplus} \pi \otimes I\, d\mu(\pi)$$

on

$$L^2(G) \cong \int_B^{\oplus} \mathcal{H}_\pi \otimes \overline{\mathcal{H}}_\pi\, d\mu(\pi).$$

The reader may refer to [1, 5, 8] for the theory of direct integrals of representations. Note that the aforementioned presentation is slightly different from (but equivalent to) the one in [5, 18.8.1]. Proposition 3.2 shows that the representation $\pi_p = \lambda(\cdot)|_{L^2(G)*\overline{p}}$ on the Hilbert space $L^2(G) * \overline{p}$ with the cyclic vector $\overline{p}$ is the Gelfand-Naimark cyclic representation of the positive-type element p. Observe, then, that

$$\rho(\overline{p}) = \int_B^{\oplus} \pi(\overline{p}) \otimes I\, d\mu(\pi),$$

so

$$\begin{aligned} L^2(G) * \overline{p} = \rho(\overline{p})L^2(G) &= \int_B^{\oplus} \pi(\overline{p})\mathcal{H}_\pi \otimes \overline{\mathcal{H}}_\pi\, d\mu(\pi) \\ &= \int_{\{\pi\in B:\pi(\overline{p})\neq 0\}}^{\oplus} \pi(\overline{p})\mathcal{H}_\pi \otimes \overline{\mathcal{H}}_\pi\, d\mu(\pi). \end{aligned} \tag{5}$$

By [5, 8.6.8 and 8.6.9], it follows that

$$\mathrm{supp}\pi_p = \mathrm{cl}\{\pi \in B : \pi(\overline{p}) \neq 0\},$$

where we have used cl$S$ to denote the closure of S, so as not to conflict with notation of conjugation. It is clear that $S(\overline{p}) \cap \widehat{G}_r$ contains $\{\pi \in B : \pi(\overline{p}) \neq 0\}$, while also that $(S(\overline{p}) \cap \widehat{G}_r) \cap (\widehat{G}_r \setminus \mathrm{cl}\{\pi \in B : \pi(\overline{p}) \neq 0\}) = \emptyset$. Interchanging p and $\overline{p}$, we obtain the desired result. $\square$

The following example shows that for a totally disconnected algebraic group, the support of an L^1-projection does not necessarily lie in the reduced dual. However, we know of no connected, unimodular, second countable, and type I group G for which the support of an L^1-projection does not lie in $\widehat{G}_r$.

Example 4.2. Let $G = \mathrm{SL}_n(\mathbb{Q}_p)$ for $(n \geq 2)$. Then G is type I, as it is a reductive p-adic group (see [3]). Note that G has an open compact subgroup $K = \mathrm{SL}_n(\mathbb{O}_p)$. Consider the projection 1_K in $L^1(G)$, and note that for the trivial character $\mathbf{1}$ on G, $\mathbf{1}(1_K) \neq 0$. But $\mathbf{1} \notin \widehat{G}_r$, as G is not amenable.

Proposition 4.1 tells us that

$$\pi_p = \int_{B \cap S(\overline{p})}^{\oplus} I \otimes \overline{\pi} d\mu(\pi).$$

In a particular case, when $S(p) \cap \widehat{G}_r$ is finite, we can describe the projection as in the following theorem.

Theorem 4.3. *Let G be unimodular, second countable, and type I, and let p be a projection in $L^1(G)$. If $S(p) \cap \widehat{G}_r = \{\pi_1, \ldots, \pi_n\}$, then $p \in A_\sigma$ where $\sigma = \bigoplus_{i=1}^n \overline{\pi_i}$, and $S(p) = \{\pi_1, \ldots, \pi_n\}$.*

Proof. Note that by (5), the measure representing π_p is supported on $\{\overline{\pi}_1, \ldots, \overline{\pi}_n\}$; hence, the Plancherel measure μ admits each $\overline{\pi}_i$ as an atom. Then, letting $\sigma = \bigoplus_{i=1}^n \overline{\pi}_i$, Proposition 3.2 shows that $p \in A_\sigma$. It is easy to see that for any proper subrepresentation σ' of σ, $p \notin A_{\sigma'}$. This follows from the orthogonality relations for square-integrable representations π_i and the fact that the support of p contains $\{\overline{\pi}_1, \ldots, \overline{\pi}_n\}$. $\square$

Corollary 4.4. *Let G be a unimodular, second countable, type I locally compact group with the property that every compact open subset of $\widehat{G}$ is a finite subset of $\widehat{G}_r$. Let p be a projection in $L^1(G)$. Then, there exist mutually inequivalent $\pi_1, \cdots, \pi_n \in \widehat{G}$ such that*

(i) *$p = \sum_{i=1}^n p_i$ where each p_i is a projection in $L^1(G) \cap A_{\pi_i}$ and $p_i * p_{i'} = 0$ for $i \neq i'$;*
(ii) *each $p_i = \sum_{j=1}^{r_i} p_{ij}$ where each p_{ij} is strongly minimal and $p_{ij} * p_{ij'} = 0$ if $j \neq j'$;*
(iii) *each π_i is integrable; and*
(iv) *$S(p) = \{\overline{\pi_1}, \ldots, \overline{\pi_n}\}$.*

Proof. Under these hypotheses, the compact open set $S(p)$ must be a finite subset of $\widehat{G}_r$. Now, combine Theorem 3.4 with Theorem 4.3. $\square$

Remark. In the second countable case, Corollary 4.4 generalizes [13, Theorem 4] since any almost connected nilpotent group is type I (see [18, p. 79] or [19, Section 12.6.30]) and unimodular. It also applies to $\mathrm{SL}_2(\mathbb{R})$ which is type I and unimodular. There is a clear description of the topology of $\widehat{\mathrm{SL}_2(\mathbb{R})}$ on pp. 246-248 of [8] from which one can check that any compact open subset of $\widehat{\mathrm{SL}_2(\mathbb{R})}$ is finite. No noncompact property (T) group enjoys this property; indeed, consider the trivial representation $\{\mathbf{1}\}$.

Remark. The referee has pointed out that the requirement that G be second countable in Theorem 4.3 and Corollary 4.4 may not be a major restriction because the following process can be applied in any particular case. If G is a unimodular type I locally compact group and p is a nonzero projection in $L^1(G)$, then there is a σ-compact open subgroup H of G which supports p. By Proposition 3.1, $p \in C_0(H)$ and thus is uniformly continuous. So, for each $n \in \mathbb{N}$ there is a neighborhood U_n of the identity in H so that p moves by less than $1/n$ in the supremum norm if it is right or left translated by elements of U_n. By the theorem of Kakutani-Kodaira (see [12, Section 8.7]), there is a compact normal subgroup K of H contained in all the U_n so that H/K is second countable. Then p will be constant on cosets of K. The passage from G to H/K maintains the unimodular property and the property of being type I (see [19, Section 12.6.30] again). What remains to be done is to verify the property that every compact open subset of $\widehat{G}$ is a finite subset of $\widehat{G}_r$ carries over to H/K. In any particular case, this can likely be checked and an explicit description of p obtained. However, general inheritance of the finiteness property for compact open subsets of $\widehat{G}$ remains to be explored.

5. Application to homomorphisms of group algebras

Let p be a projection in $L^1(G)$. Following [21], define the set

$$\mathbb{M}_p := \{\mu \in M(G) :\ \mu^* * \mu = \mu * \mu^* = p \text{ and } p * \mu = \mu\}.$$

We shall call this the *intrinsic unitary group at p*. Note that since $L^1(G)$ is an ideal in $M(G)$, we see, in fact, that $\mathbb{M}_p \subseteq L^1(G)$. One can equip $\mathbb{M}_p$ with the topology $\sigma(L^1(G), C_0(G))$ restricted to $\mathbb{M}_p$. With convolution product, identity p, and inverses $f^{-1} := f^*$, $\mathbb{M}_p$ is a semi-topological group with continuous inversion.

Let us make the assumption that G *is a unimodular, second countable, type I group, for which every compact open subset is a finite subset of* $\widehat{G}_r$. Then by Corollary 4.4, every L^1-projection p admits the form

$$p = \sum_{i=1}^{n} p_i \quad \text{and} \quad p_i = \sum_{j=1}^{r_i} k_{\pi_i} \pi_{i,\xi_j^{(i)},\xi_j^{(i)}}, \tag{6}$$

where $k_{\pi_i} > 0$ is the formal dimension of π_i, and $\xi_1^{(i)}, \ldots, \xi_{r_i}^{(i)}$ are unit vectors in $\mathcal{H}_{\pi_i}$. For notational convenience, we define $u \cdot p_i$, when u is a unitary matrix of size r_i, to be $u \cdot p_i = \sum_{k,\ell=1}^{r_i} u_{k,\ell} k_{\pi_i} \pi_{i,\xi_k^{(i)},\xi_\ell^{(i)}}$

Proposition 5.1. *With the assumptions given above, each intrinsic unitary group in $L^1(G)$ is of the form of*

$$\mathbb{M}_p = \left\{ \sum_{i=1}^{n} u_i \cdot p_i :\ u_i \in \mathrm{U}(r_i) \right\} \cong \prod_{i=1}^{n} \mathrm{U}(r_i),$$

when p is a projection in $L^1(G)$ with $p = \sum_{i=1}^{n} p_i$ as in (6), where $r_i \in \mathbb{N}$ and $U(r_i)$ is the group $r_i \times r_i$ unitary matrices, for $1 \le i \le n$.

Proof. We saw in the proof of Theorem 3.4, that a direct sum of matrix algebras $\bigoplus_{i=1}^{n} M_{r_i}(\mathbb{C})$ is $*$-isomorphic to $p * L^1(G) * p$, with the isomorphism given by $(a_i)_{i=1}^{n} \mapsto \sum_{i=1}^{n} \sum_{k,\ell=1}^{r_i} a_{k,\ell} k_{\pi_i} \pi_{i,\xi_k^{(i)},\xi_\ell^{(i)}}$. The structure of the unitary group follows immediately. $\square$

The value of the above result lies in its application to the problem of constructing homomorphisms from $L^1(F)$ to $L^1(G)$, where F is another locally compact group, as given in [21]. Let p be given as in (6), and $\mathbb{M}_p$ as in Proposition 5.1. Given any continuous homomorphism $\phi : F \to \prod_{i=1}^{n} \mathrm{U}(d_i)$, we can render a $*$-homomorphism $\Phi_p : L^1(F) \to L^1(G)$ by

$$\Phi_p(f) = \sum_{i=1}^{n} \int_F f(s)\phi(s)_i \cdot p_i \, ds. \tag{7}$$

We can use this to construct nontrivial homomorphisms from $L^1(F)$ to $L^1(G)$ where there exist no nontrivial homomorphisms from F to G. For example we may let $F = \mathrm{SU}(n)$, and let $G = \mathrm{SL}_2(\mathbb{R})$, the reduced Heisenberg group $\mathbb{H}_r$, or any finite group admitting an irreducible representation of dimension at least n. Notice that if F and G are abelian and G is compact, then each $d_i = 1$, and the $*$-homomorphism Φ_p corresponds to the piecewise affine map $\widehat{G} \to \widehat{F}$ whose domain is $\{\overline{\pi}_1, \ldots, \overline{\pi}_n\}$ and is given by $\overline{\pi}_i \mapsto \phi_i$. In the case that G is nonabelian and some $d_i > 1$, then $\|p\|_1 > 1$ and Φ_p is necessarily noncontractive; compare with [9].

Let us close with a modest characterization of homomorphisms described above.

Proposition 5.2. *Let G satisfy the conditions of Corollary* 4.4 *and let F be any locally compact group. A $*$-homomorphism $\Phi : L^1(F) \to L^1(G)$ is of the form $\Phi = \Phi_p$, as in* (7)*, if and only if* $\ker \Phi$ *is a modular ideal of* $L^1(F)$.

Proof. If $\Phi = \Phi_p$, as in (7), then $\Phi(L^1(F)) \subseteq p * L^1(G) * p$. As in the proof of Theorem 3.4, we see that $\Phi(L^1(F))$ is isomorphic to a $*$-subalgebra of a direct sum of full matrix algebras, and hence is unital, whence $\ker \Phi$ is a modular ideal of $L^1(F)$. Conversely, if $L^1(F)/\ker \Phi$ admits an identity, $q + \ker \Phi$, then $q^* + \ker \Phi$ is also the identity, so $p = \Phi(q)$ is a projection in $L^1(G)$. Furthermore, $\Phi(L^1(F)) \subseteq p * L^1(G) * p$. Hence by the method of proof of Theorem 3.8 of [21], Φ corresponds to a bounded homomorphism $\Phi_M : M(F) \to p * M(G) * p = p * L^1(G) * p$ and hence to a continuous homomorphism $\phi : F \to \mathbb{M}_p$, which, in turn, gives the form $\Phi = \Phi_p$, as in (7). $\square$

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