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FLAT EXTENSIONS IN $*$-ALGEBRAS

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Abstract. The main result of the paper is a flat extension theorem for positive linear functionals on $*$-algebras. The theorem is applied to truncated moment problems on cylinder sets, on matrices of polynomials and on enveloping algebras of Lie algebras.

1. Introduction

Given real numbers s_α, where $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq 2n$, the (real) truncated moment problem [1, 2] asks: When does there exist a positive Borel measure μ on $\mathbb{R}^d$ such that $s_\alpha = \int x^\alpha d\mu$ for all $\alpha \in \mathbb{N}_0^d, |\alpha| \leq 2n$? A fundamental result is the flat extension theorem due to R. Curto and L. Fialkow [2]; see e.g. [4] for a nice exposition and [5] for another proof. The flat extension theorem extends the corresponding linear functional on the polynomials of degree at most $2n$ to a flat positive linear functional on the whole polynomial algebra $\mathbb{C}[x_1, \ldots, x_d]$. Then the extended functional is represented by a positive measure. For this it is crucial that the extension is also flat, because then the Gelfand-Naimark-Segal representation, abbreviated as GNS representation, acts on a finite-dimensional space and the existence of the measure μ can be derived (for instance) from the finite-dimensional spectral theorem. The aim of this paper is to extend the flat extension theorem to complex unital $*$-algebras. We refer to Definition 2.3 below for the corresponding notion of flatness and to [9, Theorem 8.6.2] for the version of the GNS representation of positive linear functionals on general unital $*$-algebras.

Suppose that $\mathcal{A}$ is a unital complex $*$-algebra. Let π be a $*$-representation of $\mathcal{A}$ on a unitary space $(V, \langle\cdot,\cdot\rangle)$. For any $v \in V$, there is a positive linear functional L_v on A defined by

$$L_v(a) = \langle \pi(a)v, v\rangle, \quad a \in \mathcal{A}.$$

Functionals of this form are called *vector functionals* in the representation π.

Let L be a linear functional on a $*$-invariant linear subspace $\mathcal{B}$ of $\mathcal{A}$. A natural question arises when L has an extension to a positive linear functional $\tilde{L}$ on the whole $*$-algebra $\mathcal{A}$. By the GNS (see e.g. [9, Section 8.6]), this holds if and only if L is the restriction of a vector functional L_v in some representation π. Furthermore, the extension $\tilde{L}$ should be "nice" in order to represent it in some "well-behaved" representation π. The latter is the counterpart for requiring that the extended functional can be given by some positive measure.

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In general, it seems that these problems are difficult and no simple answers can be expected.

The main result of this paper is a general flat extension theorem for hermitian linear functionals. Roughly speaking and omitting technical details and assumptions, it says that if L is a hermitian linear functional on a subspace which is flat with respect to some appropriate smaller subspace, then L has a unique flat extension $\tilde{L}$ to $\mathcal{A}$. Furthermore, if L is positive, so is $\tilde{L}$.

This theorem contains the Curto-Fialkow theorem and its generalization in [5] as a special case when $\mathcal{A}$ is the polynomial algebra. But it also applies to certain noncommutative $*$-algebras and it allows one to treat noncommutative truncated moment problems. We mention three applications: the first concerns truncated moment problems on cylinder sets, the second is about truncated moment problems for matrices over polynomial algebras, and the third deals with enveloping algebras of finite-dimensional Lie algebras.

Flat extensions of positive functionals on path $*$-algebras are considered in [7].

2. Basic definitions and preliminaries

First we prove a proposition which contains a reformulation of flatness for hermitian matrices. It gives the justification for the flatness Definition 2.3 below.

Let $A \in M_n(\mathbb{C})$, $B \in M_{n,k}(\mathbb{C})$, $C \in M_k(\mathbb{C})$, where $A = A^*$ and $C = C^*$, and consider the hermitian block matrix X acting on the vector space $\mathbb{C}^{n+k}$:

$$X = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}.$$

Then the matrix X is called a *flat extension* of the matrix A if $\mathrm{rank} A = \mathrm{rank} X$.

Proposition 2.1. *The following statements are equivalent:*

(i) *X is a flat extension of A.*

(ii) $\mathbb{C}^{n+k} = (\mathbb{C}^n, 0) + \ker X$.

Proof. (i)$\to$(ii): Since X is a flat extension of A, a classical result of Šmul′jan [11] states that there exists a matrix $W \in M_{n,k}(\mathbb{C})$ such that $B = AW$ and $C = W^*AW$. Let $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^k$. Then we compute $X(-Wy, y) = 0$, so that $(x, y) = (x + Wy, 0) + (-Wy, y) \in \mathbb{C}^{n+k} + \ker X$, which proves (ii).

(ii)$\to$(i): By reordering the canonical basis of $\mathbb{C}^n$, we may assume that the first n' canonical basis elements of $\mathbb{C}^n$ are linearly independent of $\ker X$ and that, together with a basis of $\ker X$, they form a basis of $\mathbb{C}^{n+k}$. This implies that $\mathbb{C}^{n+k} = (\mathbb{C}^{n'}, 0) \oplus \ker X$ and we can take $n' = n$ in order to prove that (ii) implies (i).

The matrix of change of bases between the canonical basis of $\mathbb{C}^{n+k}$ and this basis is of the form

$$P = \begin{pmatrix} I & -W \\ 0 & I \end{pmatrix},$$

and the matrix of the symmetric operator in this basis is

$$\begin{aligned} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ -W^* & I \end{pmatrix} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \begin{pmatrix} I & -W \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} A & B - AW \\ B^* - W^*A & C - W^*B - B^*W + W^*AW \end{pmatrix}. \end{aligned}$$

This implies that $B = AW$, $B^* = W^*A$ and $C = W^*AW$. Thus X is a flat extension of A. $\square$

Throughout this paper we assume that $\mathcal{A}$ is a complex unital $*$-algebra. The involution of $\mathcal{A}$ is denoted by $*$ and the unit element by 1.

Let $\mathcal{C}$ be a $*$-invariant linear subspace of $\mathcal{A}$, which contains the unit element 1 of $\mathcal{A}$. A linear functional L on $\mathcal{C}$ is called *hermitian* if $L(b^*) = \overline{L(b)}$ for all $b \in \mathcal{C}$. Set

$$\mathcal{C}^2 := \text{Lin}\,\{ab; a, b \in \mathcal{C}\}.$$

Now suppose that L is a hermitian linear functional on $\mathcal{C}^2$. Then

$$\langle a, b\rangle_L := L(b^*a), a, b \in \mathcal{C},$$

defines a hermitian sesquilinear form $\langle \cdot, \cdot\rangle_L$ acting on the vector space $\mathcal{C}$. We denote by $K_L(\mathcal{C})$ the vector space

$$K_L(\mathcal{C}) := \{a \in \mathcal{C} : \langle a, b\rangle_L = 0, \forall b \in \mathcal{C}\}.$$

We verify that $K_L(\mathcal{C})$ is the kernel of the Hankel operator $H_L : \mathcal{C} \to \mathcal{C}^\star$ defined by

$$H_L : a \mapsto a \star L, \quad \text{where} \quad a \star L : b \in \mathcal{C} \mapsto L(b^*a) \in \mathbb{C}.$$

Further, for $a \in \mathcal{C}$, we define

$$\mathcal{D}_\mathcal{C}(a) = \{b \in \mathcal{C}^2 : ab \in \mathcal{C}^2\}, \quad \rho(a)b = ab \quad \text{for} \quad b \in \mathcal{D}_\mathcal{C}(a).$$

Then $\rho(a)$ is a linear operator with domain $\mathcal{D}_\mathcal{C}(a)$ on $\mathcal{C}$. It is easily verified that

$$\begin{array}{lll} \rho(ab)c & = & \rho(a)\rho(b)c \quad \text{if} \quad abc \in \mathcal{C}^2, \\ \rho(1)b & = & b \quad \text{for} \quad b \in \mathcal{C}^2, \\ L(c^*ab) & = & \langle \rho(a)b, c\rangle_L = \langle b, \rho(a^*)c\rangle_L \quad \text{if} \quad c^*ab \in \mathcal{C}^2. \end{array}$$

A linear functional L on $\mathcal{C}^2$ is called *positive* if $L(a^*a) \geqslant 0$ for all $a \in \mathcal{C}$.

Lemma 2.2. *If a functional L on $\mathcal{C}^2$ is positive, then $K_L(\mathcal{C}) = \{a \in \mathcal{C} : L(a^*a) = 0\}$.*

Proof. If $a \in K_L(\mathcal{C})$, then in particular $L(a^*a) = \langle a, a\rangle_L = 0$. Conversely, suppose that $L(a^*a) = 0$. Since L is positive, we have the Cauchy-Schwarz inequality

$$|L(b^*a)|^2 \leq L(a^*a)L(b^*b)$$

for $b \in \mathcal{C}$, which implies that $\langle a, b\rangle_L = L(b^*a) = 0$, so that $a \in K_L(\mathcal{C})$. □

The following definition contains the main concept of this paper; it is suggested by the equivalence of conditions (i) and (ii) in Proposition 2.1.

Definition 2.3. Let $\mathcal{B}$ and $\mathcal{C}$ be $*$-invariant linear subspaces of $\mathcal{A}$ such that $1 \in \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{C}$. A hermitian linear functional L on $\mathcal{C}^2$ is called a flat extension with respect to $\mathcal{B}$ if

$$\mathcal{C} = \mathcal{B} + K_L(\mathcal{C}).$$

Another important notion (see e.g. [9, Definition 8.1.9]) is the following.

Definition 2.4. Let V be a vector space equipped with a hermitian sesquilinear form $\langle \cdot, \cdot\rangle$. A $*$-representation of $\mathcal{A}$ on $(V, \langle \cdot, \cdot\rangle)$ is an algebra homomorphism ρ of $\mathcal{A}$ into the algebra of linear operators on V such that $\rho(1)v = v$ and $\langle \rho(a)v, w\rangle = \langle v, \rho(a^*)w\rangle \quad \text{for} \quad v, w \in V$.

Now suppose that L is a hermitian linear functional defined on the whole $*$-algebra $\mathcal{A}$. Then it is easily checked that $K_L(\mathcal{A})$ is a left ideal. Therefore, we can quotient out the kernel $\mathcal{I} = K_L(\mathcal{A})$ and get a *non-degenerate* hermitian form on $\mathcal{A}/\mathcal{I}$, denoted again by $\langle \cdot, \cdot \rangle_L$ by some abuse of notation, and a $*$-representation ρ on $(\mathcal{A}/\mathcal{I}, \langle \cdot, \cdot \rangle_L)$ such that

$$L(a) = \langle \rho(a)1, 1 \rangle_L \quad \text{for} \quad a \in \mathcal{A}.$$

If the functional L is positive, then this procedure coincides with the "ordinary" GNS construction; see e.g. [9, Theorem 8.6.2].

For the formulation of our main theorem, the following concept is convenient.

Definition 2.5. Let $\{a_i, i \in I\}$ be a fixed set of generators of the algebra $\mathcal{A}$. For a vector space $V \subset \mathcal{A}$, let $V^+ = \mathrm{Lin}\,\{a_i v | i \in I, v \in V\}$ be the *prolongation* of V. We shall say that V is connected to 1 if $1 \in V$ and there exists an increasing sequence of vector spaces

$$V_0 = \mathbb{C} \cdot 1 \subset V_1 \subset \cdots \subset V_l \subset \cdots \subset V,$$

such that $\bigcup_{l \in \mathbb{N}} V_l = V$ and $V_{l+1} \subset V_l^+$. For an element $v \in V$, the V-index of v is the smallest $l \in \mathbb{N}$ such that $v \in V_l$. Let $V^{[0]} = V$ and for $l \in \mathbb{N}$, we define $V^{[l+1]} = (V^{[l]})^+$. The index of an arbitrary element $a \in \mathcal{A}$ is its $\mathbb{C} \cdot 1$-index.

3. The extension theorem

Throughout this section we suppose that $\{a_i; i \in I\}$ is a fixed set of hermitian generators of the unital complex $*$-algebra $\mathcal{A}$, such that $\forall i \in I$, there exists $j \in I$ such that $a_i^* = a_j$.

Let $\mathcal{F} = \mathbb{C}\langle x_i, i \in I \rangle$ denote the free complex unital $*$-algebra in variables x_i, $i \in I$. Then there is a $*$-homomorphism $\sigma : \mathcal{F} \to \mathcal{A}$ such that $\sigma(x_i) = a_i$, $i \in I$. The kernel $\mathcal{I}$ of σ is a two-sided $*$-ideal of $\mathcal{F}$ and we have $\mathcal{A} = \mathcal{F}/\mathcal{I}$. For any set of generators S of the two-sided ideal $\mathcal{I}$, we denote by $\delta(S) \in \mathbb{N} \cup \{\infty\}$ the maximal index of the elements of S. Let $\delta(\mathcal{A})$ be the lowest $\delta(S)$ of all sets S of generators of $\mathcal{I}$.

Theorem 3.1. *Let $\mathcal{B}$ and $\mathcal{C}$ be $*$-invariant linear subspaces of $\mathcal{A}$, connected to* 1 *such that $\mathcal{B}^{[m]} \subseteq \mathcal{C}$ with $m \geq 1$, $2m \geq \delta(\mathcal{A})$. Let L be a hermitian linear functional on $\mathcal{C}^2$ which is a flat extension with respect to $\mathcal{B}$. Then there exists a unique extension of L to a hermitian linear functional $\mathcal{L}$ on $\mathcal{A}$ which is a flat extension with respect to $\mathcal{C}$.*

Let us choose a linear subspace $\mathcal{B}'$ of $\mathcal{B}$ such that $1 \in \mathcal{B}'$ and $\mathcal{C}$ is the direct sum of the vector spaces $\mathcal{B}'$ and $K_L(\mathcal{C})$. The form $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ is non-degenerate on $\mathcal{B}'$ and the projection of $\mathcal{C}$ on $\mathcal{B}'$ along $K_L(\mathcal{C})$ extends to a projection $\pi_{\mathcal{L}} : \mathcal{A} \to \mathcal{B}'$. The $*$-representation $\rho_{\mathcal{L}}$ of $\mathcal{A}$ associated with $\mathcal{L}$ acts on $(\mathcal{B}', \langle \cdot, \cdot \rangle_{\mathcal{L}})$ by $\rho_{\mathcal{L}}(a)b = \pi_{\mathcal{L}}(ab)$, $a \in \mathcal{A}, b \in \mathcal{B}'$.

Further, if L is a positive linear functional on $\mathcal{C}^2$, then $\mathcal{L}$ is a positive linear functional on $\mathcal{A}$ and $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ is a scalar product on $\mathcal{B}'$.

The remaining part of this section is devoted to the proof of this theorem. Let us recall the main assumptions. Since the vector space $\mathcal{C}$ is connected to 1, there exists a sequence of vector spaces $\mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots \subset \mathcal{C}_i \subset \cdots \subset \mathcal{C}$ such that $\mathcal{C}_{i+1} \subset \mathcal{C}_i^+$ and $\bigcup_i \mathcal{C}_i = \mathcal{C}$. That the hermitian linear functional L on $\mathcal{C}^2$ is a flat extension means that

$$\mathcal{C} = \mathcal{B} + K_L(\mathcal{C}).$$

A first consequence of these hypotheses is the following property:

$$a \in K_L(\mathcal{C}) \quad \Leftrightarrow \quad \forall b \in \mathcal{C}, L(b^* a) = 0 \tag{1}$$
$$\Leftrightarrow \quad \forall b \in \mathcal{B}, L(b^* a) = 0. \tag{2}$$

Coherence. The flat extension property induces a structure of $K_L(\mathcal{C})$, which we are going to analyze.

We first show that $K_L(\mathcal{C})$ is stable by prolongation and intersection with $\mathcal{C}$.

Lemma 3.2. *$K_L(\mathcal{C})^+ \cap \mathcal{C} \subset K_L(\mathcal{C})$.*

Proof. For any element $\kappa \in K_L(\mathcal{C})^+$, we have

$$\kappa = \sum_{j \in J} a_j \kappa'_j,$$

with $J \subset I$, $\kappa'_j \in K_L(\mathcal{C})$. For any $b \in B$,

$$L(b^* \kappa) \quad = \quad \sum_{j \in J} L(b^* a_j \kappa'_j) = 0,$$

since $b^* a_j \in \mathcal{B}^+ \subset \mathcal{C}$ and $\kappa'_j \in K_L(\mathcal{C})$. The relation (2) implies that $\kappa \in K_L(\mathcal{C})$. $\square$

Remark 3.3. There is no loss of generality to assume that $1 \in \mathcal{B}'$. Otherwise $1 \in K_L(\mathcal{C})$. By Lemma 3.2, $\mathcal{C}_1 \subset (\mathcal{C}_0)^+ \cap \mathcal{C} \subset K_L(\mathcal{C})$. By induction, we deduce that $\forall l \in \mathbb{N}, \mathcal{C}_l \subset K_L(\mathcal{C})$, which implies that $\mathcal{C} = \bigcup \mathcal{C}_l = K_L(\mathcal{C})$. Consequently, $L \equiv 0$. But in this case the assertion of Theorem 3.1 is trivial.

Let $\mathcal{K} = (\mathcal{B}')^+ \cap K_L(\mathcal{C})$. The next lemma shows that $K_L(\mathcal{C})$ contains iterated prolongations of $\mathcal{K}$ intersected with $\mathcal{C}$. As we will see, $\mathcal{K}$ generates the left ideal $K_{\mathcal{L}}(\mathcal{C})$ of the extension $\mathcal{L}$.

Lemma 3.4. *$\mathcal{K}^{[2m-1]} \cap \mathcal{C} \subset K_L(\mathcal{C})$.*

Proof. Let $\kappa \in \mathcal{K}^{[2m-1]} \cap \mathcal{C}$. It is a sum of terms of the form $a_{i_1} \cdots a_{i_l} \kappa'$ with $l \leqslant 2m-1$, $i_k \in I$, $\kappa' \in \mathcal{K}$. We decompose $a_{i_1} \cdots a_{i_l} = \mathfrak{a}_1 \mathfrak{a}_2$ as a product of a term $\mathfrak{a}_1$ of index $\leqslant m$ and a term $\mathfrak{a}_2$ of index $m-1$. Then we have $\forall \mathfrak{b} \in \mathcal{B}$, $\langle \mathfrak{a}\kappa' | \mathfrak{b} \rangle = \langle \mathfrak{a}_2 \kappa' | \mathfrak{a}_1^* \mathfrak{b} \rangle$.

By hypothesis, $\mathcal{B}^{[m]} \subset \mathcal{C}$ and $\mathcal{K} \subset \mathcal{B}^{[1]}$, and $\mathfrak{a}_2 \kappa' \in \mathcal{K}^{[m-1]} \subset \mathcal{B}^{[m]} \subset \mathcal{C}$.

Let us prove that $\mathfrak{a}_2 \kappa' \in K_L(\mathcal{C})$: $\forall \mathfrak{b}' \in \mathcal{B}$, $\mathfrak{a}_2^* \mathfrak{b}' \in \mathcal{B}^{[m]} \subset \mathcal{C}$. Relation (1) implies that

$$\langle \mathfrak{a}_2 \kappa' | \mathfrak{b}' \rangle = \langle \kappa' | \mathfrak{a}_2^* \mathfrak{b}' \rangle = 0.$$

Relation (2) implies that $\mathfrak{a}_2 \kappa' \in K_L(\mathcal{C})$.

By hypothesis, we also have $\forall \mathfrak{b} \in \mathcal{B}$, $\mathfrak{a}_1^* \mathfrak{b} \in \mathcal{B}^{[m]} \subset \mathcal{C}$ and by relation (1),

$$\langle \mathfrak{a}_2 \kappa' | \mathfrak{a}_1^* \mathfrak{b} \rangle = 0 = \langle \mathfrak{a} \kappa' | \mathfrak{b} \rangle.$$

We deduce that $\forall \mathfrak{b} \in \mathcal{B}$, $\langle \kappa | \mathfrak{b} \rangle = 0$, which implies, by relation (2), that $\kappa \in K_L(\mathcal{C})$. This ends the proof of this lemma. $\square$

The GNS construction. We describe now a GNS-type construction, which allows us to extend L on $\mathcal{A}$. For $i \in I$, we define the operator

$$\begin{aligned} X_i : \mathcal{B} &\rightarrow \mathcal{B}, \\ b &\mapsto \pi(a_i b), \end{aligned}$$

and the map

$$\begin{aligned} \phi : \mathcal{F} &\rightarrow \mathcal{B}, \\ p &\mapsto p(X)(1). \end{aligned}$$

The kernel of ϕ is denoted $\mathcal{J}$. We easily check that $\mathcal{J}$ is a left ideal of $\mathcal{F} = \mathbb{C}\langle x_i, i \in I\rangle$. Let $B, C \subset \mathcal{F}$ be $*$-invariant linear subspaces of $\mathcal{F}$, connected to 1 such that $\sigma(B) = \mathcal{B}$ and $\sigma(C) = \mathcal{C}$. We denote by S a generating set of the two-sided ideal $\mathcal{I} = \ker \sigma$ such that $\delta(S) = \delta(\mathcal{A})$ is minimal.

Lemma 3.5. *$\forall p, q \in C, \phi(pq) = \sigma(p)\phi(q) + \kappa_1 \phi(q) + \kappa_0$ with $\kappa_0 \in \mathcal{K} \subset K_L(\mathcal{C}), \kappa_1 \in K_L(\mathcal{C})$. If l is the index of p, then $\phi(pq) - \sigma(p)\phi(q) \in \mathcal{K}^{[l-1]}$.*

Proof. We prove the property by the induction on the index l of p.

The property is obviously true for $l = 0$, in which case we can take $p = 1$ and $\phi(1) = \sigma(1) = 1$.

Let us assume that it is true for $l \in \mathbb{N}$ and let $p \in C$ of index $l + 1$. Then $\sigma(p) = \sum_{j \in J} a_j \sigma(p'_j)$ with $p'_j \in C$ of index $\leqslant l$. By the induction hypothesis, we have

$$\begin{aligned} \phi(pq) &= \sum_{j \in J} \pi(a_j \phi(p'_j q)) \\ &= \sum_{j \in J} a_j \phi(p'_j q) + \kappa_0 \\ &= \sum_{j \in J} a_j (\sigma(p'_j)\phi(q) + \kappa'_j \phi(q)) + \kappa_0 \\ &= \sigma(p)\phi(q) + \kappa_1 \phi(q) + \kappa_0, \end{aligned}$$

with $\kappa_0 \in \mathcal{K} \subset K_L(\mathcal{C}), \kappa'_j \in K_L(\mathcal{C})$ and $\kappa_1 = \sum_{j \in J} a_j \kappa'_j \in K_L(\mathcal{C})^+ \cap \mathcal{C}$. By Lemma 3.2, $\kappa_1 \in K_L(\mathcal{C})$. By the induction hypothesis, $\phi(p'_j q) - \sigma(p'_j)\phi(q) \in \mathcal{K}^{[l-1]}$. Therefore

$$\phi(pq) - \sigma(p)\phi(q) = \sum_{j \in J} a_j (\phi(p'_j q) - \sigma(p'_j)\phi(q)) + \kappa_0 \in (\mathcal{K}^{[l-1]})^+ = \mathcal{K}^{[l]},$$

which proves the induction hypothesis for $l + 1$ and concludes the proof of this lemma. □

Proposition 3.6. *$\forall g \in S, \forall b, b' \in B$, $\phi(bgb') = 0$.*

Proof. By Lemma 3.5, $\phi(gb') = \sigma(g)\phi(b') + \kappa = \kappa$ with $\kappa \in \mathcal{K}^{[2m-1]}$. As $\phi(gb') \in \mathcal{B}' \subset \mathcal{B}$, by Lemma 3.4, $\kappa \in K_L(\mathcal{C}) \cap \mathcal{B}' = \{0\}$. This implies that $\phi(gb') = 0$. Consequently, $\phi(bgb') = \phi(b(X)\phi(gb')) = 0$. □

Proposition 3.7. *$\forall p \in C$, $\phi(p) = \pi(\sigma(p))$.*

Proof. Let $p \in C$. Applying Lemma 3.5 with $q = 1$, we have $\phi(p) = \sigma(p) + \kappa$, $\kappa \in K_L(\mathcal{C})$.

As $\sigma(p) \in \mathcal{C}, \phi(p) \in \mathcal{B}'$ and $\kappa \in K_L(\mathcal{C})$, $\phi(p)$ is the projection of $\sigma(p)$ on $\mathcal{B}'$ along $K_L(\mathcal{C})$. In other words, $\phi(p) = \pi(\sigma(p))$, which proves the proposition. □

As $\mathcal{I} = \ker\sigma$ is the left-right ideal generated by S, this proposition shows that ϕ factors through a map $\overline{\phi}$ on $\mathcal{A} = \mathcal{F}/\mathcal{I}$. We prove now that $\ker\overline{\phi}$ is the left ideal generated by $\mathcal{K}$.

Proposition 3.8. $\sigma(\ker\phi) = \mathcal{A}\cdot\mathcal{K}$.

Proof. Let us prove by induction on the index l of p that $\forall p \in A, \phi(p) = \sigma(p) + \kappa$, with $\kappa \in \mathcal{A}\cdot\mathcal{K}$.

If p is of index 0, then $p = 1$ and $\phi(1) = 1 = \sigma(1) = 1$.

Let us assume that it is true for $l \in \mathbb{N}$ and let $p \in A$ of index $l + 1$. Then $p = \sum_{j\in J} x_j p'_j$ with $p'_j \in A$ of index $\leqslant l$. Then we have

$$\begin{aligned}
\phi(p) &= \sum_{j\in J} \pi(a_j\phi(p'_j)) \\
&= \sum_{j\in J} a_j\phi(p'_j) + \kappa_0 \\
&= \sum_{j\in J} a_j(\sigma(p'_j) + \kappa'_j) + \kappa_0 \\
&= \sigma(p) + \kappa,
\end{aligned}$$

with $\kappa_0 \in \mathcal{K}$ and $\kappa = \sum_{j\in J} a_j\kappa'_j + \kappa_0 \in \mathcal{A}\cdot\mathcal{K}$. This proves the induction hypothesis for $l + 1$.

We deduce that if $a \in \ker\phi$, then $\phi(a)$=0 and $\sigma(a) \in \mathcal{A}\cdot\mathcal{K}$, so that $\sigma(\ker\phi) \subset \mathcal{A}\cdot\mathcal{K}$.

Conversely, $\forall \mathfrak{a} \in \mathcal{A}, \forall\kappa \in \mathcal{K}$, there exist $a \in \mathcal{F}$, $k \in C$ such that $\sigma(a) = \mathfrak{a}$ and $\sigma(k) = \kappa$. By Proposition 3.7, $\phi(k) = \pi(\kappa) = 0$ and $\phi(ak) = a(X)\phi(k) = 0$. This shows that $ak \in \ker\phi$ and $\mathfrak{a}\kappa = \sigma(ak) \in \sigma(\ker\phi)$. Therefore $\mathcal{A}\cdot\mathcal{K} \subset \sigma(\ker\phi)$. □

The extension of L**.** We define the extension $\mathcal{L}$ of L on $\mathcal{F}$ by

$$\begin{aligned}
\mathcal{L} : \mathcal{F} &\to \mathbb{C}, \\
a &\mapsto L(\phi(a)).
\end{aligned} \tag{3}$$

We recall that $\mathcal{A} = \mathcal{F}/\mathcal{I}$ where $\mathcal{I}$ is the left and right ideal degenerated by the elements of S. By Proposition 3.6, $\phi(\mathcal{I}) = 0$ and $\mathcal{L}$ induces a linear form on $\mathcal{A} = \mathcal{F}/\mathcal{I}$, that we still denote $\mathcal{L}$. If $\mathfrak{a} = \sigma(a)$ with $a \in \mathcal{F}$, then $\mathcal{L}(\mathfrak{a}) = L(\phi(a)) = L(a(X)(1))$.

We first check that $\mathcal{L}$ is an extension of L.

Proposition 3.9. $\forall\mathfrak{a},\mathfrak{b} \in \mathcal{C}, \mathcal{L}(\mathfrak{a}\mathfrak{b}) = L(\mathfrak{a}\mathfrak{b})$.

Proof. Let $\mathfrak{a} = \sigma(a), \mathfrak{b} = \sigma(b) \in \mathcal{C}$ with $a, b \in C$. By Proposition 3.7,

$$\begin{aligned}
\mathfrak{a} &= \pi(a) + \kappa = \phi(a) + \kappa, \\
\mathfrak{b} &= \pi(a) + \kappa' = \phi(b) + \kappa',
\end{aligned}$$

with $\kappa, \kappa' \in K_L(\mathcal{C})$. Relation (1) implies that we have

$$L(\mathfrak{a}\mathfrak{b}) = L((\phi(a) + \kappa)(\phi(b) + \kappa')) = L(\phi(a)\phi(b)).$$

We can therefore assume that $\mathfrak{a} = \sigma(a) = \phi(a), \mathfrak{b} = \sigma(b) = \phi(b) \in \mathcal{B}$ with $a, b \in B$ in order to prove that $\mathcal{L}(\mathfrak{ab}) = L(\phi(ab)) = L(\mathfrak{ab})$.

By Lemma 3.5, $\phi(ab) = \mathfrak{a}\phi(b) + \kappa_1\phi(b) + \kappa_0$ with $\kappa_0, \kappa_1 \in K_L(\mathcal{C})$. By Proposition 3.7, $\phi(b) = \pi(\mathfrak{b}) = \mathfrak{b} + \kappa'$ with $\kappa' \in K_L(\mathcal{C})$. We deduce that

$$\phi(ab) = \mathfrak{ab} + \mathfrak{a}\kappa' + \kappa_1\phi(b) + \kappa_0.$$

Relation (1) implies that $L(\mathfrak{a}\kappa' + \kappa_1\phi(b) + \kappa_0) = 0$, which proves that $\mathcal{L}(\mathfrak{ab}) = L(\phi(ab)) = L(\mathfrak{ab})$. □

As a consequence of the previous results, we deduce that $\mathcal{L}$ is a flat extension on $\mathcal{A}$ with respect to $\mathcal{B}$.

Proposition 3.10. *$K_{\mathcal{L}}(\mathcal{A}) = \mathcal{A} \cdot \mathcal{K}$ and $\mathcal{A} = \mathcal{B}' \oplus K_{\mathcal{L}}(\mathcal{A})$.*

Proof. Let $\kappa \in \mathcal{K}$. $\forall \mathfrak{a} \in \mathcal{A}$, there exists $a \in \mathcal{F}$, such that $\sigma(a) = \mathfrak{a}$. We deduce that

$$\mathcal{L}(\mathfrak{a}\kappa) = L(\tilde{\phi}(\mathfrak{a}\kappa)) = L(a(X)\tilde{\phi}(\kappa)) = 0,$$

since $\tilde{\phi}(\kappa) = \pi(\kappa) =0$ by Proposition 3.7. We deduce that $\kappa \in K_{\mathcal{L}}(\mathcal{A})$. As $K_{\mathcal{L}}(\mathcal{A})$ is a left ideal, we have $\mathcal{A} \cdot \mathcal{K} \subset K_{\mathcal{L}}(\mathcal{A})$.

Conversely, let $\kappa \in K_{\mathcal{L}}(\mathcal{A})$. Then $\forall \mathfrak{a} \in \mathcal{C}$, there exists $a \in \mathcal{C}$, such that $\sigma(a) = \mathfrak{a}$. By Proposition 3.7, $\tilde{\phi} \circ \tilde{\phi}(\kappa) = \pi(\tilde{\phi}(\kappa)) = \tilde{\phi}(\kappa)$ and by Proposition 3.9, we have

$$\mathcal{L}(\mathfrak{a}\kappa) = L(a(X)\tilde{\phi}(\kappa)) = L(\tilde{\phi}(\mathfrak{a}\tilde{\phi}(\kappa))) = \mathcal{L}(\mathfrak{a}\phi(\kappa)) = L(\mathfrak{a}\phi(\kappa)) = 0.$$

This implies that $\phi(\kappa) \in K_L(\mathcal{C}) \cap \mathcal{B}' = \{0\}$. We deduce that $\kappa \in \ker \tilde{\phi} = \mathcal{A} \cdot \mathcal{K}$ (by Proposition 3.8). This proves the reverse inclusion $K_{\mathcal{L}}(\mathcal{A}) \subset \mathcal{A} \cdot \mathcal{K}$.

Let $\mathfrak{a} \in \mathcal{A}$. It can be decomposed as

$$\mathfrak{a} = \tilde{\phi}(\mathfrak{a}) + \mathfrak{a} - \tilde{\phi}(\mathfrak{a}),$$

with $\tilde{\phi}(\mathfrak{a}) \in \mathcal{B}'$ and $\mathfrak{a} - \tilde{\phi}(\mathfrak{a}) \in \ker \tilde{\phi} = \mathcal{A} \cdot \mathcal{K} = K_{\mathcal{L}}(\mathcal{A})$. This shows that $\mathcal{A} = \mathcal{B}' + K_{\mathcal{L}}(\mathcal{A})$.

To prove that the sum is direct, we consider an element $\mathfrak{b} \in \mathcal{B}' \cap K_{\mathcal{L}}(\mathcal{A})$. Then $\tilde{\phi}(\mathfrak{b}) = \mathfrak{b}$ by Proposition 3.7 and $\tilde{\phi}(\mathfrak{b}) = 0$, since $\mathfrak{b} \in K_{\mathcal{L}}(\mathcal{A}) = \mathcal{A} \cdot \mathcal{K} = \ker \tilde{\phi}$. We deduce that $\mathfrak{b} = 0$ and therefore that $\mathcal{A} = B \oplus K_{\mathcal{L}}(\mathcal{A})$. □

The next proposition shows the uniqueness of the extension.

Proposition 3.11. *There exists a unique hermitian linear form $\mathcal{L} \in \mathcal{A}^*$ which extends L and is a flat extension with respect to $\mathcal{B}$.*

Proof. If $\mathcal{L}$ extends L and is a flat extension with respect to $\mathcal{B}$, then $\mathcal{A} = \mathcal{B} + K_{\mathcal{L}}(\mathcal{A})$. As $\mathcal{L}$ is hermitian, $K_{\mathcal{L}}(\mathcal{A})^* = K_{\mathcal{L}}(\mathcal{A})$. Then $\forall \kappa \in \mathcal{K}, \forall \mathfrak{a} \in \mathcal{A}$, $\mathfrak{a} = \mathfrak{b} + \kappa'$ with $\kappa' \in K_{\mathcal{L}}(\mathcal{A})$ and

$$\mathcal{L}(\mathfrak{a}\,\kappa) = \mathcal{L}(\mathfrak{b}\,\kappa) + \mathcal{L}(\kappa'\,\kappa) = \mathcal{L}(\mathfrak{b}\,\kappa) + \overline{\mathcal{L}(\kappa^*\,\kappa'^*)} = L(\mathfrak{b}\,\kappa) = 0,$$

since $\kappa'^* \in K_{\mathcal{L}}(\mathcal{A})$, $\mathfrak{b}\,\kappa \in \mathcal{C}^2$ and $\kappa \in K_L(\mathcal{C})$. This proves that $\mathcal{A} \cdot \mathcal{K} \subset K_{\mathcal{L}}(\mathcal{A})$.

For any $\mathfrak{a} \in \mathcal{A}$, let $\mathfrak{b} = \tilde{\phi}(\mathfrak{a}) \in \mathcal{B}$ so that $\mathfrak{a} - \mathfrak{b} \in \mathcal{A} \cdot \mathcal{K} \subset K_{\mathcal{L}}(\mathcal{A})$. This implies that

$$\mathcal{L}(\mathfrak{a}) = \mathcal{L}(\mathfrak{b}) = L(\mathfrak{b}) = L(\tilde{\phi}(\mathfrak{a})).$$

This shows that $\mathcal{L}$ coincides with the linear form defined in (3) and thus that the extension $\mathcal{L}$ is unique. □

Finally, we show that the positivity of L can also be extended to $\mathcal{L}$.

Proposition 3.12. *If* $\forall \mathfrak{a} \in \mathcal{C}, L(\mathfrak{a}^*\mathfrak{a}) \geqslant 0$, *then* $\forall \mathfrak{a} \in \mathcal{A}, \mathcal{L}(\mathfrak{a}^*\mathfrak{a}) \geqslant 0$.

Proof. $\forall \mathfrak{a} \in \mathcal{A}$, let $\mathfrak{b} = \tilde{\phi}(\mathfrak{a})$ so that $\mathfrak{a} - \mathfrak{b} \in \ker \tilde{\phi} = \mathcal{A} \cdot \mathcal{K} = K_{\mathcal{L}}(\mathcal{A})$. As $\tilde{\phi}(\mathfrak{a}^*) = \tilde{\phi}(\mathfrak{a})^*$ and $\forall \mathfrak{b} \in \mathcal{A}$, $\tilde{\phi}(\mathfrak{a}\mathfrak{b})^* = \tilde{\phi}(\mathfrak{b}^*\mathfrak{a}^*)$, we deduce from Proposition 3.9 that

$$\begin{aligned}\mathcal{L}(\mathfrak{a}^*\mathfrak{a}) &= \mathcal{L}((\mathfrak{b}^* + (\mathfrak{a}^* - \mathfrak{b}^*))(\mathfrak{b} + (\mathfrak{a} - \mathfrak{b}))) = \mathcal{L}(\mathfrak{b}^*\mathfrak{b}) + \mathcal{L}((\mathfrak{a}^* - \mathfrak{b}^*)\mathfrak{b}) \\ &= \mathcal{L}(\mathfrak{b}^*\mathfrak{b}) + \overline{\mathcal{L}(\mathfrak{b}^*(\mathfrak{a} - \mathfrak{b}))} = \mathcal{L}(\mathfrak{b}^*\mathfrak{b}) = L(\mathfrak{b}^*\mathfrak{b}).\end{aligned}$$

As L is positive, $\mathcal{L}(\mathfrak{a}^*\mathfrak{a}) = L(\mathfrak{b}^*\mathfrak{b}) \geqslant 0$ so that $\mathcal{L}$ is also positive. □

Representation of $\mathcal{A}$. We can now construct the representation of $\mathcal{A}$ on the vector space $\mathcal{B}'$, using the decomposition of Proposition 3.10. Let us denote by $\pi_{\mathcal{L}}$ the projection operator of $\mathcal{A}$ on $\mathcal{B}'$ along $K_{\mathcal{L}}(\mathcal{A})$. By definition of $K_{\mathcal{L}}(\mathcal{A})$, for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{A}$, we have $\langle \pi_{\mathcal{L}}(\mathfrak{a}), \pi_{\mathcal{L}}(\mathfrak{b}) \rangle_{\mathcal{L}} = \mathcal{L}(\mathfrak{b}^*\mathfrak{a})$. For $\mathfrak{a} \in \mathcal{A}$, let $\rho_{\mathcal{L}}(\mathfrak{a})$ denote the linear operator on $\mathcal{B}'$ defined by

$$\rho_{\mathcal{L}}(\mathfrak{a})\mathfrak{b} = \pi_{\mathcal{L}}(\mathfrak{a}\mathfrak{b}), \quad \text{where} \quad \mathfrak{a} \in \mathcal{A},\ \mathfrak{b} \in \mathcal{B}'.$$

Proposition 3.13. $\rho_{\mathcal{L}}$ *is the $*$-representation of $\mathcal{A}$ on $(\mathcal{B}', \langle \cdot, \cdot \rangle_{\mathcal{L}})$ which is associated with the functional $\mathcal{L}$.*

Proof. By construction, $\forall \mathfrak{a} \in \mathcal{A}$, $\rho_{\mathcal{L}}(\mathfrak{a})$ is a linear operator of $\mathcal{B}'$. We have to prove first that $\forall \mathfrak{a}, \mathfrak{b} \in \mathcal{A}$, $\forall \mathfrak{c} \in \mathcal{B}'$, $\rho_{\mathcal{L}}(\mathfrak{a}\mathfrak{b})(\mathfrak{c}) = \rho_{\mathcal{L}}(\mathfrak{a}) \circ \rho_{\mathcal{L}}(\mathfrak{b})(\mathfrak{c})$:

$$\begin{aligned}\rho_{\mathcal{L}}(\mathfrak{a}\mathfrak{b})(\mathfrak{c}) &= \pi_{\mathcal{L}}(\mathfrak{a}\mathfrak{b}\mathfrak{c}) = \pi_{\mathcal{L}}(\mathfrak{a}\,\pi_{\mathcal{L}}(\mathfrak{b}\mathfrak{c})) + \pi_{\mathcal{L}}(\mathfrak{a}\,\kappa) \quad (\kappa \in \mathcal{A} \cdot \mathcal{K}) \\ &= \pi_{\mathcal{L}}(\mathfrak{a}\,\pi_{\mathcal{L}}(\mathfrak{b}\mathfrak{c})) = \rho_{\mathcal{L}}(\mathfrak{a}) \circ \rho_{\mathcal{L}}(\mathfrak{b})(\mathfrak{c}),\end{aligned}$$

since $\mathfrak{a}\,\kappa \in \mathcal{A} \cdot \mathcal{K} = \ker \pi_{\mathcal{L}}$ (Proposition 3.10).

We have to prove secondly that $\forall \mathfrak{a} \in \mathcal{A}$, $\forall \mathfrak{b}, \mathfrak{c} \in \mathcal{B}'$, $\langle \rho_{\mathcal{L}}(\mathfrak{a})\mathfrak{b}, \mathfrak{c} \rangle_{\mathcal{L}} = \langle \mathfrak{b}, \rho_{\mathcal{L}}(\mathfrak{a}^*)\mathfrak{c} \rangle_{\mathcal{L}}$:

$$\begin{aligned}\langle \rho_{\mathcal{L}}(\mathfrak{a})\mathfrak{b}, \mathfrak{c} \rangle_{\mathcal{L}} &= \mathcal{L}(\mathfrak{c}^*\pi_{\mathcal{L}}(\mathfrak{a}\mathfrak{b})) \\ &= \mathcal{L}(\mathfrak{c}^*\mathfrak{a}\mathfrak{b}) + \mathcal{L}(\mathfrak{c}^*\,\kappa) = \mathcal{L}(\mathfrak{c}^*\mathfrak{a}\mathfrak{b}) \qquad (\kappa \in K_{\mathcal{L}}(\mathcal{A})) \\ &= \overline{\mathcal{L}(\mathfrak{b}^*\mathfrak{a}^*\mathfrak{c})} = \overline{\mathcal{L}(\mathfrak{b}^*\pi_{\mathcal{L}}(\mathfrak{a}^*\mathfrak{c}))} + \overline{\mathcal{L}(\mathfrak{b}^*\kappa')} \quad (\kappa' \in K_{\mathcal{L}}(\mathcal{A})) \\ &= \overline{\mathcal{L}(\mathfrak{b}^*\rho_{\mathcal{L}}(\mathfrak{a}^*)\mathfrak{c})} = \overline{\langle \rho_{\mathcal{L}}(\mathfrak{a}^*)\mathfrak{c}, \mathfrak{b} \rangle_{\mathcal{L}}} = \langle \mathfrak{b}, \rho_{\mathcal{L}}(\mathfrak{a}^*)\mathfrak{c} \rangle_{\mathcal{L}}.\end{aligned}$$

This concludes the proof that $\rho_{\mathcal{L}}$ is a $*$-representation of $\mathcal{A}$ on $\mathcal{B}'$. Since $\mathcal{L}(\mathfrak{a}) = \langle \rho_{\mathcal{L}}(\mathfrak{a})1, 1 \rangle_{\mathcal{L}}$ for all $\mathfrak{a} \in \mathcal{A}$, $\rho_{\mathcal{L}}$ is the $*$-representation associated with $\mathcal{L}$. □

4. Applications

Let π be a $*$-representation of $\mathcal{A}$ on a unitary space $(V, \langle \cdot, \cdot \rangle)$. For any $v \in V$, the linear functional L_v on A defined by $L_v(\cdot) = \langle \pi(\cdot)v, v \rangle$ is positive and $\mathcal{K}_v := \{a \in \mathcal{A} : \pi(a)v = 0\}$ is a two-sided $*$-ideal of $\mathcal{A}$. Let $\mathcal{B}$ and $\mathcal{C}$ be $*$-invariant linear subspaces of $\mathcal{A}$ such that $1 \in \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{C}$. Clearly, $\mathcal{K}_L(\mathcal{C}) = \mathcal{K}_v \cap \mathcal{C}$. Then the restriction of L_v to $\mathcal{C}^2$ is a flat extension with respect to $\mathcal{B}$ if and only if $\mathcal{C} = \mathcal{B} + \mathcal{K}_v \cap C$.

Suppose in addition that $V_v := \pi(\mathcal{A})v$ is finite dimensional. Then $\mathcal{A}/\mathcal{K}_v$ is finite dimensional, so we can choose a finite-dimensional $*$-invariant subspace $\mathcal{B}$ containing 1 such that $\mathcal{A} = \mathcal{B} + \mathcal{K}_v$. Then, for any $*$-invariant subspace $\mathcal{C}$ of $\mathcal{A}$ which contains $\mathcal{B}$, the restriction of L_v to $\mathcal{C}^2$ is a flat extension with respect to $\mathcal{B}$. This provides a large class of examples of flat extensions.

We now develop the three applications of Theorem 3.1 mentioned in the Introduction.

4.1. **Truncated moment problem on cylinder sets.** Let $\mathcal{A}$ be the polynomial $*$-algebra $\mathbb{C}[x_1,\dots,x_d,y]$ in $d+1$ hermitian variables $x_1,\dots,,x_d,y$. The algebra $\mathcal{A}$ is the quotient of the free $*$-algebra $\mathcal{F}=\mathbb{C}\langle x_1,\dots,x_d,y\rangle$ by the commutation relations $x_ix_j-x_jx_i=0$, $x_iy-yx_i=0$, so that $\delta(\mathcal{A})=2$. For $k\in\mathbb{N}$, let $\mathcal{A}_k$ be the linear span of $x^\alpha\mathbb{C}[y]$, where $\alpha\in\mathbb{N}_0^d$, $|\alpha|\le k$.

Proposition 4.1. *Let $m\in\mathbb{N}$, $\mathcal{B}=\mathcal{A}_m$ and $\mathcal{C}=\mathcal{A}_{m+1}$. Suppose that L is a positive linear functional on $\mathcal{C}^2$ which is a flat extension with respect to $\mathcal{B}$. Then there exist finitely points $t_1,\dots,t_k\in\mathbb{R}^d$, where $k\le\binom{d+1+m}{m}$, and a positive Borel measure μ on $\mathbb{R}^{d+1}$ supported by the set $\bigcup_{j=1}^k t_j\times\mathbb{R}$ such that*

$$L(p)=\int_{\mathbb{R}^d\times\mathbb{R}} p(x,y)\,d\mu(x,y)\quad \textit{for}\quad p\in\mathcal{C}^2=\mathcal{A}_{2m+2}.$$

Proof. Since $\delta(\mathcal{A})=2$ and $\mathcal{B}^{[1]}=\mathcal{A}_m^{[1]}=\mathcal{A}_{m+1}=\mathcal{C}$, the assumptions of Theorem 3.1 are fulfilled. Hence L has an extension to a positive linear functional $\tilde L$ on $\mathcal{A}$ which is a flat extension with respect to $\mathcal{B}$. Let L_0 and $\tilde L_0$ denote the restrictions of L and $\tilde L$, respectively, to the $*$-subalgebra $\mathbb{C}[x_1,\dots,x_d]$ of $\mathcal{A}$. Then $\tilde L_0$ is a flat extension of L_0 with respect to $\mathcal{B}\cap\mathbb{C}[x_1,\dots,x_d]$. Therefore, by the theorem of Curto and Fialkow [2], $\tilde L_0$ is given by a k-atomic measure μ_0 on $\mathbb{R}^d$, where $k\le\binom{d+1+m}{m}$. Let $t_1,\dots,t_k$ denote the atoms of μ_0. For $f\in\mathcal{A}$, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}\left|\tilde L\big(\|x-t_1\|^2\cdots\|x-t_k\|^2f\big)\right|^2 &\le \tilde L\big(\|x-t_1\|^4\cdots\|x-t_k\|^4\big)\tilde L(f^2)\\ &=\tilde L_0\big(\|x-t_1\|^4\cdots\|x-t_k\|^4\big)\tilde L(f^2)=0.\end{aligned}$$

Hence $\tilde L$ vanishes on the ideal $\mathcal{J}$ generated by the polynomial $\|x-t_1\|^2\cdots\|x-t_k\|^2$. The zero set of the ideal $\mathcal{J}$ is the set $\{t_1,\dots,t_k\}$. Hence (by [6, Note 2.1.8]) $\mathcal{J}+\sum\mathcal{A}^2$ is the preorder of the semi-algebraic set $K:=\bigcup_{j=1}^k t_j\times\mathbb{R}$ in $\mathbb{R}^{d+1}$. Since K is a cylinder with finite, hence compact, base set $\bigcup_{j=1}^k t_j$ in $\mathbb{R}^d$, it follows from Corollary 10 in [10] (see also [3]) that K has property (SMP), that is, $\tilde L$ is given by a positive Borel measure μ supported by K. □

4.2. **Truncated moment problem for matrices of polynomials.** Let $\mathcal{A}$ be the unital $*$-algebra of $n\times n$ matrices with entries from $\mathbb{C}[x_1,\dots,x_d]$. Let e_{ij} denote the corresponding matrix units. We have $e_{ij}^*=e_{ji}$. The algebra $\mathcal{A}$ is the quotient of the free unital $*$-algebra $\mathcal{F}=\mathbb{C}\langle x_1,\dots,x_d,e_{11},\dots,e_{nn}\rangle$ by the ideal generated by the commutation relations between the variables x_i, the commutation relations $x_ie_{jk}-e_{jk}x_i=0$ and the product relations $e_{ij}e_{jl}=e_{il}$, $e_{ij}e_{kl}=0$ if $k\ne l$. This implies that $\delta(\mathcal{A})=2$.

First we prove a simple well-known lemma.

Lemma 4.2. *Let ρ be an irreducible $*$-representation of $\mathcal{A}$ on a finite-dimensional unitary space V. Then there exist a unitary operator U of $\mathbb{C}^d$ onto V and a point $t_0\in\mathbb{R}^d$ such that $\rho((p_{jk}))=U((p_{jk}(t_0)))U^{-1}$ for all matrices $(p_{jk})\in\mathcal{A}$.*

Proof. Since the unit matrix E is the unit element of $\mathcal{A}$, $\rho(E)=I$ by Definition 2.4. Hence, for any $p\in\mathbb{C}[x_1,\dots,x_d]$, the operator $\rho(pE)$ belongs to the commutant of $\rho(\mathcal{A})$. Therefore, since ρ is irreducible, there is a complex number $\chi(p)$ such that $\rho(pE)=\chi(p)I$. Clearly, $p\to\rho(pE)$ is a $*$-homomorphism. Hence the map $p\to\chi(p)$

is a character on the $*$-algebra $\mathbb{C}[x_1,\dots,x_d]$, so there is a point $t_0\in\mathbb{R}^d$ such that $\chi(p)=p(t_0)$ for $p\in\mathbb{C}[x_1,\dots,x_d]$.

Let $M_d(\mathbb{C})$ denote the $*$-subalgebra of constant matrices in $\mathcal{A}$. For $(p_{jk})\in\mathcal{A}$, we have $\rho((p_{jk}))=\sum_{j,k}\rho(p_{jk}E)\rho(e_{jk})$. This implies that $\rho(\mathcal{A})$ and $\rho(M_d(\mathbb{C}))$ have the same commutants. Therefore, $\rho_{|M_d(\mathbb{C})}$ is an irreducible $*$-representation of the matrix algebra $M_d(\mathbb{C})$ and hence unitarily equivalent to the identity representation. That is, there is a unitary U of $\mathbb{C}^d$ onto V such that $\rho(e_{jk})=Ue_{jk}U^{-1}$ for $j,k=1,\dots,d$. Then, for $(p_{jk})\in\mathcal{A}$, we derive

$$\rho((p_{jk}))=\sum_{j,k=1}^{d}\rho(p_{jk}E)\rho(e_{jk})=\sum_{j,k=1}^{d}p_{jk}(t_0)Ue_{jk}U^{-1}$$
$$=U\Big(\sum_{j,k=1}^{d}p_{jk}(t_0)e_{jk}\Big)U^{-1}=U(p_{jk}(t_0))U^{-1}.$$

□

Fix $k\in\mathbb{N}$. We denote by $\mathcal{A}_k$ the span of all elements $x^\alpha e_{ij}$, where $|\alpha|\le k$ and $i,j=1,\dots,n$. Here $x^\alpha e_{ij}$ denotes the matrix with entry x^α at the (i,j)-place and zero otherwise.

Proposition 4.3. *Suppose that $m\in\mathbb{N}$, $\mathcal{B}=\mathcal{A}_m$, $\mathcal{C}=\mathcal{A}_{m+1}$ and L is a positive linear functional on $\mathcal{C}^2$ which is a flat extension with respect to $\mathcal{B}$. Then there exist points $t_i\in\mathbb{R}^d$ and vectors $u_i=(u_{li})\in\mathbb{C}^d$, $i=1,\dots,r$, $r\in\mathbb{N}$, such that*

$$L((p_{jk}))=\sum_{j,k=1}^{d}\sum_{i=1}^{r}p_{jk}(t_i)u_{ki}\overline{u_{ji}}\quad \textit{for}\quad (p_{jk})\in\mathcal{C}^2=\mathcal{A}_{2m+2}. \tag{4}$$

Proof. As $\delta(\mathcal{A})=2$ and $\mathcal{B}^{[1]}=\mathcal{A}_m^{[1]}=\mathcal{A}_{m+1}=\mathcal{C}$, by Theorem 3.1, L has a flat extension to a positive linear functional $\tilde{L}$ on the whole algebra $\mathcal{A}$. Let $\rho_{\tilde{L}}$ denote the GNS representation of $\tilde{L}$. Since $\rho_{\tilde{L}}$ acts on a subspace of the finite-dimensional space $\mathcal{B}$, it is an orthogonal direct sum of finite-dimensional irreducible $*$-representations ρ_i acting on unitary spaces V_i, $i=1,\dots,r$. Each representation ρ_i is of the form described in Lemma 4.2. For $(p_{jk})\in\mathcal{A}$, we then obtain

$$\tilde{L}((p_{jk}))=\langle\rho_{\tilde{L}}((p_{jk}))v,v\rangle_V=\sum_{i=1}^{r}\langle\rho_i((p_{jk}))v_i,v_i\rangle_{V_i}=\sum_{i=1}^{r}\sum_{j,k=1}^{d}p_{jk}(t_i)u_{ki}\overline{u_{ji}},$$

which implies (4), since $\tilde{L}$ is an extension of L. Here v is the direct sum of vectors v_i. Further, $u_i=U_i^{-1}v_i$ and t_i denotes the point and U_i is the unitary from Lemma 4.2 applied to the irreducible representation ρ_i. □

4.3. **Truncated moment problem for enveloping algebras.** Let $\mathfrak{g}$ be a real finite-dimensional Lie algebra. Recall that the universal enveloping algebra $\mathcal{A}:=\mathcal{E}(\mathfrak{g})$ of $\mathfrak{g}$ is a complex unital $*$-algebra with involution determined by the requirement $(iy)^*=iy$ for $y\in\mathfrak{g}$. Fix a basis $\{y_1,\dots,y_d\}$ of the real vector space $\mathfrak{g}$. Then there are real numbers c_{jkl}, $j,k,l=1,\dots,d$, the structure constants of the Lie algebra $\mathfrak{g}$, such that

$$[y_j,y_k]=\sum_l c_{jkl}y_l.$$

The $*$-algebra $\mathcal{E}(\mathfrak{g})$ is the quotient algebra of the free $*$-algebra $\mathcal{F}(x_1, \ldots, , x_d)$, the two-sided $*$-ideal $\mathcal{I}$ generated by the set $\mathcal{I}_G$ of elements

$$x_j x_k - x_k x_j - \sum_l c_{jkl} i x_l, \quad l = 1, \ldots, d, \tag{5}$$

where the quotient map σ is given by $\sigma(x_j) = iy_j$, $j = 1, \ldots, d$. We deduce that $\delta(\mathcal{A}) \leq 2$. Further, by the Poincare-Birkhoff-Witt theorem, the set

$$\{y_1^{n_1} \ldots y_d^{n_d}; (n_1, \ldots, n_d) \in N_0^d\}$$

is a vector space basis of $\mathcal{E}(\mathfrak{g})$. For $m \in \mathbb{N}$, let $\mathcal{A}_m$ denote the linear span of elements $y_1^{n_1} \ldots y_d^{n_d}$, where $n_1 + \ldots + n_d \leq m$.

Proposition 4.4. *Let $\mathcal{B} = \mathcal{A}_m$ and $\mathcal{C} = \mathcal{A}_{m+1}$. Suppose that L is a positive linear functional on $\mathcal{C}^2$ which is a flat extension with respect to $\mathcal{B}$. Then L has an extension to a positive linear functional $\tilde{L}$ on $\mathcal{A}$ and $\tilde{L}$ is a flat extension with respect to $\mathcal{B}$.*

Proof. Since the ideal $\mathcal{I}$ is generated by the quadratic elements in (5), $\delta(\mathcal{A}) \leq 2$ and $\mathcal{B}^{[1]} = \mathcal{A}_m^{[1]} = \mathcal{A}_{m+1} = \mathcal{C}$, the assumptions of Theorem 3.1 are fulfilled which gives the assertion. □

Let $\mathcal{R}_{\text{fin}}(\mathfrak{g})$ denote the family of all $*$-representations of the $*$-algebra $\mathcal{E}(\mathfrak{g})$ acting on *finite-dimensional* Hilbert spaces. By Theorem 3.1, the GNS representation $\rho_{\tilde{L}}$ associated with the positive functional $\tilde{L}$ on $\mathcal{E}(\mathfrak{g})$ acts on a subspace of the finite-dimensional space $\mathcal{B}$. Hence $\rho_{\tilde{L}}$ belongs to $\mathcal{R}_{\text{fin}}(\mathfrak{g})$ and there is a vector v in the representation space of $\rho_{\tilde{L}}$ such that

$$L(a) = \langle \rho_{\tilde{L}}(a)v, v\rangle \quad \text{for} \quad a \in \mathcal{C}^2 = \mathcal{A}_{2m+2}. \tag{6}$$

Let G be the simply connected Lie group which has the Lie algebra $\mathfrak{g}$. Then the $*$-representation $\rho_{\tilde{L}}$ of $\mathcal{E}(\mathfrak{g})$ exponentiates to a unitary representation U of G. That is, we have $\rho_{\tilde{L}} = dU$ and equation (6) can be considered as a solution of a truncated non-commutative moment problem for the enveloping algebra $\mathcal{E}(\mathfrak{g})$; see e.g. [8]. Note that it may happen that there is no non-trivial finite-dimensional $*$-representation of $\mathcal{E}(\mathfrak{g})$. However, if the Lie group G is *compact*, all irreducible unitary representations of G are finite dimensional, so there is a rich theory of truncated non-commutative moment problems for $\mathcal{E}(\mathfrak{g})$.

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