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AFFINE CELLULARITY OF AFFINE q-SCHUR ALGEBRAS

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Dedicated to Professor George Lusztig on his seventieth birthday

Abstract. We first present an axiomatic approach to proving that an algebra with a cell theory in Lusztig's sense is affine cellular in the sense of Koenig and Xi; then we will show that the affine q-Schur algebra $\mathfrak{U}_{r,n,n}$ is affine cellular. We also show that $\mathfrak{U}_{r,n,n}$ is of finite global dimension and its derived module category admits a stratification when the parameter $v \in \mathbb{C}^*$ is not a root of unity.

1. Introduction

Graham and Lehrer [GL] introduced the notion of cellularity to provide a unified axiomatic framework for studying the classification of the simple modules of a finite dimensional algebra. Hecke algebras of finite type have been shown to be cellular as well as various finite dimensional diagram algebras, such as Temperley-Lieb algebras, Brauer algebras, Birman-Murakami-Wenzl algebras and their cyclotomic analogs.

Recently, Koenig and Xi [KX] defined the notion of affine cellularity to generalize the notion of cellular algebras to algebras of not necessarily finite dimension over a noetherian domain k. Extended affine Hecke algebras of type A and affine Temperley-Lieb algebras were proved to be affine cellular in [KX]. Further examples of affine cellular algebras include affine Hecke algebras of rank two with generic parameters [GM], KLR algebras of finite type [KL] and BLN algebras [C] and [N].

Let $\mathbf{U}$ be a quantized enveloping algebra associated to a Kac-Moody Lie algebra $\mathfrak{g}$ and $\dot{\mathbf{U}}$ be its modified form defined by Lusztig. The cells attached to the Kazhdan-Lusztig basis of Hecke algebras associated to Coxeter groups are very useful in understanding the structures and representations of Coxeter groups and their Hecke algebras. The theory of cells attached to the canonical basis of the modified quantum algebra $\dot{\mathbf{U}}$, which has been developed by Lusztig, is also very useful in studying the algebra itself. Lusztig [L2] developed a more axiomatic approach to considering the notion of cells in an algebra, and he [L2, Section 4] described completely the structure of cells in $\dot{\mathbf{U}}$ in the case that $\mathbf{U}$ is of finite type. Moreover, Lusztig [L2, Section 5] gave a series of conjectures on the cell structure in the case

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of the level-zero modified quantum affine algebra $\widetilde{\mathbf{U}}$. These conjectures have been proved by Beck and Nakajima in [BN] (see also [Mc] for type $A_n^{(1)}$).

Lusztig [L3] geometrically defined the affine q-Schur algebra $\mathfrak{U}_{r,n,n}$ and its canonical basis $\mathfrak{B}_r$. In [VV], they have shown that $\mathfrak{U}_{r,n,n}$ is isomorphic to the affine q-Schur algebra $\widehat{\mathbf{S}}_v(n,r)$ in [G], which is algebraically defined as the endomorphism algebra of the right $\mathcal{H}_r$-module $\mathfrak{T}_r$, where $\mathcal{H}_r$ is the extended affine Hecke algebra of type A and $\mathfrak{T}_r$ is an infinite dimensional vector space.

In order to obtain the cell structure of $\widetilde{\mathbf{U}}$ for affine type A, McGerty [Mc] first investigated the structure of cells in $\mathfrak{U}_{r,n,n}$.

In [C] and [N], we have shown that the affine q-Schur algebra $\mathfrak{U}_{r,n,n}$ is an affine cellular algebra when $n > r$. In this note, we first give an axiomatic approach to proving that an algebra with a cell theory in Lusztig's sense is affine cellular in the sense of Koenig and Xi; then we will show that the affine q-Schur algebra $\mathfrak{U}_{r,n,n}$ is affine cellular. Our proof relies heavily on McGerty's explicit descriptions of two-sided cells of $\mathfrak{U}_{r,n,n}$ and the asymptotic algebra of $\mathfrak{U}_{r,n,n}$. When the parameter $v \in \mathbb{C}^*$ is not a root of unity, we show that all the affine cell ideals are idempotent and have nonzero idempotent elements. In this case, applying [KX, Theorem 4.4], we then obtain that $\mathfrak{U}_{r,n,n}$ is of finite global dimension and its derived module category admits a stratification whose sections are the derived categories of the representation ring of products of general linear groups.

The organization of this note is as follows. In Section 2, we introduce affine cellular algebras. In Section 3, we give an axiomatic approach to studying affine cellular algebras. In Section 4, we prove our main results, Theorems 4.1 and 4.4.

2. Affine cellular algebras

In this section, we recall Koenig and Xi's [KX] definition and results on affine cellular algebras. Throughout, we assume that k is a noetherian domain.

For two k-modules V and W, let τ be the switch map: $V \otimes W \to W \otimes V$ defined by $\tau(v \otimes w) = w \otimes v$ for $v \in V$ and $w \in W$. A k-linear anti-automorphism i of a k-algebra A which satisfies $i^2 = id_A$ will be called a k-involution on A. A commutative k-algebra B is called an affine k-algebra if it is a quotient of a polynomial ring $k[x_1, \dots, x_r]$ in finitely many variables $x_1, \dots, x_r$ by some ideal I.

Definition 2.1 (See [KX, Definition 2.1])**.** Let A be a unitary k-algebra with a k-involution i. A two-sided ideal J in A is called an affine cell ideal if and only if the following data are given and the following conditions are satisfied:

(1) $i(J) = J$.

(2) There exist a free k-module V of finite rank and an affine k-algebra B with a k-involution σ such that $\Delta := V \otimes_k B$ is an A-B-bimodule, where the right B-module structure is induced by the right regular B-module B_B.

(3) There is an A-A-bimodule isomorphism $\alpha : J \to \Delta \otimes_B \Delta'$, where $\Delta' := B \otimes_k V$ is a B-A-bimodule with the left B-module induced by the left regular B-module ${}_BB$ and with the right A-module structure defined by $(b \otimes v)a := \tau(i(a)(v \otimes b))$ for $a \in A$, $b \in B$ and $v \in V$, such that the following diagram is commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_B \Delta' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle u\otimes b\otimes_B b'\otimes v \mapsto v\otimes\sigma(b')\otimes_B\sigma(b)\otimes u} \\ J & \xrightarrow{\alpha} & \Delta \otimes_B \Delta'. \end{array}$$

The algebra A together with the k-involution i is called affine cellular if and only if there is a k-module decomposition $A = J'_1 \oplus J'_2 \oplus \cdots \oplus J'_n$ (for some n) with $i(J'_k) = J'_k$ for $1 \le k \le n$, such that, setting $J_l := \bigoplus_{k=1}^{l} J'_k$, we have a chain of two-sided ideals of A:

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$$

so that each $J'_l = J_l/J_{l-1}$ $(1 \le l \le n)$ is an affine cell ideal of A/J_{l-1} (with respect to the involution induced by i on the quotient).

Given a free k-module V of finite rank, an affine k-algebra B and a k-bilinear form $\rho : V \otimes_k V \to B$, we define an associative algebra $\mathbf{A}(V, B, \rho)$ as follows: $\mathbf{A}(V, B, \rho) := V \otimes_k B \otimes_k V$ as a k-module, and the multiplication on $\mathbf{A}(V, B, \rho)$ is defined by

$$(u_1 \otimes_k b_1 \otimes_k v_1)(u_2 \otimes_k b_2 \otimes_k v_2) := u_1 \otimes_k b_1\rho(v_1, u_2)b_2 \otimes_k v_2$$

for all $u_1, u_2, v_1, v_2 \in V$ and $b_1, b_2 \in B$. Moreover, if B admits a k-involution σ satisfying $\sigma\rho(v_1, v_2) = \rho(v_2, v_1)$, then $\mathbf{A}(V, B, \rho)$ admits a k-involution ϱ which sends $u \otimes b \otimes v$ to $v \otimes \sigma(b) \otimes u$ for all $u, v \in V$ and $b \in B$.

An equivalent description of this construction is as follows: Given V, B, ρ as above, we define the generalized matrix algebra $(M_n(B), \rho)$ over B with respect to ρ in the following way. It equals the ordinary matrix algebra $M_n(B)$ of $n \times n$ matrices over B as a k-space, but the multiplication is given by

$$\widetilde{x} \cdot \widetilde{y} = x\Psi y$$

for all $x, y \in M_n(B)$, where $\widetilde{x}$ and $\widetilde{y}$ are elements of $(M_n(B), \rho)$ corresponding to x and y, respectively, and Ψ is the matrix describing the bilinear form ρ with respect to some basis of V. Moreover, if B admits a k-involution σ satisfying $\sigma\rho(v_1, v_2) = \rho(v_2, v_1)$, then $(M_n(B), \rho)$ admits a k-involution κ which sends $E_{jl}(b)$ to $E_{lj}(\sigma(b))$, where $E_{jl}(b)$ denotes a square matrix whose (j, l)-entry is $b \in B$ and all the other entries are zero.

From the above discussion, we can easily get the following proposition about the description of affine cell ideals, which we will use in Section 3.

Proposition 2.2 (See [KX, Proposition 2.2]). *Let k be a noetherian domain, A a unitary k-algebra with a k-involution i. A two-sided ideal J in A is an affine cell ideal if and only if $i(J) = J$, J is isomorphic to some generalized matrix algebra $(M_n(B), \rho)$ for some affine k-algebra B with a k-involution σ, a free k-module V of finite rank and a k-bilinear form $\rho : V \otimes_k V \to B$. Under this isomorphism, if a basis element a of J corresponds to $E_{jl}(b')$ for some $b' \in B$, then $i(a)$ corresponds to $E_{lj}(\sigma(b'))$.*

The following theorem plays an important role in investigating homological properties of affine cellular algebras.

Theorem 2.3 (See [KX, Theorem 4.4]). *Let A be an affine cellular algebra with a cell chain $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ such that $J_l/J_{l-1} = V_l \otimes_k B_l \otimes_k V_l$ as in Definition 2.1. Suppose that each B_l satisfies $\operatorname{rad}(B_l) = 0$. Moreover, suppose that* *each J_l/J_{l-1} is idempotent and contains a nonzero idempotent element in A/J_{l-1}. Then*:

(1) *The parameter set of simple A-modules equals the parameter set of simple modules of the asymptotic algebra, that is, a finite union of affine spaces* (*one for each B_l*).

(2) *The unbounded derived category* $D(A\text{-Mod})$ *of A admits a stratification, that is, an iterated recollement whose strata are the derived categories of the various affine k-algebras B_l.*

(3) *The global dimension* $\mathrm{gldim}(A)$ *is finite if and only if* $\mathrm{gldim}(B_l)$ *is finite for all l.*

Remark 2.4. Koenig [Koe, p. 531] called an affine cellular algebra with the assumptions stated as in Theorem 2.3 an affine quasi-hereditary algebra, since it implies the crucial homological properties analogous to known results about quasi-hereditary algebras and highest weight categories; see also [Kle] for the graded version of affine quasi-heredity.

3. An axiomatic approach

In this section, we will present an axiomatic approach to affine cellular algebras, and show which conditions in [L2, Section 1] are required to prove that an algebra with a cell theory is affine cellular in the sense of Koenig and Xi.

3.1. Let $k = \mathbb{Z}[v, v^{-1}]$ (v an indeterminate), and let A be an associative k-algebra with a k-involution i. Given a set X, we say that an embedding $X \to A$ ($\lambda \mapsto 1_\lambda$) is a generalized unit for A if $1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'} 1_\lambda$ for $\lambda, \lambda' \in X$ and $A = \sum_{\lambda,\lambda' \in X} 1_\lambda A 1_{\lambda'}$.

We assume that we are given a basis B of A as a k-module which is compatible with the generalized unit in the following sense: the elements 1_λ ($\lambda \in X$) lie in B and any $b \in B$ is contained in $1_\lambda A 1_{\lambda'}$ for some uniquely determined $\lambda, \lambda' \in X$.

We assume that the structure constants $c_{b,b'}^{b''} \in k$ are given by

$$bb' = \sum_{b''} c_{b,b'}^{b''} b'' \ (b, b', b'' \in B).$$

3.2. We say that a two-sided ideal of A is based if it is the span of a subset of B. For $b, b' \in B$, we say that $b \preceq_{LR} b'$ if b lies in every based two-sided ideal which contains b'. We say that $b \sim_{LR} b'$ if $b \preceq_{LR} b'$ and $b' \preceq_{LR} b$. The equivalence classes for $\sim_{LR}$ are called 2-cells. If we replace the "two-sided ideal" with the "left ideal" (resp. "right ideal"), we get the corresponding notion of left cells (resp. right cells).

For each 2-cell $\mathbf{c}$ in B, let $A_{\mathbf{c}}$ be the k-subspace of A spanned by $\mathbf{c}$. There is an associative algebra structure on $A_{\mathbf{c}}$ in which the product of $b, b' \in \mathbf{c}$ is equal to $\sum_{b'' \in \mathbf{c}} c_{b,b'}^{b''} b''$. Note that the algebra $A_{\mathbf{c}}$ is naturally a subquotient of A.

3.3. In the following we define a function $a : B \to \mathbb{N} \cup \{\infty\}$. Let $\mathbf{c}$ be a 2-cell and let L be the $\mathbb{Z}[v^{-1}]$-submodule of $A_{\mathbf{c}}$ generated by $\{b' \mid b' \in \mathbf{c}\}$.

Let $b \in \mathbf{c}$. If there is $n \in \mathbb{Z}_{\geq 0}$ such that $v^{-n} bL \subset L$, then we define $a(b)$ to be the smallest such n. If there is no such n, we set $a(b) = \infty$.

We say that B has property P_1 if

(a) the number of 2-cells in B is countable, which is indexed by a partially ordered set; further, the partial order $\preceq$ is compatible with the partial order $\preceq_{LR}$ and for a given 2-cell $\mathbf{c}$, the set $\{\mathbf{c}' \mid \mathbf{c}' \preceq \mathbf{c}\}$ is finite.

(b) $a(b)$ is finite for all $b \in B$.

(c) For each 2-cell $\mathbf{c}$ and each $\lambda_1 \in X$, the restriction of a to $\mathbf{c}1_{\lambda_1}$ is constant.

Assume that B has property P_1. We can define for each 2-cell $\mathbf{c}$ a ring $A^\infty_{\mathbf{c}}$ with a $\mathbb{Z}$-basis $t_{\mathbf{c}} = \{t_b \mid b \in \mathbf{c}\}$, and the multiplication is defined by

$$t_b t_{b'} = \sum_{b'' \in \mathbf{c}} \gamma^{b''}_{b,b'} t_{b''},$$

where $\gamma^{b''}_{b,b'} \in \mathbb{Z}$ is given by $v^{-a(b)} c^{b''}_{b,b'} = \gamma^{b''}_{b,b'} \mod v^{-1}\mathbb{Z}[v^{-1}]$.

3.4. We say that B, equipped with property P_1, has property P_2 if

(a) for any 2-cell $\mathbf{c}$, the $\mathbb{Z}$-algebra $A^\infty_{\mathbf{c}}$ admits a generalized unit $\mathcal{D}_{\mathbf{c}} \to A^\infty_{\mathbf{c}}$, where $\mathcal{D}_{\mathbf{c}}$ is a finite set, and the basis $t_{\mathbf{c}}$ is compatible with this generalized unit.

We will identify $\mathcal{D}_{\mathbf{c}}$ with a subset of $\mathbf{c}$, so that the embedding $\mathcal{D}_{\mathbf{c}} \to t_{\mathbf{c}}$ is $d \mapsto t_d$. In this case, the asymptotic ring $A^\infty_{\mathbf{c}}$ has 1, namely $1 = \sum_{d \in \mathcal{D}_{\mathbf{c}}} t_d$.

Assume that B has property P_1. We say that B has property P_3 if we have the following equations for any $b_1, b_2, b_3, b' \in B$ with b', b_2 belonging to the same 2-cell $\mathbf{c}$:

$$\sum_{b \in \mathbf{c}} c^b_{b_1,b_2}(v) \gamma^{b'}_{b,b_3} = \sum_{b \in \mathbf{c}} c^{b'}_{b_1,b}(v) \gamma^b_{b_2,b_3},$$

$$\sum_{b \in \mathbf{c}} \gamma^b_{b_1,b_2} c^{b'}_{b,b_3}(v) = \sum_{b \in \mathbf{c}} \gamma^{b'}_{b_1,b} c^b_{b_2,b_3}(v).$$

3.5. Given a 2-cell $\mathbf{c}$, let $T_{\mathbf{c}}$ be the set of triples (d, d', s), where $d, d' \in \mathcal{D}_{\mathbf{c}}$ and $s \in \mathrm{Irr}\ G_{\mathbf{c}}$ with $G_{\mathbf{c}}$ a reductive group over $\mathbb{C}$. Let $J_{\mathbf{c}}$ be the free abelian group on $T_{\mathbf{c}}$ with a ring structure defined by

$$(d_1, d_1', s)(d_2, d_2', s') = \delta_{d_1',d_2} \sum_{s'' \in \mathrm{Irr}\ G_{\mathbf{c}}} c^{s''}_{s,s'}(d_1, d_2', s''),$$

where $c^{s''}_{s,s'}$ is the multiplicity of s'' in the tensor product $s \otimes s'$.

Since $\mathcal{D}_{\mathbf{c}}$ is a finite set, we will use $\{1, 2, \dots, n_{\mathbf{c}}\}$ to label these elements in it, where $n_{\mathbf{c}} = |\mathcal{D}_{\mathbf{c}}|$. From now on we will always use this fixed label.

We say that B, endowed with properties P_1 and P_2, has property P_4 if the following conditions hold:

(a) For each 2-cell $\mathbf{c}$, there is a bijection between $\mathbf{c}$ and the set $C = \{(j, l, s) \mid 1 \leq j, l \leq n_{\mathbf{c}}, s \in \mathrm{Irr}\ G_{\mathbf{c}}\}$. Moreover, if $b \in \mathbf{c}$ corresponds to (j, l, s), we have $i(b)$ corresponds to $(l, j, \sigma(s))$, where $\sigma(s)$ denotes the dual representation of s.

Hereafter, we will identify $\mathbf{c}$ with the set C.

(b) There exists a ring isomorphism $A^\infty_{\mathbf{c}} \to J_{\mathbf{c}}$, that is, the asymptotic ring $A^\infty_{\mathbf{c}}$ is isomorphic to an $n_{\mathbf{c}} \times n_{\mathbf{c}}$ matrix algebra over $B_{\mathbf{c}}$, where $B_{\mathbf{c}}$ is the representation ring of $G_{\mathbf{c}}$. The isomorphism is given by $t_b \mapsto E_{jl}(s)$ for $b = (j, l, s) \in \mathbf{c}$.

3.6. From now on we will assume that B has properties P_1, P_2, P_3 and P_4. For each 2-cell $\mathbf{c}$, $B_{\mathbf{c}}$ is an affine commutative $\mathbb{Z}$-algebra ([Se, §3.6]). By property P_4(b) each element in $A^\infty_{\mathbf{c}}$ is a matrix over $B_{\mathbf{c}}$, so we identify t_b with $E_{jl}(s)$ for each $b = (j, l, s) \in \mathbf{c}$. We may also label the basis element b, that is, we write $\widetilde{E}_{jl}(s)$ for b with $b = (j, l, s) \in \mathbf{c}$.

Lemma 3.1. *Let $A^{\infty}_{\mathbf{c},k} = k \otimes_{\mathbb{Z}} A^{\infty}_{\mathbf{c}}$. Then $A^{\infty}_{\mathbf{c},k}$ is an $A^{\infty}_{\mathbf{c},k}$-$A_{\mathbf{c}}$-bimodule via*

$$t_b \circ b' = \sum_{b'' \in \mathbf{c}} c^{b''}_{b,b'}(v) t_{b''}.$$

Proof. It follows from property P_3. □

Lemma 3.2. *In $A^{\infty}_{\mathbf{c},k}$, for all $b, b' \in \mathbf{c}$ we have*

$$t_b \circ b' = t_b\Big(\Big(\sum_{d\in\mathcal{D}_{\mathbf{c}}} t_d\Big) \circ \Big(\sum_{d\in\mathcal{D}_{\mathbf{c}}} d\Big)\Big) t_{b'}.$$

Proof. Since $\sum_{d\in\mathcal{D}_{\mathbf{c}}} t_d$ is the identity of $A^{\infty}_{\mathbf{c}}$, we have by Lemma 3.1 the following equalities:

$$t_b\Big(\Big(\sum_{d\in\mathcal{D}_{\mathbf{c}}} t_d\Big) \circ \Big(\sum_{d\in\mathcal{D}_{\mathbf{c}}} d\Big)\Big) t_{b'} = \Big(t_b \circ \Big(\sum_{d\in\mathcal{D}_{\mathbf{c}}} d\Big)\Big) t_{b'} = \sum_{\substack{d\in\mathcal{D}_{\mathbf{c}}\\ a,b''\in\mathbf{c}}} c^{a}_{b,d}\gamma^{b''}_{a,b'} t_{b''}.$$

Now, we use property P_3 to replace $\sum_a c^{a}_{b,d}\gamma^{b''}_{a,b'}$ by $\sum_a c^{b''}_{b,a}\gamma^{a}_{d,b'}$, and we get

$$t_b\Big(\Big(\sum_{d\in\mathcal{D}_{\mathbf{c}}} t_d\Big) \circ \Big(\sum_{d\in\mathcal{D}_{\mathbf{c}}} d\Big)\Big) t_{b'} = \sum_{\substack{d\in\mathcal{D}_{\mathbf{c}}\\ a,b''\in\mathbf{c}}} c^{b''}_{b,a}\gamma^{a}_{d,b'} t_{b''} = \sum_{a,b''\in\mathbf{c}} c^{b''}_{b,a}\Big(\sum_{d\in\mathcal{D}_{\mathbf{c}}} \gamma^{a}_{d,b'}\Big) t_{b''}.$$

Since $\sum_{d\in\mathcal{D}_{\mathbf{c}}} t_d$ is the identity of $A^{\infty}_{\mathbf{c}}$, we have $\sum_{d\in\mathcal{D}_{\mathbf{c}}} \gamma^{a}_{d,b'} = \delta_{a,b'}$. Then we get

$$t_b\Big(\Big(\textstyle\sum_{d\in\mathcal{D}_{\mathbf{c}}} t_d\Big) \circ \Big(\sum_{d\in\mathcal{D}_{\mathbf{c}}} d\Big)\Big) t_{b'} = \displaystyle\sum_{b''\in\mathbf{c}} c^{b''}_{b,b'} t_{b''} = t_b \circ b'.$$ □

When Lemma 3.2 gets translated into matrix language, the left-hand side of the equation expresses the multiplication $\widetilde{E}_{jl}(s) \cdot \widetilde{E}_{pq}(s')$ in $A_{\mathbf{c}}$ for $b = (j, l, s) \in \mathbf{c}$ and $b' = (p, q, s') \in \mathbf{c}$, and the right-hand side is just the product $E_{jl}(s)\Psi_{\mathbf{c}}E_{pq}(s')$ in the usual matrix algebra $A^{\infty}_{\mathbf{c},k}$, where $\Psi_{\mathbf{c}}$ is the matrix representing $(\sum_{d\in\mathcal{D}_{\mathbf{c}}} t_d) \circ (\sum_{d\in\mathcal{D}_{\mathbf{c}}} d)$.

From property P_4(a), there exists a k-involution i on $A_{\mathbf{c}}$, which is given by $b \mapsto i(b)$ for any $b \in \mathbf{c}$.

Summarizing, we get the following result.

Proposition 3.3. *Let $\mathbf{c}$ be a 2-cell. Then there is a matrix $\Psi_{\mathbf{c}}$ in $A^{\infty}_{\mathbf{c},k}$ such that $A_{\mathbf{c}}$ can be identified with the generalized matrix algebra $(M_{n_{\mathbf{c}}}(B_{\mathbf{c}}),\ \Psi_{\mathbf{c}})$. Moreover, if b is identified with $\widetilde{E}_{jl}(s)$ for $b = (j, l, s) \in \mathbf{c}$, then we obtain that $i(b)$ is identified with $\widetilde{E}_{lj}(\sigma(s))$. The multiplication in $(M_{n_{\mathbf{c}}}(B_{\mathbf{c}}),\ \Psi_{\mathbf{c}})$ is given by*

$$\widetilde{E}_{jl}(s) \cdot \widetilde{E}_{pq}(s') = E_{jl}(s)\Psi_{\mathbf{c}}E_{pq}(s').$$

3.7. From property P_1(a), we can easily get a linear order $\leq$ on the set of 2-cells $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_f, \ldots$ such that $\mathbf{c}_j \preceq_{LR} \mathbf{c}_l$ implies that $j \leq l$.

For each j, we define $\mathscr{C}'_j$ to be the k-submodule generated by all b with $b \in \mathbf{c}_j$, and set $\mathscr{C}_j = \bigoplus_{l=1}^{j} \mathscr{C}'_l$. Then $\mathscr{C}'_j$ is invariant under the involution i, $\mathscr{C}_j$ is a two-sided ideal in A with $\mathscr{C}_j/\mathscr{C}_{j-1} = A_{\mathbf{c}_j}$, and the chain

$$\mathscr{C}_1 \subset \mathscr{C}_2 \subset \cdots \subset \mathscr{C}_f \subset \cdots \subset A$$

is a cell chain for A by Propositions 2.2 and 3.3. Thus we have proved the following theorem.

Theorem 3.4. *Let $k = \mathbb{Z}[v, v^{-1}]$ (v an indeterminate). If a k-basis B of A with a k-involution i satisfies properties P_1, P_2, P_3 and P_4, then A is an affine cellular $\mathbb{Z}$-algebra with respect to i.*

4. Affine cellularity of affine q-Schur algebras

We first give a definition of the affine q-Schur algebra $\mathfrak{U}_{r,n,n}$ following [L3] (see also [GV]). Thus, let V_ϵ be a free $\mathbf{k}[\epsilon, \epsilon^{-1}]$-module of rank r, where $\mathbf{k}$ is a finite field with q elements, and ϵ is an indeterminate.

A sequence $\mathbf{L} = (L_i)_{i\in\mathbb{Z}}$ of lattices in V_ϵ is called an n-step periodic lattice if $L_i \subset L_{i+1}$ and $L_{i-n} = \epsilon L_i$. Let $\mathcal{F}^n$ be the space of n-step periodic lattices, which is naturally endowed with an action of $\mathrm{G} = \mathrm{Aut}(V_\epsilon)$. Let $\mathfrak{S}_{r,n}$ be the set of nonnegative integer sequences $(a_i)_{i\in\mathbb{Z}}$ such that $a_i = a_{i+n}$ and $\sum_{i=1}^n a_i = r$, and let $\mathfrak{S}_{r,n,n}$ be the set of $\mathbb{Z}\times\mathbb{Z}$ matrices $A = (a_{i,j})_{i,j\in\mathbb{Z}}$ with nonnegative entries such that $a_{i,j} = a_{i+n,j+n}$ and $\sum_{i\in[1,n],j\in\mathbb{Z}} a_{i,j} = r$. The orbits of G on $\mathcal{F}^n$ are indexed by $\mathfrak{S}_{r,n}$, where $\mathbf{L}$ is in the orbit $\mathcal{F}_{\mathbf{a}}$ corresponding to $\mathbf{a}$ if $a_i = \dim_{\mathbf{k}}(L_i/L_{i-1})$. The orbits of G on $\mathcal{F}^n \times \mathcal{F}^n$ are indexed by $\mathfrak{S}_{r,n,n}$, where a pair $(\mathbf{L}, \mathbf{L}')$ is in the orbit $\mathcal{O}_A$ corresponding to A if

$$a_{i,j} = \dim\left(\frac{L_i \cap L'_j}{(L_{i-1}\cap L'_j) + (L_i \cap L'_{j-1})}\right).$$

For $A \in \mathfrak{S}_{r,n,n}$, let $r(A), c(A) \in \mathfrak{S}_{r,n}$ be given by $r(A)_i = \sum_{j\in\mathbb{Z}} a_{i,j}$ and $c(A)_j = \sum_{i\in\mathbb{Z}} a_{i,j}$.

Let $\mathfrak{U}_{r,q}$ be the span of the characteristic functions of the G-orbits on $\mathcal{F}^n \times \mathcal{F}^n$. Convolution makes $\mathfrak{U}_{r,q}$ an algebra. For $A \in \mathfrak{S}_{r,n,n}$, set

$$d_A = \sum_{i\geq k; j<l; 1\leq i\leq n} a_{i,j}a_{k,l}.$$

Let $\{[A] \mid A \in \mathfrak{S}_{r,n,n}\}$ be the basis of $\mathfrak{U}_{r,q}$ given by $q^{-d_A/2}$ times the characteristic function of the orbit corresponding to A.

This space of functions is the specialization at $v = \sqrt{q}$ of modules over $k = \mathbb{Z}[v, v^{-1}]$, which we denote by $\mathfrak{U}_r = \mathfrak{U}_{r,n,n}$ (here v is an indeterminate). The k-algebra $\mathfrak{U}_r$ is just the affine q-Schur algebra.

For each $A \in \mathfrak{S}_{r,n,n}$, there exists a unique element $\{A\} \in \mathfrak{U}_r$ defined by

$$\{A\} = \sum_{A_1; A_1\leq A} \Pi_{A_1,A}[A_1],$$

where $\leq$ is a natural partial order on $\mathfrak{S}_{r,n,n}$ and the $\Pi_{A_1,A}$ are certain polynomials in $\mathbb{Z}[v^{-1}]$ (see [L3, Section 4]). The elements $\{A\}$ ($A \in \mathfrak{S}_{r,n,n}$) form a k-basis $\mathfrak{B}_r$ of $\mathfrak{U}_r$. The algebra $\mathfrak{U}_r$ has a natural k-linear anti-automorphism i which sends $\{A\}$ to $\{A^t\}$.

The fact that $\mathfrak{B}_r$ has property P_1 follows from [Sh] and [Mc, Proposition 4.4 and Lemma 4.12]. We can easily get that $\mathfrak{B}_r$ has property P_2. The fact that $\mathfrak{B}_r$ has property P_3 can be deduced from [Mc, Lemma 2.2] and that for the affine Hecke algebra. The fact that $\mathfrak{B}_r$ has property P_4 is shown in [Mc, Proposition 4.13].

Thus, the basis $\mathfrak{B}_r$ of $\mathfrak{U}_r$ satisfies properties P_1, P_2, P_3 and P_4 of Section 3. From Theorem 3.4 we can get the following result.

Theorem 4.1. *Let $k = \mathbb{Z}[v, v^{-1}]$ (v an indeterminate). The affine q-Schur algebra $\mathfrak{U}_r$ over k is an affine cellular $\mathbb{Z}$-algebra with respect to the k-involution i.*

By specializing at $v^{\pm 1} = 1$, we get the affine Schur algebra $\mathfrak{u}_r$. From [KX, Lemma 2.4] we can also obtain the following theorem.

Theorem 4.2. *The affine Schur algebra $\mathfrak{u}_r$ over $\mathbb{Z}$ is an affine cellular $\mathbb{Z}$-algebra.*

In the remainder of this section we will consider the affine q-Schur algebra $\mathfrak{U}_{r,\mathbb{C}} = \mathbb{C} \otimes_k \mathfrak{U}_r$ over $\mathbb{C}$, where $\mathbb{C}$ is regarded as a k-module by specializing v to z and $z \in \mathbb{C}^*$ is not a root of unity.

Given $a \in \mathbb{C}^*$, a segment $\mathbf{s}$ with center a is defined as an ordered sequence

$$\mathbf{s} = (az^{-k+1}, az^{-k+3}, \ldots, az^{k-1}) \in (\mathbb{C}^*)^k,$$

where k is called the length of the segment, denoted by $|\mathbf{s}|$. If $\mathbf{s} = \{\mathbf{s}_1, \ldots, \mathbf{s}_p\}$ is an unordered collection of segments, we define $\wp(\mathbf{s})$ to be the partition associated with the sequence $(|\mathbf{s}_1|, \ldots, |\mathbf{s}_p|)$. That is, $\wp(\mathbf{s}) = (|\mathbf{s}_{i_1}|, \ldots, |\mathbf{s}_{i_p}|)$ with $|\mathbf{s}_{i_1}| \geq \cdots \geq |\mathbf{s}_{i_p}|$, where $|\mathbf{s}_{i_1}|, \ldots, |\mathbf{s}_{i_p}|$ is a permutation of $|\mathbf{s}_1|, \ldots, |\mathbf{s}_p|$. We also call $|\mathbf{s}| := |\mathbf{s_1}| + \cdots + |\mathbf{s_p}|$ the length of $\mathbf{s}$. Let $\mathscr{S}_r$ be the set of unordered collections of segments $\mathbf{s}$ with $|\mathbf{s}| = r$, and let $\mathscr{S}_r^{(n)} = \bigcup_{\lambda \in \mathcal{P}_r^n} \mathscr{S}_{r,\lambda}$, where $\mathcal{P}_r^n$ is the set of partitions of r with at most n parts and $\mathscr{S}_{r,\lambda} = \{\mathbf{s} \in \mathscr{S}_r | \ \wp(\mathbf{s}) = \lambda\}$.

Let $\mathbf{c}_\lambda$ be a 2-cell corresponding to a partition $\lambda \in \mathcal{P}_r^n$ by [Mc, Proposition 4.4], and let $\mathfrak{U}_{r,\mathbf{c}_\lambda}$ be the free $\mathbb{C}$-submodule of $\mathfrak{U}_{r,\mathbb{C}}$, which is spanned by $\{A\}$ with $\{A\} \in \mathbf{c}_\lambda$.

Lemma 4.3. *When $z \in \mathbb{C}^*$ is not a root of unity, $\mathfrak{U}_{r,\mathbf{c}_\lambda}$ is idempotent, and moreover, is generated by a nonzero idempotent element.*

Proof. Let $J_{\mathbb{C}} = \bigoplus_{\lambda \in \mathcal{P}_r^n} \mathbb{C} \otimes_{\mathbb{Z}} J_{\mathbf{c}_\lambda}$ be the asymptotic ring for $\mathfrak{U}_{r,\mathbb{C}}$. Since $\mathfrak{B}_r$ has property P_4(b), it follows that the set of isomorphism classes of simple $\mathbb{C} \otimes_{\mathbb{Z}} J_{\mathbf{c}_\lambda}$-modules is in bijection with the semisimple conjugacy classes of $G_\lambda = \prod_{i=1}^n GL_{\lambda(i)}(\mathbb{C})$ with $\lambda(i) = \lambda_i - \lambda_{i+1}$ (where $\lambda_{n+1} = 0$). It follows from [Ro], [Z] and [L1, Corollary 3.6] that the set of isomorphism classes of simple $J_{\mathbb{C}}$-modules is in bijection with the set $\mathscr{S}_r^{(n)}$ when $z \in \mathbb{C}^*$ is not a root of unity. According to [DDF, Theorems 4.3.4 and 4.5.3], when $z \in \mathbb{C}^*$ is not a root of unity, the set of nonisomorphic simple $\mathfrak{U}_{r,\mathbb{C}}$-modules is also parameterized by the set $\mathscr{S}_r^{(n)}$. So we have obtained a bijection between the parameter set of nonisomorphic simple $J_{\mathbb{C}}$-modules and that of nonisomorphic simple $\mathfrak{U}_{r,\mathbb{C}}$-modules. Applying [KX, Theorem 4.1(1)], we get the idempotence of $\mathfrak{U}_{r,\mathbf{c}_\lambda}$.

We can describe an element A of $\mathfrak{S}_{r,n,n}$ uniquely by a triple consisting of a pair $\mathbf{a}, \mathbf{b} \in \mathfrak{S}_{r,n}$ together with an element $w_A \in \mathrm{W}$, where W is the extended affine Weyl group of type A. Indeed, $\mathbf{a}, \mathbf{b}$ are just $r(A)$ and $c(A)$, respectively, and w_A is the element of maximal length in the finite double coset of $S_{\mathbf{a}} \backslash \mathrm{W} / S_{\mathbf{b}}$. If we set $A = (\lambda, w_\lambda, \lambda)$, where w_λ is the longest element in the standard Young subgroup $W_\lambda := \mathfrak{S}_{(\lambda_1, \ldots, \lambda_n)}$, we can easily get that $A \in \mathbf{c}_\lambda$ and $\{A\}\{A\} = \{A\}$. By [KX, Theorem 4.3(1)], we get that $\mathfrak{U}_{r,\mathbf{c}_\lambda}$ is generated by the idempotent element $\{A\}$. We have proved the claims. ☐

Applying Theorem 2.3, from Theorem 4.1 and Lemma 4.3 we can get the following theorem.

Theorem 4.4. *Assume that $z \in \mathbb{C}^*$ is not a root of unity. Then all cells in the cell chain of $\mathfrak{U}_{r,\mathbb{C}}$ correspond to idempotent ideals, which all have idempotent generators. Moreover, $\mathfrak{U}_{r,\mathbb{C}}$ is of finite global dimension and its derived module* *category admits a stratification whose sections are the derived module categories of the various algebras* $B_{\mathbf{c}_\lambda}$.

Remark 4.5. Based on an algebraic study of extension algebras, S. Kato [Ka] has obtained that the affine q-Schur algebra $\mathfrak{U}_{r,\mathbb{C}}$ has finite global dimension. Derived module categories and stratifications are not considered in [Ka]. When $n > r$, we have proved in [C] and [N] that the affine q-Schur algebra $\mathfrak{U}_{r,k}$ over a noetherian domain k always has finite global dimension provided that k has that. From Theorem 4.4, we immediately get that the affine q-Schur algebra $\mathfrak{U}_{r,\mathbb{C}}$ is an affine quasi-hereditary algebra when the parameter $z \in \mathbb{C}^*$ is not a root of unity.

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