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MEASURES ON HYPERSPACES

WŁODZIMIERZ J. CHARATONIK AND MATT INSALL

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Abstract. From a measure space, $(X, \mu^{\mathbb{X}})$ we define a measure $\mu^{\mathbb{P}(\mathbb{X})}$ on the power set of X. If (X, τ) is a compactum, whose topology τ is compatible with the measure $\mu^{\mathbb{X}}$ on X, then the measure $\mu^{\mathbb{P}(\mathbb{X})}$ restricts to a natural measure on the hyperspace of closed sets of that given compactum. Surprisingly, under very mild conditions, $\mu^{\mathbb{P}(\mathbb{X})}$ is always supported on the hyperspace of finite subsets.

1. Introduction and preliminaries

1.1. **The problem.** One of the main difficulties encountered in hyperspace theory is the problem of "lifting" structure from the underlying space up to its hyperspaces. The Vietoris [13] and Fell [5] topologies, and some others, provide ways to do this for topological structure, so that important topological properties, like compactness, connectedness, etc., are preserved. The Hausdorff metric, and several other metrics (see [2], [3], and [4]), provide for a "lifting" of metrics to hyperspaces that is compatible with these topologies. Here, we will deal with the corresponding measure extension problem that arises when a given space carries a measure.

For more information about continua and their hyperspaces, see [10] or [6].

For more information about measures and measure theory, see the classical works [1], [7], [8], [9], or the more modern one [12].

2. Construction of the measure

The following definition is inspired by the definition of basis elements for the Vietoris topology; however, it is not restricted to open sets in a base topological space.

Definition 2.1. Let X be a set, and let $A_1, \ldots, A_n \subseteq X$. Then

$$\langle A_1, \ldots, A_n \rangle = \{ P \subseteq A_1 \cup \cdots \cup A_n | \, j \in \{1, \ldots, n\} \Rightarrow P \cap A_j \neq \emptyset \} .$$

Let $\mathbb{X} = \left(X, \mathcal{M}^{\mathbb{X}}, \mu^{\mathbb{X}}\right)$ be a measure space. That is, $\mathcal{M}^{\mathbb{X}}$ is a σ-subalgebra of the power set of X that supports a given measure, $\mu^{\mathbb{X}}$, on the set X. Then

$$\mathbb{P}(\mathbb{X}) = \left(\mathcal{P}(X), \mathcal{M}^{\mathbb{P}(\mathbb{X})}, \mu^{\mathbb{P}(\mathbb{X})}\right),$$

where

(1) $\mathcal{P}(X)$ is the power set of X: $\mathcal{P}(X) = \{A | A \subseteq X\}$.

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(2) $\mu^{\mathbb{P}(\mathbb{X})}$ is the measure defined by the formula

$$\mu^{\mathbb{P}(\mathbb{X})}\left(\langle A_1,\ldots,A_n\rangle\right)=\left(2^{\mu^{\mathbb{X}}(A_1)}-1\right)\ldots\left(2^{\mu^{\mathbb{X}}(A_n)}-1\right),$$

for any pairwise disjoint finite list $A_1,\ldots,A_n$, each member of which is a $\mu^{\mathbb{X}}$-measurable set, such that either all are of finite $\mu^{\mathbb{X}}$ measure or all are of positive $\mu^{\mathbb{X}}$ measure.

(3) $\mathcal{M}^{\mathbb{P}(\mathbb{X})}$ is the set (i.e. the σ-algebra) of all $\mu^{\mathbb{P}(\mathbb{X})}$-measurable subsets of $\mathcal{P}(X)$. That is, a subset P of $\mathcal{P}(X)$ is a member of the set $\mathcal{M}^{\mathbb{P}(\mathbb{X})}$ precisely if the inner and outer measures determined by $\mu^{\mathbb{P}(\mathbb{X})}$, namely ${\mu^{\mathbb{P}(\mathbb{X})}}_*$ and ${\mu^{\mathbb{P}(\mathbb{X})}}^*$, agree on P:

$${\mu^{\mathbb{P}(\mathbb{X})}}_*(P)={\mu^{\mathbb{P}(\mathbb{X})}}^*(P).$$

Remark 2.2. Definition 2.1 yields counting measure in the case that the set X is finite. In fact, this was our motivation for crafting the formula for this extension of the measure.

Let us call a subset of $\mathcal{P}(X)$ a *basic* $\mu^{\mathbb{P}(\mathbb{X})}$-measurable set if it is of the form $\langle A_1,\ldots,A_n\rangle$, where $A_1,\ldots,A_n$ are μ-measurable subsets of X.

To verify that we actually have defined a measure on $\mathcal{P}(X)$, we need to check that it does not lead to a contradiction, i.e., if $\mathcal{A}$, $\mathcal{A}_1$, $\ldots$, $\mathcal{A}_n$ are basic $\mu^{\mathbb{P}(\mathbb{X})}$-measurable sets, $\mathcal{A}=\mathcal{A}_1,\cup\cdots\cup\mathcal{A}_n$, and $\mathcal{A}_i\cap\mathcal{A}_j=\emptyset$ for $i\neq j$, then $\mu^{\mathbb{P}(\mathbb{X})}\left(\mathcal{A}\right)=\sum_1^n\mathcal{A}_j$.

The simplest such case is presented in a theorem below. The remaining cases follow from this one by induction.

2.1. Preliminary observations.

Theorem 2.3. *Let X be a set, and let $A,B,C_1,\ldots,C_n$ be pairwise disjoint subsets of X. Then*

$$\begin{gathered}\langle A\cup B,C_1,\ldots,C_n\rangle=\langle A,C_1,\ldots,C_n\rangle\\ \cup\langle B,C_1,\ldots,C_n\rangle\cup\langle A,B,C_1,\ldots,C_n\rangle,\end{gathered}$$

and

$$\begin{gathered}\mu^{\mathbb{P}(\mathbb{X})}\left(\langle A\cup B,C_1,\ldots,C_n\rangle\right)=\mu^{\mathbb{P}(\mathbb{X})}\left(\langle A,C_1,\ldots,C_n\rangle\right)\\ +\mu^{\mathbb{P}(\mathbb{X})}\left(\langle B,C_1,\ldots,C_n\rangle\right)+\mu^{\mathbb{P}(\mathbb{X})}\left(\langle A,B,C_1,\ldots,C_n\rangle\right).\end{gathered}$$

Proof. The first equation is an easy set-theoretic consequence of our definitions and notation. To see that the second equation holds, set $\alpha=2^{\mu^{\mathbb{X}}(A)}$, $\beta=2^{\mu^{\mathbb{X}}(B)}$, and $\gamma=\mu^{\mathbb{P}(\mathbb{X})}\left(\langle C_1,\ldots,C_n\rangle\right)$. Then we have

$$\begin{aligned}&\mu^{\mathbb{P}(\mathbb{X})}\left(\langle A\cup B,C_1,\ldots,C_n\rangle\right)\\ &=(\alpha\beta-1)\gamma\\ &=(\alpha-1)\gamma+(\beta-1)\gamma+(\alpha-1)(\beta-1)\gamma\\ &=\mu^{\mathbb{P}(\mathbb{X})}\left(\langle A,C_1,\ldots,C_n\rangle\right)+\mu^{\mathbb{P}(\mathbb{X})}\left(\langle B,C_1,\ldots,C_n\rangle\right)\\ &\quad+\mu^{\mathbb{P}(\mathbb{X})}\left(\langle A,B,C_1,\ldots,C_n\rangle\right),\end{aligned}$$

as desired. □

Remark 2.4. If $\mu^{\mathbb{X}}(X)$ is infinite, then so is $\mu^{\mathbb{P}(\mathbb{X})}(\{X\})$.

Definition 2.5. A set A in $\mathcal{M}^{\mathbb{X}}$ is called an atom if $\mu^{\mathbb{X}}(A) > 0$ and for any measurable subset B of A with $\mu^{\mathbb{X}}(B) < \mu^{\mathbb{X}}(A)$ one has $\mu^{\mathbb{X}}(B) = 0$. A measure which has no atoms is called non-atomic.

The following relevant result is well known, and is due to Sierpiński [11].

Theorem 2.6. *Let $(X, \mu^{\mathbb{X}})$ be a non-atomic measure space, and let $\varepsilon \leq \mu^{\mathbb{X}}(X)$ be positive and finite. If $p \in X$, then there is a $\mu^{\mathbb{X}}$-measurable set $E \subseteq X$ such that $p \in E$ and $\mu^{\mathbb{X}}(E) = \varepsilon$.*

As an immediate consequence, we get the following helpful result.

Lemma 2.7. *Let the measure space $(X, \mu^{\mathbb{X}})$ be non-atomic. If $\mu^{\mathbb{X}}(X) = 1$, then X has a family, $\mathcal{C} = \{\mathcal{C}_m | 1 \leq m < \infty\}$, of coverings by disjoint non-empty measurable subsets, such that for each positive integer m, $A \in \mathcal{C}_m$ implies that $\mu^{\mathbb{X}}(A) = \frac{1}{m}$.*

Notation 2.8. Let $\mathcal{C} = \{\mathcal{C}_m | 1 \leq m < \infty\}$ be a family of coverings of X by disjoint non-empty subsets. For each pair, m, k, of positive integers,

$$\mathcal{F}_{m,k}(\mathcal{C}) = \bigcup \left\{ \langle U_1, \ldots, U_k \rangle \,|U_1, \ldots, U_k \in \mathcal{C}_m \text{ and } |\{U_1, \ldots, U_k\}| = k \right\}.$$

In other words, $\mathcal{F}_{m,k}(\mathcal{C})$ is the family of all sets that can be covered by exactly k elements of $\mathcal{C}_m$.

Lemma 2.9. *Let the measure space $(X, \mu^{\mathbb{X}})$ be non-atomic with $\mu^{\mathbb{X}}(X) = 1$, and let $\mathcal{C} = \{\mathcal{C}_m | 1 \leq m < \infty\}$ be a family of coverings of X by disjoint non-empty subsets, such that, for each positive integer m, $A \in \mathcal{C}_m$ implies that $\mu^{\mathbb{X}}(A) = \frac{1}{m}$. Given a positive integer k, the collection*

$$\mathcal{F}_k(\mathcal{C}) = \bigcap \left\{ \bigcup \{\mathcal{F}_{m,k}(\mathcal{C}) | M \leq m < \infty\} \,|1 \leq M < \infty \right\}$$

consists of sets of measure zero. Moreover, we have

$$\mu^{\mathbb{P}(\mathbb{X})} \left(\mathcal{F}_k(\mathcal{C})\right) = \frac{1}{k!} \left(\ln(2)\right)^k.$$

Proof. Let A be an element of $\mathcal{F}_k(\mathcal{C})$. Then, for any $M > 0$, there is $m \geq M$ such that A can be covered by exactly k elements of $\mathcal{C}_m$. That each member of $\mathcal{F}_k(\mathcal{C})$ has measure zero follows. Finally,

$$\begin{aligned}
\mu^{\mathbb{P}(\mathbb{X})} \left(\mathcal{F}_k(\mathcal{C})\right) &= \lim_{m \to \infty} \left[\binom{m}{k} \left(2^{\frac{1}{m}} - 1\right)^k \right] \\
&= \lim_{m \to \infty} \left[\frac{1}{k!} \left(2^{\frac{1}{m}} - 1\right)^k \right] \\
&= \frac{1}{k!} \left(\ln(2)\right)^k,
\end{aligned}$$

as required. □

Remark 2.10. If $X = [0, 1]^n$ with Lebesgue measure, then we may choose the coverings in the family $\mathcal{C}$ so that the diameters of the elements of $\mathcal{C}_k$ tend to zero as $k \to \infty$. Then the collection $\mathcal{F}_k(\mathcal{C})$ is the hyperspace of all k-element subsets of X.

Theorem 2.11 (Main theorem)**.** *Let the measure space $(X, \mu^{\mathbb{X}})$ be non-atomic with $\mu^{\mathbb{X}}$ a finite measure. Then the measure $\mu^{\mathbb{P}(\mathbb{X})}$ is supported on the hyperspace $\mathcal{F}(X) = \{A \subseteq X | \mu^{\mathbb{X}}(A) = 0\}$. In other words, in hyperspaces, under these hypotheses, a typical set is a set of measure zero.*

Proof. For simplicity, we assume, without loss of generality, that the measure on X is normalized, i.e., that $\mu^{\mathbb{X}}(X) = 1$. Since $\mathcal{F}(X) \supseteq \bigcup_{k=1}^{\infty} (\mathcal{F}_k(\mathcal{C}))$, we have that

$$\begin{aligned} \mu^{\mathbb{P}(\mathbb{X})}(\mathcal{F}(X)) &\geq \sum_{k=1}^{\infty} \left[\mu^{\mathbb{P}(\mathbb{X})}(\mathcal{F}_k(\mathcal{C}))\right] \\ &= \sum_{k=1}^{\infty} \left[\frac{1}{k!}(\ln(2))^k\right] = 1, \end{aligned}$$

the measure $\mu^{\mathbb{P}(\mathbb{X})}$ is supported on the hyperspace $\mathcal{F}(X) = \{A \subseteq X | \mu^{\mathbb{X}}(A) = 0\}$, as claimed. □

3. Connections to topology

The proof of the following theorem is immediate from the definitions.

Theorem 3.1. *The measure $\mu^{\mathbb{P}(\mathbb{X})}$ has the following properties, when the space $\mathbb{X}$ carries a given topology, τ, with respect to which each Borel set is $\mu^{\mathbb{X}}$-measurable:*

(1) *The set 2^X of closed subsets of (X, τ) has full measure, so that restriction of $\mu^{\mathbb{P}(\mathbb{X})}$ to the measurable subsets of the hyperspace 2^X yields a measure.*
(2) *If the original measure has the property that every non-empty open set has positive measure, then the same holds for 2^X, with the Vietoris or Fell topology.*

Consequently, we have the following interesting corollary.

Corollary 3.2. *In the hyperspace of subcontinua of a continuum, a typical set is degenerate.*

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Department of Mathematics and Statistics, Missouri University of Science and Technology, 400 W. 12^{th} Street, Rolla, Missouri 65409-0020

E-mail address: wjcharat@mst.edu

Department of Mathematics and Statistics, Missouri University of Science and Technology, 400 W. 12^{th} Street, Rolla, Missouri 65409-0020

E-mail address: insall@mst.edu