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CHARACTERIZING τ-TILTING FINITE ALGEBRAS WITH RADICAL SQUARE ZERO

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ABSTRACT. In this paper, we give a characterization of τ-tilting finite algebras with radical square zero in terms of the separated quivers, which is an analog of a famous characterization of representation-finite algebras with radical square zero due to Gabriel.

1. Introduction

The Auslander-Reiten translation τ is a basic operation in representation theory (see Section 2). For a given finite dimensional algebra Λ, it gives a bijection from indecomposable non-projective Λ-modules to indecomposable non-injective Λ-modules. Using this notion, Auslander and Smalø [AS] (see also [Sk, ASS]) studied a class of modules, which are nowadays called τ-rigid, as those Λ-modules X satisfying $\mathrm{Hom}_\Lambda(X, \tau X) = 0$. From the perspective of tilting mutation theory, the authors in [AIR] introduced the notion of (support) τ-tilting modules as a special class of τ-rigid modules. They correspond bijectively with many important objects in representation theory, *i.e.*, functorially finite torsion classes, two-term silting complexes and cluster-tilting objects in a special cases.

It is important to classify τ-tilting finite algebras, that is, algebras having only finitely many indecomposable τ-rigid modules, or equivalently basic support τ-tilting modules (see [DIJ]). Recently, τ-rigid modules were studied by several authors (Jasso [Ja], Mizuno [Mi] and Malicki, de la Peña, and Skowroński [MPS]). Also a paper by Zhang [Zh] appeared on arXiv after we obtained our key Theorem C. His results and methods are quite different from ours.

In this paper, we study τ-rigid modules over algebras with radical square zero, which provide one of the most fundamental classes of algebras in representation theory (*e.g.*, work of Yoshii [Yo] in 1956 and Gabriel [Ga] in 1972). For an algebra Λ with radical square zero, an important role is played by the path algebra $\Delta = KQ^{\mathrm{s}}$ of the separated quiver Q^{s}, which is a quiver defined by the vertex set $Q_0^{\mathrm{s}} = \{i^+, i^- \mid i \in Q_0\}$ and the arrow set $Q_1^{\mathrm{s}} = \{i^+ \to j^- \mid i \to j \text{ in } Q_1\}$ for the vertex set Q_0 and the arrow set Q_1 of the quiver Q for Λ. For example, for a given quiver $Q = (\, 1 \rightleftarrows 2 \,)$, we have the separated quiver $Q^s = (\, 1^+ \longrightarrow 2^- \qquad 2^+ \longrightarrow 1^- \,)$. The representation theory of Λ is very close to that of Δ. Precisely speaking, there exists an equivalence

$$F : \underline{\mathsf{mod}}\Lambda \to \underline{\mathsf{mod}}\Delta,$$

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where $\underline{\mathsf{mod}}\Lambda$ is the stable category, which is the quotient category of $\mathsf{mod}\Lambda$ modulo the ideal generated by all projective Λ-modules. In particular, we have the following famous theorem characterizing representation-finiteness.

Theorem A ([Ga, ARS]). *Let Λ be a finite dimensional algebra with radical square zero. Then the following are equivalent:*

(1) *Λ is representation-finite, that is, admits only finitely many isomorphism classes of indecomposable modules.*
(2) *The separated quiver for Λ is a disjoint union of Dynkin quivers.*

The following main theorem of this paper is an analog of this result for τ-tilting finiteness. A full subquiver of Q^{s} is called a *single subquiver* if, for any $i \in Q_0$, the vertex set contains at most one of i^+ or i^-.

Theorem B (Theorem 3.1). *Let Λ be a finite dimensional algebra with radical square zero. Then the following are equivalent:*

(1) *Λ is τ-tilting finite.*
(2) *Every single subquiver of the separated quiver for Λ is a disjoint union of Dynkin quivers.*

The following result plays a crucial role in the proof of Theorem B.

Theorem C (Theorem 3.2). *Let X be a Λ-module. Let $P_1^X \to P_0^X \to X \to 0$ be a minimal projective presentation of X. The following are equivalent:*

(1) *X is a τ-rigid Λ-module.*
(2) *FX is a τ-rigid KQ^{s}-module and $\mathsf{add}\, P_0^X \cap \mathsf{add}\, P_1^X = 0$.*

Moreover, we have the following bijection. For an algebra Λ, we denote by $\tau\text{-rigid}\,\Lambda$ the set of isomorphism classes of indecomposable τ-rigid Λ-modules and let $\tau\text{-rigid}^{\circ}\Lambda := \{X \in \tau\text{-rigid}\,\Lambda \mid \mathsf{add}\,(P_0^X \oplus P_1^X) = \mathsf{add}\,\Lambda\}$, where $P_1^X \to P_0^X \to X \to 0$ is a minimal projective presentation of X. We denote by $\mathcal{S}^+$ the set of all connected single subquivers of Q^{s} except the quivers with exactly one vertex i^- for any $i \in Q_0$.

Corollary D (Corollary 3.5). *There is a natural bijection*

$$\tau\text{-rigid}\,\Lambda \longrightarrow \coprod_{Q' \in \mathcal{S}^+} \tau\text{-rigid}^{\circ} KQ'.$$

We also give applications of our results. First, we give a positive answer to a question given by Zhang [Zh]. Namely, for an algebra Λ with radical square zero, if every indecomposable Λ-module is τ-rigid, then Λ is representation-finite. Secondly, we give an example of τ-tilting finite algebras which are not representation-finite. Let Λ be a multiplicity-free Brauer cyclic graph algebra with n vertices. Then Λ is not representation-finite. Moreover, it is τ-tilting finite if and only if n is odd. In this case, the cardinality of $\tau\text{-rigid}\,\Lambda$ is

$$|\tau\text{-rigid}\,\Lambda| = 2n^2 - n.$$

Throughout this paper, we use the following notation. Let K be an algebraically closed field and $D := \mathrm{Hom}_K(-, K)$. By an algebra we mean a basic and finite dimensional K-algebra, and by a module we mean a finite dimensional right module. For an algebra Λ, we denote by $\mathsf{mod}\Lambda$ the category of Λ-modules, by $\underline{\mathsf{mod}}\Lambda$ the stable category, and by τ the Auslander-Reiten translation of Λ. For a Λ-module X, we denote by $\mathsf{add}\, X$ the full subcategory of $\mathsf{mod}\Lambda$ consisting of direct summands of finite direct sums of copies of X. We call a quiver *Dynkin* (respectively, *Euclidean*) if the underlying graph is one of the Dynkin (respectively, Euclidean) graphs of type A, D and E (respectively, $\widetilde{A}, \widetilde{D}$ and $\widetilde{E}$). We refer to [ASS, ARS] for background on representation theory.

2. Preliminaries

In this section, we collect some results which are necessary in this paper. Let Λ be an algebra and $J := J_\Lambda$ a Jacobson radical of Λ. For a Λ-module X, we denote by

$$P_1^X \xrightarrow{p} P_0^X \xrightarrow{q} X \to 0$$

a minimal projective presentation of X. We define τX in $\operatorname{\mathsf{mod}}\Lambda$ by an exact sequence

$$0 \to \tau X \to \nu P_1^X \xrightarrow{\nu p} \nu P_0^X,$$

where $\nu := D\operatorname{Hom}_\Lambda(-, \Lambda)$ is the Nakayama functor. We call τ the Auslander-Reiten translation of Λ.

2.1. **τ-rigid modules.** We recall basic properties of τ-rigid modules.

Definition 2.1. A Λ-module X is called *τ-rigid* if $\operatorname{Hom}_\Lambda(X, \tau X) = 0$. We denote by $\tau\text{-rigid}\,\Lambda$ the set of isomorphism classes of indecomposable τ-rigid Λ-modules. An algebra Λ is called *τ-tilting finite* if $\tau\text{-rigid}\,\Lambda$ is a finite set. Note that τ-tilting finiteness is equivalent to that there are only finitely many τ-tilting modules (see [DIJ] for details.).

By Auslander-Reiten duality $\operatorname{Ext}^1_\Lambda(X, Y) \simeq D\overline{\operatorname{Hom}}_\Lambda(Y, \tau X)$, every τ-rigid Λ-module X is *rigid* (*i.e.* $\operatorname{Ext}^1_\Lambda(X, X) = 0$), and the converse is true if Λ is hereditary (*e.g.*, the path algebra KQ of an acyclic quiver Q).

The following proposition plays an important role in this paper.

Proposition 2.2 ([AIR, Proposition 2.4 and 2.5])**.** *For a Λ-module X, the following hold:*

1. *X is τ-rigid if and only if the map $(p, X) : \operatorname{Hom}_\Lambda(P_0^X, X) \to \operatorname{Hom}_\Lambda(P_1^X, X)$ is surjective.*
2. *If X is τ-rigid, then we have $\operatorname{\mathsf{add}} P_0^X \cap \operatorname{\mathsf{add}} P_1^X = 0$.*

For an idempotent $e \in \Lambda$, we consider two K-linear functors,

$$L_e(-) := - \otimes_{e\Lambda e} e\Lambda : \operatorname{\mathsf{mod}}(e\Lambda e) \longrightarrow \operatorname{\mathsf{mod}}\Lambda, \quad R_e(-) := (-)e : \operatorname{\mathsf{mod}}\Lambda \longrightarrow \operatorname{\mathsf{mod}}(e\Lambda e).$$

Then (L_e, R_e) is an adjoint pair. Moreover, the following result gives a connection between τ-rigid $(e\Lambda e)$-modules and τ-rigid Λ-modules.

Lemma 2.3 ([ASS, I.6.8])**.** *Let Λ be an algebra and $e \in \Lambda$ an idempotent.*

1. *The functor L_e is fully faithful and there exists a functorial isomorphism $R_e L_e \simeq 1_{\operatorname{\mathsf{mod}} eAe}$. In particular, L_e and R_e induce mutually quasi-inverse equivalences between categories $\operatorname{\mathsf{mod}}(e\Lambda e)$ and $\operatorname{Im} L_e := \{L_e(X) \mid X \in \operatorname{\mathsf{mod}}(e\Lambda e)\}$.*
2. *A Λ-module X is in the category $\operatorname{Im} L_e$ if and only if $P_0^X \oplus P_1^X \in \operatorname{\mathsf{add}} e\Lambda$.*

We have the following result.

Proposition 2.4. *Let Λ be an algebra and $e \in \Lambda$ an idempotent. Assume that a Λ-module X is in $\operatorname{Im} L_e$. Then X is τ-rigid if and only if the $(e\Lambda e)$-module Xe is τ-rigid. In particular, L_e and R_e induce mutually inverse bijections*

$$\tau\text{-rigid}(e\Lambda e) \longleftrightarrow \tau\text{-rigid}\Lambda \cap \operatorname{Im} L_e.$$

Proof. By Lemma 2.3(2), we have $P_0^X \oplus P_1^X \in \mathsf{add}\, e\Lambda$. Hence the sequence

$$P_1^X e \xrightarrow{pe} P_0^X e \xrightarrow{qe} Xe \to 0$$

is a projective presentation. By Lemma 2.3(1), the projective presentation is minimal, and moreover we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_A(P_0^X, X) & \xrightarrow{(p,X)} & \operatorname{Hom}_A(P_1^X, X) \\ \downarrow \simeq & & \downarrow \simeq \\ \operatorname{Hom}_{eAe}(P_0^X e, Xe) & \xrightarrow{(pe,Xe)} & \operatorname{Hom}_{eAe}(P_1^X e, Xe) \end{array}$$

where the vertical maps are isomorphisms. By using Proposition 2.2(1), we have that X is a τ-rigid Λ-module if and only if Xe is a τ-rigid $(e\Lambda e)$-module. □

The following proposition is a well-known result for path algebras. Refer to [ASS, VII.5.10, VIII.2.7 and VIII.2.9].

Proposition 2.5. *Let Q be a connected acyclic quiver and $\Lambda := KQ$ the path algebra of Q. Then the following hold:*

(1) [Ga] Λ *is representation-finite if and only if Q is a Dynkin quiver. In this case, every indecomposable Λ-module is rigid.*
(2) *If Λ is not representation-finite, then there exist infinitely many isomorphism classes of indecomposable rigid Λ-modules. Moreover, there exists an indecomposable Λ-module which is not rigid.*

Immediately, we have the following characterization of τ-tilting finiteness for path algebras of acyclic quivers.

Theorem 2.6. *Let Q be a connected acyclic quiver and $\Lambda := KQ$ the path algebra of Q. Then the following are equivalent:*

(1) Λ *is representation-finite.*
(2) Λ *is τ-tilting finite.*
(3) *Q is a Dynkin quiver.*

Proof. It follows from Proposition 2.5 because rigid modules are exactly τ-rigid modules for any hereditary algebra. □

2.2. **Algebras with radical square zero.** Throughout this subsection, we assume that Λ is an algebra with radical square zero (*i.e.*, $J^2 = 0$). For algebras with radical square zero, the following triangular matrix algebra plays an important role:

$$\Delta := \Delta(\Lambda) := \begin{bmatrix} \Lambda/J & J \\ 0 & \Lambda/J \end{bmatrix}.$$

Each Δ-module is given by a triplet $(X', X''; \varphi)$, where X', X'' are (Λ/J)-modules and φ is a morphism

$$\varphi : X' \otimes_{\Lambda/J} J \to X''$$

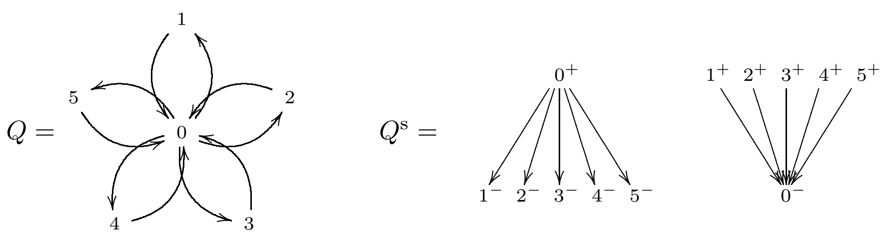
in $\mathsf{mod}(\Lambda/J)$. A morphism $f : (X', X''; \varphi) \to (Y', Y''; \psi)$ in $\mathsf{mod}\,\Delta$ is given by a pair (f', f''), where $f' : X' \to Y'$ and $f'' : X'' \to Y''$ are morphisms in $\mathsf{mod}(\Lambda/J)$ such that $\psi f' = f''\varphi$ (see [ASS, A.2.7] and [ARS, III.2] for details). Thus we have

$$\begin{array}{ccc} X' \otimes_{\Lambda/J} J & \xrightarrow{\ \varphi\ } & X'' \\ {\scriptstyle f'\otimes J}\downarrow & & \downarrow{\scriptstyle f''} \\ Y' \otimes_{\Lambda/J} J & \xrightarrow{\ \psi\ } & Y'' \end{array}$$

We recall some properties of the triangular matrix algebra Δ. Let $Q = (Q_0, Q_1)$ be a quiver, where Q_0 is the vertex set and Q_1 is the arrow set. Then we define a new quiver $Q^{\mathrm{s}} = (Q_0^{\mathrm{s}}, Q_1^{\mathrm{s}})$, called a *separated quiver*, as follows: Let $Q_0^+ := \{i^+ \mid i \in Q_0\}$ and $Q_0^- := \{i^- \mid i \in Q_0\}$ be copies of Q_0. Then

$$Q_0^{\mathrm{s}} := Q_0^+ \coprod Q_0^-, \quad Q_1^{\mathrm{s}} := \{i^+ \to j^- \mid i \to j \text{ in } Q_1\}.$$

Note that the separated quiver Q^{s} is not necessarily connected even if Q is connected. For example, the separated quiver Q^{s} of the following quiver Q is not connected:



We call a quiver *bipartite* if each vertex is either a sink or a source.

Proposition 2.7 ([ARS, III.2.5]). *Let Q be the quiver of Λ. The following hold:*

(1) *The separated quiver Q^{s} is bipartite.*
(2) *The algebra Δ is isomorphic to the path algebra of Q^{s}. In particular, Δ is a hereditary algebra with radical square zero.*
(3) *Each simple Δ-module is one of the form $(S, 0; 0)$ or $(0, S; 0)$, where S is a simple Λ-module.*
(4) *Each indecomposable projective Δ-module is one of the form $(P/PJ, PJ; 1_{PJ})$ or $(0, P/PJ; 0)$, where P is an indecomposable projective Λ-module.*

Now we recall results on representation theory of algebras with radical square zero. We define a functor $F : \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Delta$ as follows: For any Λ-module X, we let

$$F(X) := (X/XJ, XJ; \varphi_X),$$

where the map $\varphi_X : X/XJ \otimes_{\Lambda/J} J \to XJ$ is naturally induced by the natural multiplication morphism $X \otimes_\Lambda J \to XJ$ since $J^2 = 0$. For any morphism $g : X \to Y$, we let

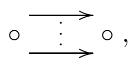
$$F(g) := (g', g''),$$

where $g' : X/XJ \to Y/YJ$ is induced by g and $g'' : XJ \to YJ$ is the restriction to XJ.

Proposition 2.8 ([ARS, X.2.1, X.2.2, X.2.4 and X.2.6]). *The following hold:*

(1) *The functor F is full and induces an equivalence of categories $\underline{\mathsf{mod}}\Lambda \to \underline{\mathsf{mod}}\Delta$.*
(2) *A Λ-module X is indecomposable (respectively, projective) if and only if FX is an indecomposable (respectively, a projective) Δ-module.*
(3) *The following are equivalent:*
  (a) *Λ is representation-finite.*
  (b) *The separated quiver of the quiver for Λ is a disjoint union of Dynkin quivers.*

Remark 2.9. Clearly a stable equivalence preserves representation-finiteness. However a stable equivalence does not preserve τ-tilting finiteness in general. Indeed, let Λ be an algebra with radical square zero whose quiver consists of one vertex and n loops with $n \geq 2$. Since Λ is local, every indecomposable τ-rigid Λ-module is projective, and in particular Λ is τ-rigid-finite. On the other hand, since the separated quiver is the n-Kronecker quiver



Δ is not τ-tilting finite by Theorem 2.6.

3. Main results

Throughout this section, Λ is an algebra with radical square zero (*i.e.*, $J^2 = 0$), and Δ, F are as in Subsection 2.2. Let Q be the quiver of Λ and Q^{s} the separated quiver of Q. A full subquiver Q' of Q^{s} is called a *single subquiver* if, for any $i \in Q_0$, the vertex set Q'_0 contains at most one of i^+ or i^-. We denote by $\mathcal{S}$ the set of all single subquivers of Q^{s}.

The following theorem is our main result of this paper.

Theorem 3.1. *Let Λ be an algebra with radical square zero and Q^{s} the separated quiver of the quiver Q for Λ. Then the following are equivalent:*

(1) *Λ is τ-tilting finite.*
(2) *Every single subquiver of Q^{s} is a disjoint union of Dynkin quivers.*

The proof of Theorem 3.1 will be given in the rest of this section. A key result is the following criterion for indecomposable Λ-modules to be τ-rigid in terms of the triangular matrix algebra Δ.

Theorem 3.2. *Let X be a Λ-module. The following are equivalent:*

(1) *X is a τ-rigid Λ-module.*
(2) *FX is a τ-rigid Δ-module and $\mathsf{add}\, P_0^X \cap \mathsf{add}\, P_1^X = 0$.*

We prove Theorem 3.2 by comparing a minimal projective presentation of a Λ-module X with that of the Δ-module FX. We start with an easy lemma.

Lemma 3.3. *Let $f : X \to Y$ be a non-zero morphism in $\mathsf{mod}\,\Lambda$. If $\mathrm{Im}\, f$ is contained in YJ, then there exists a unique morphism $\tilde{f} : X/XJ \to YJ$ such that*

$$f = (X \xrightarrow{\pi} X/XJ \xrightarrow{\tilde{f}} YJ \xrightarrow{\iota} Y),$$

where π and ι are natural morphisms.

Proof. This is clear since $f(XJ) \subseteq (YJ)J = 0$ holds by $J^2 = 0$. □

The following lemma gives a construction of a minimal projective presentation of FX from that of X.

Lemma 3.4. *Let $P_1^X \xrightarrow{p} P_0^X \xrightarrow{q} X \to 0$ be a minimal projective presentation of a Λ-module X. Then*

$$(*) \qquad 0 \longrightarrow (0, P_1^X/P_1^X J; 0) \xrightarrow{(0,\tilde{p})} FP_0^X \xrightarrow{Fq} FX \longrightarrow 0$$

is a minimal projective resolution of the Δ-module FX.

Proof. Since $J^2 = 0$ holds, $\ker q$ is semisimple. Hence $\ker q = P_1^X/P_1^X J$ holds. Since $\ker q$ is contained in $P_0^X J$, by Lemma 3.3, we have a decomposition

$$p = (P_1^X \xrightarrow{\pi} P_1^X/P_1^X J \xrightarrow{\tilde{p}} P_0^X J \xrightarrow{\iota} P_0^X),$$

where π and ι are natural morphisms. Thus, we have the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & P_1^X/P_1^X J & = & \ker q & & & & \\
 & & \downarrow{\scriptstyle \tilde{p}} & & \downarrow & & & & \\
0 & \longrightarrow & P_0^X J & \xrightarrow{\iota} & P_0^X & \longrightarrow & P_0^X/P_0^X J & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle q''} & & \downarrow{\scriptstyle q} & & \Vert{\scriptstyle q'} & & \\
0 & \longrightarrow & XJ & \longrightarrow & X & \longrightarrow & X/XJ & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $Fq = (q', q'')$. Thus the exact sequence

$$0 \to (0, P_1^X/P_1^X J; 0) \xrightarrow{(0,\tilde{p})} (P_0^X/P_0^X J, P_0^X J; 1_{P_0^X J}) \xrightarrow{(q',q'')} (X/XJ, XJ; \varphi_X) \to 0$$

in $\mathsf{mod}\,\Delta$ is a minimal projective resolution of FX, because Δ is hereditary and $(0, \tilde{p})$ is in the radical of $\mathsf{mod}\,\Lambda$. $\square$

Proof of Theorem 3.2. Let $\iota : P_0^X J \to P_0^X$, $\iota' : XJ \to X$ and $\pi : P_1^X \to P_1^X/P_1^X J$ be natural morphisms.

(1)$\Rightarrow$(2): Assume that X is τ-rigid. By Proposition 2.2(2), we have $\mathsf{add}\,P_0^X \cap \mathsf{add}\,P_1^X = 0$. Now we show that FX is a τ-rigid Δ-module. We have a minimal projective resolution $(*)$ in Lemma 3.4. By Proposition 2.2(1), we have only to show that

$$(\natural) \qquad ((0, \tilde{p}), FX) : \mathrm{Hom}_\Delta(FP, FX) \to \mathrm{Hom}_\Delta((0, P_1^X/P_1^X J; 0), FX)$$

is surjective. Let

$$(0, h) : (0, P_1^X/P_1^X J; 0) \to FX = (X/XJ, XJ; \varphi_X)$$

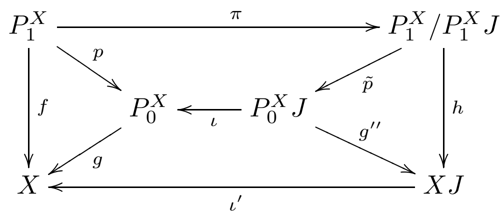
be a morphism in $\mathsf{mod}\,\Delta$ and $f := \iota' h \pi : P_1^X \to X$ a morphism in $\mathsf{mod}\,\Lambda$. Since X is τ-rigid, there exists a morphism $g : P_0^X \to X$ such that $f = gp$. Let

$$Fg := (g', g'') : (P_0^X/P_0^X J, P_0^X J; 1_{P_0^X J}) \to (X/XJ, XJ; \varphi_X).$$

Then we have

$$\iota' h \pi = f = gp = g\iota\tilde{p}\pi = \iota' g'' \tilde{p}\pi,$$

and hence we have $h = g''\tilde{p}$. Thus we have $(0, h) = (g', g'')(0, \tilde{p})$. Consequently, the map (♮) is surjective. Thus we have



(2)⇒(1): Assume that FX is τ-rigid and $\mathsf{add}\, P_0^X \cap \mathsf{add}\, P_1^X = 0$. We have only to show that

$$(\sharp) \qquad (p, X) : \mathrm{Hom}_\Lambda(P_0^X, X) \to \mathrm{Hom}_\Lambda(P_1^X, X)$$

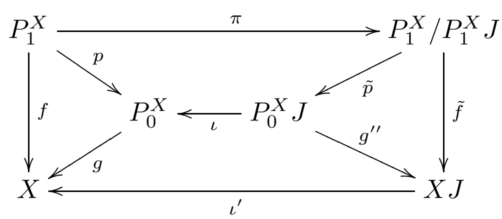
is surjective by Proposition 2.2(1). Let $f : P_1^X \to X$ be a morphism in $\mathsf{mod}\,\Lambda$. Since $\mathsf{add}\, P_0^X \cap \mathsf{add}\, P_1^X = 0$ holds, $\mathrm{Im}\, f$ is contained in XJ. Thus, by Lemma 3.3, there exists a morphism $\tilde{f} : P_1^X/P_1^X J \to XJ$ in $\mathsf{mod}\,\Lambda$ such that $f = \iota' \tilde{f} \pi$. Now we consider a morphism

$$(0, \tilde{f}) : (0, P_1^X/P_1^X J; 0) \to FX = (X/XJ, XJ; \varphi_X)$$

in $\mathsf{mod}\,\Delta$. Since $(*)$ in Lemma 3.4 gives a minimal projective resolution and FX is τ-rigid, there exists a morphism $(g', g'') : FP_0^X \to FX$ in $\mathsf{mod}\,\Delta$ such that $(0, \tilde{f}) = (g', g'')(0, \tilde{p})$. In particular, we have $\tilde{f} = g''\tilde{p}$. Since F is full by Proposition 2.8(1), there exists a morphism $g : P_0^X \to X$ such that $Fg = (g', g'')$. Then g'' is a restriction of g by construction of F, and we have

$$gp = g\iota\tilde{p}\pi = \iota' g''\tilde{p}\pi = \iota'\tilde{f}\pi = f.$$

Consequently, the map (♯) is surjective. Thus we have



This finishes the proof. □

For any indecomposable Λ-module X, we decompose the terms P_0^X and P_1^X in a minimal projective presentation of X as

$$P_0^X := \bigoplus_{i \in Q_0} (e_i\Lambda)^{n_i}, \; P_1^X := \bigoplus_{i \in Q_0} (e_i\Lambda)^{m_i},$$

where n_i and m_i are multiplicities of the indecomposable projective Λ-module corresponding to $i \in Q_0$. We denote by Q^X the full subquiver of Q^{s} with

$$Q_0^X := \{i^+ \in Q_0^{\mathrm{s}} \mid n_i \neq 0\} \coprod \{i^- \in Q_0^{\mathrm{s}} \mid m_i \neq 0\}.$$

Then, the condition $\mathsf{add}\, P_0^X \cap \mathsf{add}\, P_1^X = 0$ is satisfied if and only if Q^X is a single subquiver of Q^{s}. In particular, if X is τ-rigid, then Q^X is a single subquiver of Q^s by Proposition 2.2(2).

Now we are ready to prove Theorem 3.1. For any full subquiver Q' of Q^{s}, let

$$\begin{aligned}\tau\text{-rigid}(\Delta, Q') :&= \tau\text{-rigid}\,\Delta \cap \operatorname{Im} L_{e_{Q'}} \\ &= \{X \in \tau\text{-rigid}\,\Delta \mid \mathsf{add}(P_0^X \oplus P_1^X) \subseteq \mathsf{add}(\bigoplus_{i \in Q'_0} e_i\Delta)\}\end{aligned}$$

$$\tau\text{-rigid}^{\circ}(\Delta, Q') = \{X \in \tau\text{-rigid}\,\Delta \mid \mathsf{add}(P_0^X \oplus P_1^X) = \mathsf{add}(\bigoplus_{i \in Q'_0} e_i\Delta)\},$$

where $e_{Q'} := \sum_{i \in Q'_0} e_i$ and $L_{e_{Q'}}$ is the functor in Subsection 2.1. We denote by $\tau\text{-rigid}_{\mathrm{np}}\Lambda$ the subset of $\tau\text{-rigid}\,\Lambda$ consisting of non-projective modules.

Proof of Theorem 3.1. (1)⇔(2): First we claim that the functor $F : \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Delta$ induces a bijection

$$(\flat) \qquad \tau\text{-rigid}_{\mathrm{np}}\Lambda \to \bigcup_{Q' \in \mathcal{S}} \tau\text{-rigid}_{\mathrm{np}}(\Delta, Q').$$

Indeed, by Proposition 2.8(2) and Theorem 3.2, X is an indecomposable non-projective τ-rigid Λ-module if and only if FX is an indecomposable non-projective τ-rigid Δ-module satisfying $\mathsf{add}\,P_0^X \cap \mathsf{add}\,P_1^X = 0$, or equivalently $Q^X \in \mathcal{S}$. In this case, since $P_0^{FX} \oplus P_1^{FX} \in \mathsf{add}\,e_{Q^X}\Delta$ holds by Lemma 3.4, we have $FX \in \operatorname{Im} L_{e_{Q^X}}$ by Lemma 2.3(2). Hence the claim follows from the fact that the functor F induces a stable equivalence $\underline{\mathsf{mod}}\,\Lambda \to \underline{\mathsf{mod}}\,\Delta$ by Proposition 2.8(1).

Next, by Proposition 2.4, for any full subquiver Q' of Q^{s}, we have bijections

$$\tau\text{-rigid}(\Delta, Q') \leftrightarrow \tau\text{-rigid}(e_{Q'}\Delta e_{Q'}).$$

Since Q' is bipartite, there is an isomorphism $e_{Q'}\Delta e_{Q'} \simeq KQ'$. Since there are only finitely many single subquivers of Q^{s}, we have that Λ is τ-tilting finite if and only if KQ' is τ-tilting finite for every single subquiver Q' of Q^{s}. Hence the assertion follows from Theorem 2.6. □

By the proof of Theorem 3.1, we have the following corollary. We denote by $\mathcal{S}^+$ the set of all connected single subquivers of Q^{s} except the quivers with exactly one vertex i^- for any $i \in Q_0$. Let

$$\tau\text{-rigid}^{\circ}KQ' := \{M \in \tau\text{-rigid}\,KQ' \mid \mathsf{add}(P_0^M \oplus P_1^M) = \mathsf{add}(KQ')\}$$

for a quiver Q'.

Corollary 3.5. *There is a bijection*

$$\tau\text{-rigid}\,\Lambda \to \coprod_{Q' \in \mathcal{S}^+} \tau\text{-rigid}^{\circ}KQ'$$

given by $X \mapsto (FX)e_{Q^X}$. *In particular, the cardinality of* $\tau\text{-rigid}\,\Lambda$ *is*

$$|\tau\text{-rigid}\,\Lambda| = \sum_{Q' \in \mathcal{S}^+} |\tau\text{-rigid}^{\circ}KQ'|.$$

Proof. Since the map $e_i\Lambda \mapsto e_{i^+}\Delta$ gives a bijection between the set of isomorphism classes of indecomposable projective Λ-modules and that of indecomposable Δ-modules corresponding to the vertex $i^+ \in Q_0^+$, the functor F induces a bijection

$$\tau\text{-rigid}\,\Lambda \to \bigcup_{Q' \in \mathcal{S}} \tau\text{-rigid}(\Delta, Q') \setminus \{e_{i^-}\Delta \mid i \in Q_0\}$$

by ($\flat$). Then, for any $X \in \tau\text{-rigid}\,\Lambda$, we have $FX \in \tau\text{-rigid}^{\circ}(\Delta, Q^X)$ with $Q^X \in \mathcal{S}^+$. Moreover, we have

$$\bigcup_{Q' \in \mathcal{S}} \tau\text{-rigid}(\Delta, Q') \setminus \{e_i\text{-}\Delta \mid i \in Q_0\} = \coprod_{Q' \in \mathcal{S}} \tau\text{-rigid}^{\circ}(\Delta, Q') \setminus \{e_i\text{-}\Delta \mid i \in Q_0\}$$
$$= \coprod_{Q' \in \mathcal{S}^+} \tau\text{-rigid}^{\circ}(\Delta, Q').$$

Hence the assertion follows from the fact that the map $M \mapsto M_{e_{Q'}}$ gives a bijection

$$\tau\text{-rigid}^{\circ}(\Delta, Q') \longrightarrow \tau\text{-rigid}^{\circ} KQ'$$

by Proposition 2.4. □

4. Applications and examples

In this section, we give applications and examples of results in the previous section. As an immediate consequence of Theorem 3.2, we have the following two corollaries.

Corollary 4.1. *Let Λ be a representation-finite algebra with radical square zero and X a Λ-module. Then X is a τ-rigid Λ-module if and only if* $\mathsf{add}\, P_0^X \cap \mathsf{add}\, P_1^X = 0$.

Proof. The 'only if' part follows from Proposition 2.2(2). We show the 'if' part. Since Λ is representation-finite, Δ is a finite product of path algebras with Dynkin quivers by Proposition 2.8(3). Hence FX is a τ-rigid Δ-module by Proposition 2.5(1). Thus, if $\mathsf{add}\, P_0^X \cap \mathsf{add}\, P_1^X = 0$ holds, then X is τ-rigid by Theorem 3.2. □

We give a positive answer to a question posed by Zhang [Zh].

Corollary 4.2. *Let Λ be an algebra with radical square zero. If every indecomposable Λ-module is τ-rigid, then Λ is representation-finite.*

Proof. Assume that Λ is not representation-finite. By Proposition 2.8(3), the separated quiver contains a non-Dynkin quiver as a subquiver. By Proposition 2.5(2), there exists an indecomposable Δ-module M which is not rigid. By Proposition 2.8(1), there exists an indecomposable non-projective Λ-module X such that $FX \simeq M$. The Λ-module X is not τ-rigid by Theorem 3.2. □

At the end of this paper, we apply our main results to the following algebras which associate with Brauer graph algebras. We start with the following observation.

Proposition 4.3 ([Ad, Corollary 3.7])**.** *Let Λ be a ring-indecomposable non-semisimple symmetric algebra and $\overline{\Lambda} := \Lambda/\operatorname{soc}\Lambda$. Then there is a bijection*

$$\tau\text{-rigid}\,\Lambda \to \tau\text{-rigid}\,\overline{\Lambda}$$

given by $X \mapsto X \otimes_{\Lambda} \overline{\Lambda}$. In particular, Λ is τ-tilting finite if and only if $\overline{\Lambda}$ is τ-rigid-finite.

Note that, for every indecomposable projective Λ-module P, the module $P/\operatorname{soc} P$ is a τ-rigid $\overline{\Lambda}$-module but not a τ-rigid Λ-module.

First, we give a classification of indecomposable τ-rigid modules over a multiplicity-free Brauer line algebra. This is a special case of results in [AZ] and [AAC]. We denote by $\operatorname{ind}\Lambda$ the set of isomorphism classes of indecomposable Λ-modules.

Corollary 4.4. *Let Q be the following quiver:*

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} \cdots \underset{\beta_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} n-1 \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} n.$$

(1) *Let Λ be an algebra with radical square zero whose quiver is Q. Then Λ is a representation-finite algebra with*

$$\tau\text{-rigid}\,\Lambda = \operatorname{ind}\Lambda.$$

(2) *Let Γ be a multiplicity-free Brauer line algebra, that is, $\Gamma \simeq KQ/I$, where*

$$I = \langle \alpha_1\beta_1\alpha_1, \beta_{n-1}\alpha_{n-1}\beta_{n-1}, \alpha_i\alpha_{i+1}, \beta_{i+1}\beta_i, \beta_i\alpha_i - \alpha_{i+1}\beta_{i+1} \mid i = 1, 2, \ldots, n-2\rangle.$$

Then we have

$$\tau\text{-rigid}\,\Gamma = \operatorname{ind}\Gamma \setminus \{e_i\Gamma/\operatorname{soc}(e_i\Gamma) \mid i \in Q_0\}.$$

(3) *The cardinalities of* τ-rigid Λ *and* τ-rigid Γ *are*

$$|\tau\text{-rigid}\,\Lambda| = |\tau\text{-rigid}\,\Gamma| = n^2.$$

Proof. (1) Since the underlying graph of the separated quiver Q^{s} is a disjoint union of the following two Dynkin graphs of type A:

$$1^+ \;—\; 2^- \;—\; \cdots \;—\; n{-}1^{-\epsilon} \;—\; n^{\epsilon}$$
$$1^- \;—\; 2^+ \;—\; \cdots \;—\; n{-}1^{\epsilon} \;—\; n^{-\epsilon}$$

where $\epsilon = -$ if n is even and $\epsilon = +$ if n is odd, Λ is representation-finite by Proposition 2.8. Moreover, for any indecomposable Λ-module X, the Δ-module FX is rigid by Proposition 2.5(1), or equivalently τ-rigid. Moreover, we have $\mathsf{add}\,P_0^X \cap \mathsf{add}\,P_1^X = 0$ by Lemma 3.4. Hence, every indecomposable Λ-module is always τ-rigid by Theorem 3.2.

(2) Since Γ is a symmetric algebra, there is a bijection

$$\tau\text{-rigid}\,\Gamma \to \tau\text{-rigid}\,\overline{\Gamma}$$

by Proposition 4.3. Since $\overline{\Gamma}$ is isomorphic to Λ, the assertion follows from (1).

(3) By Corollary 3.5, we have

$$|\tau\text{-rigid}\,\Lambda| = |\tau\text{-rigid}\,\Gamma| = \sum_{Q' \in \mathcal{S}^+} |\tau\text{-rigid}^{\circ} KQ'|.$$

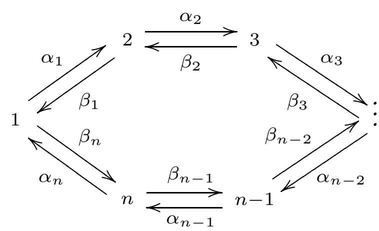
By (1), all single subquivers in $\mathcal{S}^+$ with $l > 1$ (respectively, $l = 1$) vertices are exactly $2(n-l+1)$ (respectively, n) Dynkin quivers of type A. Since, for each Dynkin quiver Q' of type A, we have $|\tau\text{-rigid}^{\circ} KQ'| = 1$, the cardinalities are

$$|\tau\text{-rigid}\,\Lambda| = |\tau\text{-rigid}\,\Gamma| = n + 2\sum_{l=2}^{n}(n-l+1) = n^2.$$

□

Finally, we give an example of τ-tilting finite algebras which is not representation-finite.

Corollary 4.5. *Let Q be the following quiver:*



(1) *Let Λ be an algebra with radical square zero whose quiver is Q. Then the following hold:*
  (a) *Λ is not representation-finite.*
  (b) *Λ is τ-tilting finite if and only if n is odd.*

(2) *Let Γ be a multiplicity-free Brauer cyclic graph algebra, that is, $\Gamma \simeq KQ/I$, where*

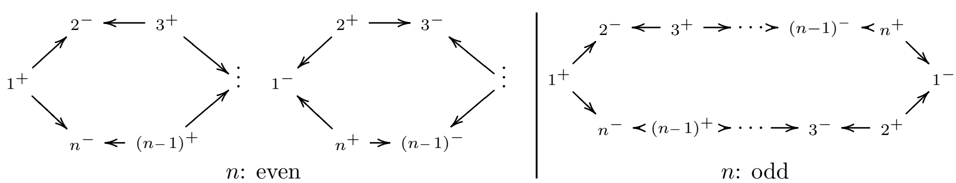
$$I=\langle \alpha_n\alpha_1, \beta_1\beta_n, \beta_n\alpha_n-\alpha_1\beta_1, \alpha_i\alpha_{i+1}, \beta_{i+1}\beta_i, \beta_i\alpha_i-\alpha_{i+1}\beta_{i+1} \mid i=1,2,\dots,n-1\rangle.$$

Then Γ is τ-tilting finite if and only if n is odd.

(3) *Assume that n is odd. Then the cardinalities of $\tau\text{-rigid}\,\Lambda$ and $\tau\text{-rigid}\,\Gamma$ are*

$$|\tau\text{-rigid}\,\Lambda| = |\tau\text{-rigid}\,\Gamma| = 2n^2-n.$$

Proof. (1) The separated quiver Q^{s} is one of the following quivers: Thus Λ is not



representation-finite by Proposition 2.8(3).

If n is odd, then every single subquiver is a disjoint union of Dynkin quivers. Thus Λ is τ-tilting finite by Theorem 3.1. On the other hand, if n is even, then two connected components are non-Dynkin single subquivers. Thus Λ is not τ-tilting finite by Theorem 3.1.

(2) Since Γ is a symmetric algebra, by Proposition 4.3, we have only to claim that $\overline{\Gamma}$ is τ-tilting finite if and only if n is odd. Indeed, since $\overline{\Gamma}$ is isomorphic to Λ, the claim follows from (1).

(3) By Corollary 3.5, we have

$$|\tau\text{-rigid}\,\Lambda| = |\tau\text{-rigid}\,\Gamma| = \sum_{Q'\in\mathcal{S}^+} |\tau\text{-rigid}^{\circ} KQ'|.$$

By (1), all single subquivers in $\mathcal{S}^+$ with $l>1$ (respectively, $l=1$) vertices are exactly $2n$ (respectively, n) Dynkin quivers of type A. Since, for each Dynkin quiver Q' of type A, we have $|\tau\text{-rigid}^{\circ} KQ'|=1$, the cardinalities are

$$|\tau\text{-rigid}\,\Lambda| = |\tau\text{-rigid}\,\Gamma| = n+2n(n-1) = 2n^2-n.$$

□

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