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THE BOUNDEDNESS OF THE WEIGHTED COXETER GROUP WITH COMPLETE GRAPH

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Dedicated to Professor George Lusztig on his 70th birthday.

ABSTRACT. We prove that a weighted Coxeter group (W, S, L) is bounded with $\mathbf{a}(W) = \mathbf{b}'(W) := \max\{L(u), L(w_{s,t}) \mid u, s, t \in S, |W_{s,t}| < \infty\}$ if the Coxeter graph of W is complete and $\mathbf{b}'(W) < \infty$, where $W_{s,t}$ is the parabolic subgroup of W generated by $s \neq t$ in S and $w_{s,t}$ is the longest element in $W_{s,t}$ whenever $W_{s,t}$ is finite.

Let W be a Coxeter group with S its Coxeter generator set and ℓ its length function. In [2], Lusztig defined a weight function L on (W, S). Call (W, S, L) a weighted Coxeter group. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials in an indeterminate v with integer coefficients. In [2], Lusztig introduced an $\mathcal{A}$-basis $\{c_w \mid w \in W\}$ in the Iwahori-Hecke algebra $\mathcal{H}$ associated to (W, S, L). For any $x, y, z \in W$, define $h_{x,y,z} \in \mathcal{A}$ by the expression $c_x c_y = \sum_{z \in W} h_{x,y,z} c_z$. Call (W, S, L) *bounded* if $\mathbf{a}(W) := \max\{\deg h_{x,y,z} \mid x, y, z \in W\} < \infty$. In [6, Theorem 2.1], Xi asserted that (W, S, ℓ) is bounded if its Coxeter graph $\Gamma(W)$ is complete and the cardinalities of its finite parabolic subgroups have a common upper bound. In [4], we proved that the weighted universal Coxeter group is bounded if the set $L(S) := \{L(s) \mid s \in S\}$ is finite. For any $s \neq t$ in S, denote by $W_{s,t}$ the subgroup of W generated by s, t and by w_{st} the longest element in $W_{s,t}$ whenever $W_{s,t}$ is finite. In this paper, we shall prove in Theorem 3.2 that a weighted Coxeter group (W, S, L) is bounded if (W, S, L) satisfies the condition $(*)$ below: $(*)$ $\Gamma(W)$ is complete and

$$\mathbf{b}'(W) := \max\{L(u), L(w_{st}) \mid u, s, t \in S, |W_{s,t}| < \infty\} < \infty,$$

verifying a conjecture of Lusztig in our case (see [2, Conjecture 13.4]).

Now we give an outline of the paper. In Section 1, we fix our notation concerning bc-expressions of W. Then in Section 2, we introduce the concept of a simple bc-expression and deduce some of its properties which is crucial in the proof of our main result (i.e., Theorem 3.2) in Section 3.

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1. BC-EXPRESSIONS

1.1. Let $\mathbb{Z}$ (resp., $\mathbb{N}$, $\mathbb{P}$) denote the set of integers (resp., non-negative integers, positive integers). For any $i \leqslant j$ in $\mathbb{Z}$, denote by $[i,j]$ the set $\{i, i+1, ..., j\}$ and denote $[1,j]$ simply by $[j]$.

Let W be a Coxeter group with S its Coxeter generator set such that the Coxeter graph $\Gamma(W)$ of W is complete, that is, the order m_{st} of the product st is either infinite or greater than 2 for any $s \neq t$ in S. Let ℓ be the length function and $\leqslant$ the Bruhat-Chevalley order on (W,S).

Throughout this paper, we always assume $\mathbf{b}'(W) < \infty$. Note that we do not assume the set S finite in general.

1.2. An expression $w = s_1 s_2 \cdots s_r \in W$ with $s_i \in S$ is called *reduced* if $r = \ell(w)$. For $w, x, y \in W$, we use the notation $w = x \cdot y$ to mean $w = xy$ and $\ell(w) = \ell(x) + \ell(y)$. We use the notation $s_1 s_2 \cdots s_r \equiv t_1 t_2 \cdots t_u$ with $s_i, t_j \in S$ to mean $r = u$ and $s_c = t_c$ for any $c \in [r]$. For any $s \neq t$ in S and any $k \in \mathbb{N}$, denote by $[sts\cdots]_k$, $[\cdots sts]_k$ the expressions $sts\cdots$, $\cdots sts$ (k factors for each) respectively. A transformation $s_1 \cdots [sts\cdots]_{m_{st}} \cdots s_r \mapsto s_1 \cdots [tst\cdots]_{m_{st}} \cdots s_r$ is called a *braid-move* (or a *b-move* for short) if the order m_{st} of st is finite for some $s \neq t$ in S. Let $\mathrm{Red}(z)$ be the set of all reduced expressions of $z \in W$. By a result of Tits in [5], we know that any $\zeta, \zeta' \in \mathrm{Red}(z)$ can be transformed from one to the other by successively applying some b-moves.

1.3. Let $\zeta \equiv s_1 s_2 \cdots s_r \in \mathrm{Red}(z)$ with $s_i \in S$. Call $\zeta_{i,j} \equiv s_i s_{i+1} \cdots s_j$ a *segment* of ζ if $i \leqslant j$ in $[r]$. A segment $\zeta_{i,j}$ of ζ is called *a braid factor*, if $\zeta_{i,j} \equiv [sts\cdots]_k$ for some $s \neq t$ in S with $m_{st} < \infty$ and $k = j+1-i \in \{m_{st} - c \mid c \in \{0,1,2\}\}$ such that $s_{i-1} \notin \{s,t\}$ if $i > 1$ and that $s_{j+1} \notin \{s,t\}$ if $j < r$. $\{s,t\}$ is called the *associated pair* in S for $\zeta_{i,j}$. A braid factor $\zeta_{i,j} \equiv [sts\cdots]_k$ of ζ is called *full* if $k = m_{st}$.

Let $\zeta_{i,j} \equiv [sts\cdots]_{j+1-i}$, $\zeta_{p,q} \equiv [s't's'\cdots]_{q+1-p}$ be two braid factors of ζ with $i < p$ and $j < q$. Call $\zeta_{i,j}$, $\zeta_{p,q}$ *neighboring* if $p \in \{j, j+1\}$. In this case, call $\zeta_{p,q}$ the *right-neighbor* of $\zeta_{i,j}$ and call $\zeta_{i,j}$ the *left-neighbor* of $\zeta_{p,q}$. Call $\zeta_{i,j}$, $\zeta_{p,q}$ *braid-connected* if there exists some $\zeta' \in \mathrm{Red}(z)$ satisfying one of the following conditions:

(i) $j = p$, and $\zeta'_{i'j}$, $\zeta'_{j,q'}$ are full braid factors of ζ' for some $i' \leqslant i$ and $q \leqslant q'$ in $[r]$;

(ii) $p = j+1$, and $\zeta'_{i'j'}$, $\zeta'_{j',q'}$ are full braid factors of ζ' for some $i', j', q' \in [r]$ with $i' \leqslant i$ and $j \leqslant j' \leqslant j+1$ and $q \leqslant q'$.

If two neighboring braid factors $\zeta_{i,j}$, $\zeta_{p,q}$ of ζ are braid connected, then the associated pairs $\{s,t\}$, $\{s',t'\}$ of $\zeta_{i,j}$, $\zeta_{p,q}$ respectively in S satisfy $|\{s,t\} \cap \{s',t'\}| = 1$.

1.4. For $z \in W$, $\zeta \equiv s_1 s_2 \cdots s_r \in \mathrm{Red}(z)$ with $s_k \in S$ is called *a braid-connected expression* (or *a bc-exp* for short) if there exists a sequence of braid factors τ : $\zeta_{i_1,j_1}, \zeta_{i_2,j_2}, ..., \zeta_{i_a,j_a}$ of ζ with some $a \in \mathbb{P}$ and $i_1 < i_2 < \cdots < i_a$ and $(i_1, j_a) = (1, r)$ such that either $a = 1$ with ζ_{i_1,j_1} full, or $a > 1$ with ζ_{i_c,j_c}, $\zeta_{i_{c+1},j_{c+1}}$ braid-connected for any $c \in [a-1]$. It is known by [3, Lemmas 2.1–2.2] that the number a of the terms in a braid sequence τ of a bc-exp ζ only depends on z but not on the choice of ζ and τ. So we can denote a simply by $l_b(z)$. Call τ *a braid sequence* of ζ. Call ζ_{i_c,j_c} the cth braid factor of ζ and call its associated pair in S the *cth associated pair* of ζ.

1.5. It is known that if there exists some bc-exp in $\mathrm{Red}(z)$, then all the expressions in $\mathrm{Red}(z)$ are bc-exps (see [3, Lemma 2.4]). Let $\zeta_{i_c,j_c} :\equiv [s_c t_c s_c \cdots s'_c]_{m_{s_c t_c}}$ be the

cth braid factor of $\zeta \in \mathrm{Red}(z)$ which is full, where $s'_c = s_c$ if $m_{s_c t_c}$ is odd and $s'_c = t_c$ if $m_{s_c t_c}$ is even. If $\zeta_{i_{c-1},j_{c-1}}$ with $c > 1$ (resp., $\zeta_{i_{c+1},j_{c+1}}$ with $c < a$) is also full, then $j_{c-1} = i_c$ (resp., $i_{c+1} = j_c$); if $\tau' : \zeta'_{i'_1,j'_1}, \zeta'_{i'_2,j'_2}, ..., \zeta'_{i'_a,j'_a}$ is a braid sequence of some $\zeta' \in \mathrm{Red}(z)$, then $i'_c \in \{i_c, i_c+1\}$ and $j'_c \in \{j_c, j_c-1\}$ (see [3, Lemma 2.1]). We say that $\zeta'_{i'_c,j'_c}$ is *left-full* if $i'_c = i_c$ and *right-full* if $j'_c = j_c$. Hence $\zeta'_{i'_c,j'_c}$ is full if and only if it is both left-full and right-full.

By the definition, it is easily seen that $\zeta \equiv s_1 s_2 \cdots s_r$ is a bc-exp in W if and only if such is $\zeta^{-1} \equiv s_r \cdots s_2 s_1$. A segment ξ of a reduced expression ζ is called a *bc-segment*, if ξ itself is a bc-exp.

2. Simple bc-expressions

2.1. A bc-exp ζ is called *simple*, if ζ has one of the following two forms:

(i) The left-most braid factor is full, and all the other braid factors are right-full.

(ii) The right-most braid factor is full, and all the other braid factors are left-full.

A simple bc-exp in the case (i) (resp., (ii)) is called *l-simple* (resp., *r-simple*). Clearly, a full braid factor is a simple bc-exp which is both l-simple and r-simple. So a simple bc-exp can be regarded as a generalization of a full braid factor.

The following are some properties for $z \in W$ having a simple bc-exp:

(2.1.1) If $z \in W$ has an l-simple (resp., r-simple) bc-exp $\zeta :\equiv s_1 s_2 \cdots s_r$ with $s_i \in S$, then $\zeta_{2,r} s_{r-1}$ (resp., $s_2 \zeta_{1,r-1}$) is in $\mathrm{Red}(z)$ and is an r-simple (resp., l-simple) bc-exp; $\zeta_{i,j} \neq \zeta_{i+1,j} s_{j-1}$ for any $1 < i < j$ (resp., $\zeta_{i,j} \neq s_{i+1} \zeta_{i,j-1}$ for any $i < j < r$) in $[r]$.

(2.1.2) If $z \in W$ has a simple bc-exp, then $\mathrm{Red}(z)$ contains a unique l-simple (resp., r-simple) bc-exp with $|\mathrm{Red}(z)| = l_b(z) + 1$ (see [3, Lemma 4.4]).

Lemma 2.2. *For $z \in W$, let $\zeta :\equiv s_1 s_2 \cdots s_r \in \mathrm{Red}(z)$ with $s_i \in S$ for any $i \in [r]$.*

(1) *$|\mathcal{L}(z)| = 2$ (resp., $|\mathcal{R}(z)| = 2$) if and only if there exists some $k \in [r]$ such that $\zeta_{1,k} \equiv s_1 s_2 \cdots s_k$ (resp., $\zeta_{k,r} \equiv s_k s_{k+1} \cdots s_r$) is an r-simple (resp., l-simple) bc-segment of ζ. When the equivalent conditions hold, we have $\mathcal{L}(z) = \{s_1, s_2\}$ and $\zeta_{1,k} = s_2 \zeta_{1,k-1}$ (resp., $\mathcal{R}(z) = \{s_r, s_{r-1}\}$ and $\zeta_{k,r} = \zeta_{k+1,r} s_{r-1}$).*

(2) *Suppose that $z = x \cdot w_I \cdot y$ for some $x, y \in W$ and $I \subseteq S$ with $|I| = 2$ and $|W_I| < \infty$. Then $\mathcal{L}(z) = \mathcal{L}(x \cdot w_I)$ and $\mathcal{R}(z) = \mathcal{R}(w_I \cdot y)$.*

Proof. Suppose $|\mathcal{L}(z)| = 2$. Then there must exist some full braid factor in ζ. Let $\zeta_{h,k} :\equiv s_h s_{h+1} \cdots s_k$ be the left-most full braid factor of ζ for some $h < k$ in $[r]$. Apply induction on $h \geqslant 1$. If $h = 1$, then $\zeta_{h,k}$ is an r-simple bc-segment of ζ. If $h > 1$, then ζ can be transformed to some $\zeta' \in \mathrm{Red}(z)$ by successively applying some b-moves such that ζ' has a full braid factor as its left-most segment. Since any b-move at the segment $\zeta_{k,r}$ of ζ keeps the segment $\zeta_{1,k-1}$ unchanged, we see that to "create" a full braid factor to the left of $\zeta_{h,k}$ by applying some b-moves on ζ, we must apply a b-move at the segment $\zeta_{h,k}$ in some step (say the resulting expression in $\mathrm{Red}(z)$ is ζ'' after this step). Thus there must exist some left-full braid factor (say $\zeta_{j,h-1}$ for some $j \in [h-2]$) of ζ which is the left-neighbor of $\zeta_{h,k}$. Hence $\zeta''_{j,h}$ is a full braid factor of ζ''. Since $j < h$, we see by inductive hypothesis that $\zeta''_{1,h}$ is an r-simple bc-segment of ζ'', hence $\zeta_{1,k}$ is an r-simple bc-segment of ζ. On the other hand, suppose that $\zeta_{1,k}$ is an r-simple bc-segment of ζ. Then there are some $0 = i_0 < i_1 < \cdots < i_c = k$ in $\mathbb{P}$ such that $\zeta_{i_{c-1}+1,i_c}$ is a full braid factor and that $\zeta_{i_{j-1}+1,i_j}$, $j \in [c-1]$, are left-full braid factors of ζ. By applying b-moves on ζ at the segments $\zeta_{i_{c-1}+1,i_c}$, $\zeta_{i_{c-2}+1,i_{c-1}+1}$, ..., ζ_{i_1+1,i_2+1} in turn, the left-most segment

ζ'_{1,i_1+1} of the resulting expression $\zeta' \in \mathrm{Red}(z)$ is a full braid factor. This implies $|\mathcal{L}(z)| = 2$. Similarly, we can prove that $|\mathcal{R}(z)| = 2$ if and only if there exists some $k \in [r]$ such that $\zeta_{k,r}$ is an l-simple bc-segment of ζ. The second assertion of (1) follows by (2.1.1). So (1) is proved. Then (2) follows by (1). □

Next we give a description for an expression $\zeta :\equiv s_1 s_2 \cdots s_r$ to be a simple bc-exp.

Lemma 2.3. *Let $\zeta :\equiv s_1 s_2 \cdots s_r$ be with $s_i \in S$ and $s_j \neq s_{j+1}$ for any $i \in [r]$ and $j \in [r-1]$. Denote $\zeta_{i,j} :\equiv s_i s_{i+1} \cdots s_j$ for any $i \leqslant j$ in $[r]$. Then*

(1) *ζ is an l-simple bc-exp if and only if*

(1a) $\zeta = \zeta_{2,r} s_{r-1}$ *and* $\zeta_{i,j} \neq \zeta_{i+1,j} s_{j-1}$ *for any* $1 < i < j$ *in* $[r]$.

(2) *ζ is an r-simple bc-exp if and only if*

(2a) $\zeta = s_2 \zeta_{1,r-1}$ *and* $\zeta_{i,j} \neq s_{i+1} \zeta_{i,j-1}$ *for any* $i < j < r$ *in* $[r]$.

Proof. By symmetry, we need only to prove (1). The implication "$\Longrightarrow$" follows by (2.1.1). Now let us prove the implication "$\Longleftarrow$". Assume ζ satisfies the condition (1a). First we claim that ζ is a reduced expression. For otherwise, there should exist some $p < q$ in $[r]$ such that

$$s_1 s_2 \cdots s_r = s_1 \cdots \widehat{s}_p \cdots \widehat{s}_q \cdots s_r, \tag{2.3.1}$$

by the deletion condition on a Coxeter system (see [1, Corollary 5.8]), where the notation $\widehat{s}$ means the deletion of the factor s. This is equivalent to the relation

$$\zeta_{p,q-1} = \zeta_{p+1,q}. \tag{2.3.2}$$

We may take such a pair p, q with $q-p$ the smallest possible. Then the expressions on both sides of (2.3.2) are reduced. Since $\mathcal{R}(\zeta_{p+1,q}) = \{s_q, s_{q-1}\}$, there must exist some $j \in [p+1, q-1]$ such that $\zeta_{j,q}$ is an l-simple bc-segment of $\zeta_{p+1,q}$ with $\zeta_{j,q} = \zeta_{j+1,q} s_{q-1}$ by Lemma 2.2, contradicting the condition (1a) on ζ. The claim is proved. Since $\zeta = \zeta_{2,r} s_{r-1}$, we have $\mathcal{R}(\zeta) = \{s_r, s_{r-1}\}$. By Lemma 2.2, there exists some $j \in [r-1]$ such that $\zeta_{j,r}$ is an l-simple segment of ζ. So $\zeta_{j,r} = \zeta_{j+1,r} s_{r-1}$ again by Lemma 2.2. By the condition (1a) on ζ, this forces $j = 1$, that is, ζ is an l-simple bc-exp. □

Lemma 2.4. *Let $\zeta :\equiv s_1 s_2 \cdots s_r$ be with $s_k \in S$ for any $k \in [r]$ such that $\zeta_{i,j}$ is a full braid factor of ζ for some $i < j$ in $[r]$. Then the expression ζ is reduced if and only if both segments $\zeta_{1,j}$ and $\zeta_{i,r}$ of ζ are reduced.*

Proof. The implication "$\Longrightarrow$" is obvious. We must prove the implication "$\Longleftarrow$". Assume that both $\zeta_{1,j}$ and $\zeta_{i,r}$ are reduced. Suppose ζ is not reduced. Then by the deletion condition on a Coxeter system, there exist some $p \in [i-1]$ and $q \in [j+1, r]$ such that

$$s_1 s_2 \cdots s_r = s_1 \cdots \widehat{s}_p \cdots s_i s_{i+1} \cdots s_j \cdots \widehat{s}_q \cdots s_r. \tag{2.4.1}$$

This is equivalent to the relation

$$\zeta_{p,q-1} = \zeta_{p+1,q}. \tag{2.4.2}$$

We may take such a pair p, q with $q - p$ the smallest possible. Thus $\zeta_{p+1,q}$ is a reduced expression with $\mathcal{L}(\zeta_{p+1,q}) = \{s_p, s_{p+1}\}$. By Lemma 2.2 (2), we have $\mathcal{L}(\zeta_{p+1,j}) = \{s_p, s_{p+1}\}$. By Lemma 2.2 (1) and the fact $s_{p+2} = s_p$, there exists an r-simple bc-segment $\zeta_{p+1,c}$ of $\zeta_{p+1,j}$ with some $c \in [p+1, j]$ such that $\mathcal{L}(\zeta_{p+1,c}) = \{s_p, s_{p+1}\}$, hence $\zeta_{p,c-1} = \zeta_{p+1,c}$ by (2.1.1). But this would imply $s_1 s_2 \cdots s_j = s_1 \cdots \widehat{s}_p \cdots \widehat{s}_c \cdots s_j$, contradicting the assumption of $\zeta_{1,j}$ being reduced. □

Lemma 2.5. *Let $x', y' \in W$ and $z \in W_I$ be with $I = \{s, t\} \subseteq S$ and $2 < m_{st} < \infty$ such that $\mathcal{R}(x'), \mathcal{L}(y') \subseteq S - I$ and $\ell(z) \geqslant 2$.*
(1) $\ell(x'zy') < \ell(x') + \ell(z) + \ell(y')$ if and only if one of the following cases (1a)-(1b) occurs:

(1a) $z = st$ *and* $x' = x'' \cdot [\cdots t's t']_{m_{st'}-1}$ *and* $y' = [t'tt' \cdots]_{m_{tt'}-1} \cdot y''$;

(1b) $z = ts$ *and* $x' = x'' \cdot [\cdots t'tt']_{m_{tt'}-1}$ *and* $y' = [t'st' \cdots]_{m_{st'}-1} \cdot y''$,

where $x'', y'' \in W$ and, $t' \in S - \{s, t\}$ satisfy $m_{st'}, m_{tt'} < \infty$.
(2) When the equivalent conditions in (1) hold, we have $\ell(x'zy') + 2 = \ell(x') + \ell(z) + \ell(y')$.

Proof. For (1), the implication "$\Longleftarrow$" is obvious. Now we prove the implication "$\Longrightarrow$". Suppose $\ell(x'zy') < \ell(x') + \ell(z) + \ell(y')$. By Lemma 2.4, the assumptions $\mathcal{R}(x'), \mathcal{L}(y') \subseteq S - I$ and $z \in W_I$, we have

$$\ell(x'z) = \ell(x') + \ell(z), \quad \ell(zy') = \ell(z) + \ell(y') \quad \text{and} \quad z \neq w_I. \tag{2.5.1}$$

By symmetry of s, t in W_I, we may assume $z = [sts \cdots s']_k$ with some $2 \leqslant k < m_{st}$ for the sake of definiteness, where $s' = s$ if k is odd and $s' = t$ if k is even. We shall prove that the case (1a) of the lemma occurs. Let

$$s_r s_{r-1} \cdots s_1 \in \mathrm{Red}(x') \quad \text{and} \quad t_1 t_2 \cdots t_u \in \mathrm{Red}(y') \tag{2.5.2}$$

with $s_i, t_j \in S$. Then we see by (2.5.1) that there are some $p \in [r]$ and $q \in [u]$ such that

$$s_r s_{r-1} \cdots s_1 [sts \cdots s']_k t_1 t_2 \cdots t_u = s_r \cdots \widehat{s}_p \cdots s_1 [sts \cdots s']_k t_1 \cdots \widehat{t}_q \cdots t_u \tag{2.5.3}$$

by the deletion condition on a Coxeter system. We may take such a pair p, q with $p+q$ the smallest possible, as the expressions in (2.5.2) range over $\mathrm{Red}(x')$, $\mathrm{Red}(y')$, respectively. This implies that

$$s_p s_{p-1} \cdots s_1 [sts \cdots s']_k t_1 \cdots t_{q-1} = s_{p-1} \cdots s_1 [sts \cdots s']_k t_1 \cdots t_{q-1} t_q. \tag{2.5.4}$$

We see by Lemma 2.3 that the expression ζ on the LHS of (2.5.4) is an l-simple bc-exp and that on the RHS of (2.5.4) it is an r-simple bc-exp. Since

$$\mathcal{L}(\zeta) = \{s_p, s_{p-1}\} \quad \text{and} \quad \mathcal{R}(\zeta) = \{t_q, t_{q-1}\}, \tag{2.5.5}$$

each of the expressions on both sides of (2.5.4) contains some full braid factor.

We claim that there exists no full braid factor in any of the expressions

$$\zeta_1 :\equiv s_{p-1} s_{p-2} \cdots s_1 [sts \cdots s']_k t_1 \cdots t_{q-1}, \ \zeta_2 :\equiv s_p s_{p-1} \cdots s_1, \ \zeta_3 :\equiv t_1 t_2 \cdots t_q. \tag{2.5.6}$$

For otherwise, say ζ_1' is the left-most full braid factor of ζ_1, say $\zeta_1 \equiv \zeta_1'' \zeta_1' \zeta_1'''$. Then $\zeta_1'' \zeta_1'$ is an r-simple bc-exp with $\mathcal{L}(\zeta_1'' \zeta_1') = \{s_p, s_{p-1}\}$ by Lemmas 2.4, 2.2, (2.5.5) and by regarding $\zeta_1'' \zeta_1'$ as a left-most segment of $\zeta_1 t_q$, which would imply that $s_p \zeta_1$ is not reduced, a contradiction. Next, if ζ_2 contains some full braid factor ζ_2', then $\zeta_2' \equiv s_p s_{p-1} \cdots s_j$ for some $j \in [p-1]$ by the above proof, hence $\zeta_2' = s_{p-1} s_{p-2} \cdots s_j s_j'$, where s_j' is determined by the condition $\{s_j, s_j'\} = \{s_p, s_{p-1}\}$. By replacing the expression $x' = s_r s_{r-1} \cdots s_1$ by $x' = s_r s_{r-1} \cdots s_{p+1} \cdot s_{p-1} s_{p-2} \cdots s_j s_j' \cdot s_{j-1} s_{j-2} \cdots s_1$, we can replace p, q by j, q in (2.5.3). Since $j + q < p + q$, this contradicts the minimality of $p+q$. Similarly, we can show that there exists no full braid factor in ζ_3. The claim is proved.

Since each of the expressions on both sides of (2.5.4) contains some full braid factor, we must have $s_ps_{p-1}\cdots s_1s \equiv [\cdots ss_1s]_{m_{ss_1}}$ and $s't_1t_2\cdots t_u \equiv [s't_1s'\cdots]_{m_{s't_1}}$ by the above claim and the facts that the elements s,t,s_1 (resp., s,t,t_1) are pairwise distinct. So (2.5.4) becomes

$$(2.5.7)\qquad \begin{aligned}&[\cdots ss_1s]_{m_{ss_1}}[tst\cdots s'']_{k-2}[s't_1s'\cdots]_{m_{s't_1}}\\ &= [\cdots ss_1s]_{m_{ss_1}-1}[tst\cdots s'']_{k-2}[s't_1s'\cdots]_{m_{s't_1}-1},\end{aligned}$$

where $s''\in S$ is determined by the condition $\{s',s''\}=\{s,t\}$. Since the expression on the top of (2.5.7) is not reduced and $2\leqslant k<m_{st}$, this forces $k=2$ and $s_1=t_1$, so we are in the case (1a) of the lemma.

Next consider (2). By symmetry, we may assume to be in the case (1a). By the assumption $\mathcal{R}(x'),\mathcal{L}(y')\subseteq S-I$, we have $s,t'\notin\mathcal{R}(x'')$ and $t,t'\notin\mathcal{L}(y'')$. Let $x'_0 := x''\cdot[\cdots t'st']_{m_{st'}-2}$ and $y'_0 := [tt't\cdots]_{m_{tt'}-2}\cdot y''$. Then $x'_0zy'_0 = x'zy'$. We have $\ell(x'_0zy'_0)=\ell(x'_0)+\ell(z)+\ell(y'_0)$ by (1) with x'_0,y'_0 in the places of x',y', respectively. This implies that $\ell(x'zy')=\ell(x'_0zy'_0)=\ell(x'_0)+\ell(z)+\ell(y'_0)=\ell(x')+\ell(z)+\ell(y')-2$, so (2) is proved. □

3. The boundedness of (W,S,L)

Recall the notation $\mathbf{a}(W)$, $\mathbf{b}'(W)$, the condition $(*)$ and the boundedness of a weighted Coxeter group defined at the beginning of the paper. In this section, we always assume (W,S,L) is a weighted Coxeter group satisfying the condition $(*)$. We shall prove that (W,S,L) is bounded.

3.1. Denote $v_w := v^{L(w)}$ for $w\in W$. The Iwahori-Hecke algebra $\mathcal{H} := \mathcal{H}(W,S,L)$ of (W,S,L) is by definition the associative $\mathcal{A}$-algebra with an $\mathcal{A}$-basis $\{T_w \mid w\in W\}$, subject to the multiplication rule:

$$(3.1.1)\qquad \begin{aligned}T_s^2 &= (v_s-v_s^{-1})T_s+T_e &&\text{for } s\in S,\\ T_xT_y &= T_{xy} &&\text{for } x,y\in W \text{ with } \ell(xy)=\ell(x)+\ell(y),\end{aligned}$$

where e is the identity element of W. In [2, Chapter 5], Lusztig defined another $\mathcal{A}$-basis $\{c_w \mid w\in W\}$ of $\mathcal{H}$, where each $c_w=\sum_{y\leqslant w}p_{y,w}T_y$ is fixed by a certain ring involution (called the *bar operation* by Lusztig) of $\mathcal{H}$, and $p_{y,w}\in\mathbb{Z}[v^{-1}]$ satisfies $p_{y,w}=0$ if $y\nleqslant w$, $p_{w,w}=1$ and $p_{y,w}\in v^{-1}\mathbb{Z}[v^{-1}]$ if $y<w$.

The main result of the paper is as follows.

Theorem 3.2. *Let (W,S,L) be a weighted Coxeter group satisfying the condition $(*)$. Then (W,S,L) is bounded. More precisely, we have $\mathbf{a}(W)=\mathbf{b}'(W)$.*

For $x,y,z\in W$, define $f_{x,y,z}\in\mathcal{A}$ by $T_xT_y=\sum_{z\in W}f_{x,y,z}T_z$. Define $\mathbf{b}(W) := \max\{\deg f_{x,y,z} \mid x,y,z\in W\}$. By [4, Lemma 2.1], we have

$$(3.2.1)\qquad \mathbf{a}(W)=\mathbf{b}(W).$$

By (3.2.1), to prove Theorem 3.2, we need only to prove $\mathbf{b}(W)=\mathbf{b}'(W)$.

3.3. Fix $x,y,z\in W$ and a reduced expression $x=s_rs_{r-1}\cdots s_1$ with $s_i\in S$ for $i\in[r]$. Let $\Delta(x,y;z)$ be the set of all sequences

$$(3.3.1)\qquad \zeta : y_0=y, y_1, ..., y_r=z \quad \text{in } W$$

such that for any $i \in [r]$, $y_i = s_i y_{i-1}$ if $s_i \notin \mathcal{L}(y_{i-1})$ and $y_i \in \{y_{i-1}, s_i y_{i-1}\}$ if $s_i \in \mathcal{L}(y_{i-1})$. Let $E_\zeta = \{i \in [r] \mid y_i = y_{i-1}\}$ and $f_\zeta = \prod_{i \in E_\zeta} \xi_{s_i}$ with $\xi_s := v_s - v_s^{-1}$ for $s \in S$.

Lemma 3.4. *In the above setup, we have* $f_{x,y,z} = \sum_{\zeta \in \Delta(x,y;z)} f_\zeta$.

Proof. This follows simply by (3.1.1). □

3.5. Lemma 3.4 shows that the sum $\sum_{\zeta \in \Delta(x,y;z)} f_\zeta$ is independent of the choice of a reduced expression of x although the set $\Delta(x,y;z)$ might depend on it. Since $f_{x,y,z}$ is a polynomial in ξ_s, $s \in S$, with non-negative integer coefficients, we have

(3.5.1) $$\deg f_{x,y,z} = \max\{\deg f_\zeta \mid \zeta \in \Delta(x,y;z)\}.$$

By (3.1.1), it is easy to prove (3.5.2)-(3.5.3) below.

(3.5.2) $\deg f_{x,y,z} \leqslant \min\{L(x), L(y), L(z)\}$ for any $x, y, z \in W$.

(3.5.3) For any $I \subseteq S$ with $|W_I| < \infty$, let w_I be the longest element in W_I. Then $\deg f_{w_I, w_I, w} = L(w)$ for any $w \in W_I$.

Taking $w = w_I$ in (3.5.3), we obtain the following result:

Lemma 3.6. $\mathbf{b}'(W) \leqslant \mathbf{b}(W)$.

3.7. By Lemma 3.6, to prove the equality $\mathbf{b}(W) = \mathbf{b}'(W)$, we need only to prove the inequality $\deg f_\zeta \leqslant \mathbf{b}'(W)$ for any $x, y, z \in W$ and any $\zeta \in \Delta(x,y;z)$.

We see in (3.3.1) that for $i \in [r-1]$, if $s_i \notin \mathcal{L}(y_{i-1})$, then $s_i \in \mathcal{L}(y_i)$; in this case, we have $s_{i+1} \in \mathcal{L}(y_i)$ if and only if $\mathcal{L}(y_i) = \{s_i, s_{i+1}\}$ (note that $|\mathcal{L}(w)| \leqslant 2$ for any $w \in W$).

$\zeta \in \Delta(x,y;z)$ in (3.3.1) is said to be *small* if there exists at most one $i \in [r]$ with $y_i = y_{i-1}$, and *large* otherwise.

Lemma 3.8. *Let $x, y, z \in W$ be with $\Delta(x,y;z) \neq \emptyset$. If all the elements of $\Delta(x,y;z)$ are small, then $|\Delta(x,y;z)| = 1$. Say $\Delta(x,y;z) = \{\zeta\}$. Then $f_\zeta \in \{1, \xi_s \mid s \in S\}$.*

Proof. Let $\zeta \in \Delta(x,y;z)$ be as that in (3.3.1) with $x = s_r s_{r-1} \cdots s_1$ a reduced expression with $s_k \in S$ for $k \in [r]$. Then one of the following cases must happen:

(1) $y_i = s_i y_{i-1}$ for any $i \in [r]$. Hence $z = xy$ and $f_\zeta = 1$.

(2) There exists some $i \in [r]$ such that $y_i = y_{i-1}$ and $y_j = s_j y_{j-1}$ for any $j \in [r] - \{i\}$. Hence $z = s_r s_{r-1} \cdots \widehat{s}_i \cdots s_1 y$ and $f_\zeta = \xi_{s_i}$.

We have $xy \neq s_r s_{r-1} \cdots \widehat{s}_i \cdots s_1 y$ for $i \in [r]$. So $|\Delta(x,y;z)| = 1$ and $f_{x,y,z} = 1$ in the case (1). If ζ is in the case (2), then i is determined uniquely by z, hence $|\Delta(x,y;z)| = 1$ and $f_{x,y,z} = f_\zeta = \xi_{s_i}$ in this case. □

In order to prove Theorem 3.2, we need one more result.

Lemma 3.9. *Suppose that $x = s_r s_{r-1} \cdots s_1 t$ and $y = t t_1 t_2 \cdots t_u$ are two reduced expressions in W with $r, u \in \mathbb{P}$ and $s_i, t_j \in S$ for any $i \in [r]$ and $j \in [u]$ such that $m_{s_1 t} = m_{t t_1} = \infty$. Then the expression*

(3.9.1) $$s_r s_{r-1} \cdots s_1 t t_1 t_2 \cdots t_u$$

is reduced.

Proof. If the expression (3.9.1) is not reduced, then there would exist some $p \in [r]$ and $q \in [u]$ such that

(3.9.2) $$s_r s_{r-1} \cdots s_1 t t_1 t_2 \cdots t_u = s_r \cdots \widehat{s}_p \cdots s_1 t t_1 \cdots \widehat{t}_q \cdots t_u$$

by the deletion condition on a Coxeter system. We may take such a pair p, q with $p+q$ the smallest possible. Then

$$s_{p-1}s_{p-2}\cdots s_1tt_1t_2\cdots t_q = s_ps_{p-1}\cdots s_1tt_1t_2\cdots t_{q-1} \tag{3.9.3}$$

with the convention that $s_0 = t_0 = t$. In (3.9.3), we see by the minimum assumption of $p+q$ that the expressions on both sides are reduced, and by Lemmas 2.2, 2.3 that the expression (denoted by ζ) on the LHS is an r-simple bc-exp, and that the one (denoted by ζ') on the RHS is an l-simple exp. So any braid factor of ζ (resp., ζ') is either left-full (resp., right-full) or full. By the assumption $m_{s_1t} = m_{tt_1} = \infty$, the factor t can't belong to any braid factor of ζ, ζ', a contradiction. This proves our result. □

3.10. *Proof of Theorem* 3.2. Suppose there exists some large element in $\Delta(z) := \bigcup_{x,y\in W}\Delta(x,y;z)$. Then $\delta(z) := \max\{\deg f_{\zeta'} \mid \zeta' \in \Delta(z)\} > 1$. Let $\Delta_0(z) := \{\zeta \in \Delta(z) \mid \deg f_\zeta = \delta(z)\}$. Take $\zeta : y_0 = y, y_1, ..., y_r = z$ in $\Delta_0(z)$ with $x = s_rs_{r-1}\cdots s_1$ the corresponding reduced expression as in Subsection 3.3 for some $x, y \in W$ (ζ must be large with $\ell(x), \ell(y) \geqslant 2$ by the condition $\deg f_\zeta = \delta(z) > 1$) with r the smallest possible. The first claim is that $y_0 = y_1$. For otherwise, the smallest $k \in [r]$ with $y_k = y_{k-1}$ is in $[2, r]$. Let $x_0 = s_rs_{r-1}\cdots s_k$, $y'_0 = y_{k-1}$ and define $\zeta' : y'_0, y'_1, ..., y'_{r+1-k}$ by setting $y'_j = y_{k-1+j}$ for $j \in [0, r+1-k]$. Then $\zeta' \in \Delta(x_0, y'_0; z)$ and $f_{\zeta'} = f_\zeta$, hence $\zeta' \in \Delta_0(z)$. Since $r+1-k < r$, it contradicts our assumption of ζ being the shortest in $\Delta_0(z)$. The first claim is proved. The second claim is either that there exists some reduced expression $x = s_rs_{r-1}\cdots s_1$ with $s_i \in S$ for any $i \in [r]$ and $3 \leqslant m_{s_1s_2} < \infty$ and $s_1 \in \mathcal{L}(y)$ or that there exists some reduced expression $y = t_1t_2\cdots t_u$ with $t_j \in S$ for any $j \in [u]$ and $3 \leqslant m_{t_1t_2} < \infty$ and $t_1 \in \mathcal{R}(x)$. For otherwise, $m_{s_1s_2} = m_{t_1t_2} = \infty$ for any reduced expressions $x = s_rs_{r-1}\cdots s_1$ and $y = t_1t_2\cdots t_u$ with $s_i, t_j \in S$ and $s_1 \in \mathcal{L}(y)$ and $t_1 \in \mathcal{R}(x)$ for any $i \in [r]$ and $j \in [u]$. We may write $x = x' \cdot st$ and $y = ts' \cdot y'$ for some $x', y' \in W$ and $s, t, s' \in S$. Then $z = x' \cdot sts' \cdot y'$ by Lemma 3.9 and the fact $m_{st} = m_{ts'} = \infty$, hence $f_\zeta = \xi_t$, contradicting the assumption of ζ being large. This proves the second claim.

Hence there exist some $x', y' \in W$, $c, d \in \mathbb{P}$, $x_1 = [\cdots sts]_c \equiv s'_cs'_{c-1}\cdots s'_1$, $y_1 = [sts\cdots]_d$ and $z_1 \in W_I - (I \cup \{e\})$ such that

(1) $x = x' \cdot x_1$ and $y = y_1 \cdot y'$ with $\mathcal{R}(x'), \mathcal{L}(y') \subseteq S - \{s, t\}$.

(2) There exists some $\zeta' : y_{10} = y_1, y_{11}, ..., y_{1c} = z_1$ in W_I satisfying that

(2a) $y_{1i} = s'_iy_{1,i-1}$ if $s'_i \notin \mathcal{L}(y_{1,i-1})$ and $y_{1i} \in \{y_{1,i-1}, s'_iy_{1,i-1}\}$ if $s'_i \in \mathcal{L}(y_{1,i-1})$.

(2b) $y_{10} = y_{11}$.

(2c) $y_{10}y', y_{11}y', ..., y_{1c}y'$ is the subsequence of ζ consisting of the first $c+1$ terms.

A necessary condition for f_ζ reaching its highest possible degree $\delta(z)$ is that $f_{\zeta'}$ reaches its highest possible degree $L(z_1)$ by (3.5.2)-(3.5.3). Thus we may take $x_1 = y_1 = w_I$ (hence $c = d = \ell(w_I)$). Since $w_I = z_1w_I \cdot w_Iz_1^{-1}w_I$, we can write $w_I \equiv [\cdots sts]_c \equiv s'_cs'_{c-1}\cdots s'_1$ such that $z_1w_I \equiv s'_cs'_{c-1}\cdots s'_{p+1}$ and $w_Iz_1^{-1}w_I \equiv s'_ps'_{p-1}\cdots s'_1$ with $p = \ell(z_1)$ and that $y_{1j} = y_{1,j-1}$ and $y_{1i} = s'_iy_{1,i-1}$ for $j \in [p]$ and $i \in [p+1, c]$.

Consider the condition (2d) below.

(2d) $z_1 = st$, $x' = x'' \cdot [\cdots t'st']_{m_{st'}-1}$ and $y' = [t'tt'\cdots]_{m_{tt'}-1} \cdot y''$ for some $x'', y'' \in W$ and $t' \in S - \{s, t\}$ with $m_{st'}, m_{tt'} < \infty$.

If we are not in the case (2d), then $z = x' \cdot z_1 \cdot y'$ and $\deg f_\zeta = L(z_1) \leqslant L(w_I)$ by Lemma 2.5 and (3.5.2). If we are in the case (2d), let $x'_0 = x'' \cdot$

$[\cdots st's]_{m_{st'}-1}$ and $y_0' = [tt't\cdots]_{m_{tt'}} \cdot y''$; then $z = x_0' \cdot y_0'$ and $\deg f_\zeta = L(st)+L(t') \leqslant \max\{L(w_{s,t}), L(w_{s,t'}), L(w_{t,t'})\}$ by (3.5.2) and the fact $m_{st}, m_{st'}, m_{tt'} \geqslant 3$.

This, together with Lemma 3.8, completes the proof of the equality $\mathbf{b}(W) = \mathbf{b}'(W)$, hence Theorem 3.2 is proved. □

3.11. Let $x, y, z \in W$ satisfy $\deg f_{x,y,z} = \delta(z) > 1$. In the proof of Theorem 3.2, we see that if we are not in the case (2d) for any pairwise distinct $s, t, t' \in S$ with $3 \leqslant m_{st}, m_{st'}, m_{tt'} < \infty$, then $x = x' \cdot x_1 \cdot u$ and $y = u^{-1} \cdot y_1 \cdot y'$ and $z = x' \cdot z_1 \cdot y'$ and $\deg f_{x,y,z} = \deg f_{x_1,y_1,z_1} = L(z_1)$ for some $x', y', u \in W$, $x_1, y_1, z_1 \in W_I$ and $I = \{s,t\} \subseteq S$ with $3 \leqslant m_{st} < \infty$. If we are in the case (2d) for some pairwise distinct $s, t, t' \in S$ with $3 \leqslant m_{st}, m_{st'}, m_{tt'} < \infty$, then $\deg f_{x,y,z} = L(st) + L(t')$.

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