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WEIGHTED ENDPOINT ESTIMATES FOR COMMUTATORS OF CALDERÓN-ZYGMUND OPERATORS

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ABSTRACT. Let $\delta \in (0,1]$ and T be a δ-Calderón-Zygmund operator. Let w be in the Muckenhoupt class $A_{1+\delta/n}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \frac{w(x)}{1+|x|^n}\,dx < \infty$. When $b \in \mathrm{BMO}(\mathbb{R}^n)$, it is well known that the commutator $[b,T]$ is not bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ if b is not a constant function. In this article, the authors find out a proper subspace $\mathcal{BMO}_w(\mathbb{R}^n)$ of $\mathrm{BMO}(\mathbb{R}^n)$ such that, if $b \in \mathcal{BMO}_w(\mathbb{R}^n)$, then $[b,T]$ is bounded from the weighted Hardy space $H^1_w(\mathbb{R}^n)$ to the weighted Lebesgue space $L^1_w(\mathbb{R}^n)$. Conversely, if $b \in \mathrm{BMO}(\mathbb{R}^n)$ and the commutators of the classical Riesz transforms $\{[b,R_j]\}_{j=1}^n$ are bounded from $H^1_w(\mathbb{R}^n)$ to $L^1_w(\mathbb{R}^n)$, then $b \in \mathcal{BMO}_w(\mathbb{R}^n)$.

1. Introduction

Given a function b locally integrable on $\mathbb{R}^n$ and a classical Calderón-Zygmund operator T, we consider the linear commutator $[b,T]$ defined by setting, for all smooth, compactly supported functions f,

$$[b,T](f) := bT(f) - T(bf).$$

A classical result of Coifman et al. [5] states that the commutator $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$ for $p \in (1,\infty)$, when $b \in \mathrm{BMO}(\mathbb{R}^n)$. Moreover, their proof does not rely on a weak type $(1,1)$ estimate for $[b,T]$. Indeed, this operator is more singular than the associated Calderón-Zygmund operator since it fails, in general, to be of weak type $(1,1)$, when b is in $\mathrm{BMO}(\mathbb{R}^n)$. Moreover, Harboure et al. [9, Theorem (3.1)] showed that $[b,T]$ is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ if and only if b is equal to a constant almost everywhere. Although the commutator $[b,T]$ does not map continuously, in general, $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, following Pérez [13], one can find a subspace $\mathcal{H}^1_b(\mathbb{R}^n)$ of $H^1(\mathbb{R}^n)$ such that $[b,T]$ maps continuously $\mathcal{H}^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Very recently, Ky [12] found the *largest subspace of* $H^1(\mathbb{R}^n)$ such that all commutators $[b,T]$ of Calderón-Zygmund operators are bounded from

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this subspace into $L^1(\mathbb{R}^n)$. More precisely, it was shown in [12] that there exists a bilinear operator $\mathfrak{R} := \mathfrak{R}_T$ mapping continuously $H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ such that, for all $(f, b) \in H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n)$, we have

$$[b, T](f) = \mathfrak{R}(f, b) + T(\mathfrak{S}(f, b)), \tag{1.1}$$

where $\mathfrak{S}$ is a bounded bilinear operator from $H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ which is independent of T. The bilinear decomposition (1.1) allows one to give a general overview of all known endpoint estimates; see [12] for the details.

For the weighted case, when $b \in \mathrm{BMO}(\mathbb{R}^n)$, Álvarez et al. [1] proved that the commutator $[b, T]$ is bounded on the weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ with $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$, where $A_p(\mathbb{R}^n)$ denotes the class of Muckenhoupt weights. Similar to the unweighted case, $[b, T]$ may not be bounded from the weighted Hardy space $H^1_w(\mathbb{R}^n)$ into the weighted Lebesgue space $L^1_w(\mathbb{R}^n)$ if b is not a constant function. Thus, a natural question is whether or not there exists a non-trivial subspace of $\mathrm{BMO}(\mathbb{R}^n)$ such that, when b belongs to this subspace, the commutator $[b, T]$ is bounded from $H^1_w(\mathbb{R}^n)$ to $L^1_w(\mathbb{R}^n)$.

The purpose of the present paper is to give an answer to the above question. To this end, we first recall the definition of the Muckenhoupt weights. A non-negative measurable function w is said to belong to the *class of Muckenhoupt weights,* $A_q(\mathbb{R}^n)$ with $q \in [1, \infty)$, denoted by $w \in A_q(\mathbb{R}^n)$ if, when $q \in (1, \infty)$,

$$[w]_{A_q(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B w(x)\, dx \left\{ \frac{1}{|B|} \int_B [w(y)]^{-q'/q}\, dy \right\}^{q/q'} < \infty, \tag{1.2}$$

where $1/q + 1/q' = 1$, or, when $q = 1$,

$$[w]_{A_1(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B w(x)\, dx \left\{ \operatorname*{ess\,sup}_{y \in B} [w(y)]^{-1} \right\} < \infty. \tag{1.3}$$

Here the suprema are taken over all balls $B \subset \mathbb{R}^n$. Let

$$A_\infty(\mathbb{R}^n) := \bigcup_{q \in [1, \infty)} A_q(\mathbb{R}^n).$$

Let $w \in A_\infty(\mathbb{R}^n)$ and $q \in (0, \infty]$. If $q \in (0, \infty)$, then we let $L^q_w(\mathbb{R}^n)$ be the space of all measurable functions f such that

$$\|f\|_{L^q_w(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} |f(x)|^q w(x)\, dx \right\}^{1/q} < \infty. \tag{1.4}$$

When $q = \infty$, $L^\infty_w(\mathbb{R}^n)$ is defined to be the same as $L^\infty(\mathbb{R}^n)$ and, for any $f \in L^\infty_w(\mathbb{R}^n)$, let

$$\|f\|_{L^\infty_w(\mathbb{R}^n)} := \|f\|_{L^\infty(\mathbb{R}^n)}.$$

Let ϕ be a function in the Schwartz class, $\mathcal{S}(\mathbb{R}^n)$, satisfying $\phi(x) = 1$ for all $x \in B(\vec{0}, 1)$, here and hereafter, $\vec{0} := (\overbrace{0, \ldots, 0}^{n \text{ times}})$. The *maximal function* of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$\mathcal{M}_\phi f(x) := \sup_{t \in (0, \infty)} |f * \phi_t(x)|, \tag{1.5}$$

where $\phi_t(\cdot) := \frac{1}{t^n}\phi(t^{-1}\cdot)$ for all $t \in (0, \infty)$. Then the *weighted Hardy space* $H^1_w(\mathbb{R}^n)$

is defined as the space of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H^1_w(\mathbb{R}^n)} := \|\mathcal{M}_\phi f\|_{L^1_w(\mathbb{R}^n)} < \infty;$$

see [6].

Notice that $\|\cdot\|_{H^1_w(\mathbb{R}^n)}$ defines a norm on $H^1_w(\mathbb{R}^n)$, whose size depends on the choice of ϕ, but the space $H^1_w(\mathbb{R}^n)$ is independent of this choice.

Definition 1.1. Let $w \in A_\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \frac{w(x)}{1+|x|^n}\,dx < \infty$. A locally integrable function b is said to be in $\mathcal{BMO}_w(\mathbb{R}^n)$ if
(1.6)

$$\|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \left\{ \frac{1}{w(B)} \left[\int_{B^{\complement}} \frac{w(x)}{|x-x_B|^n}\,dx\right] \left[\int_B |b(y)-b_B|\,dy\right]\right\} < \infty,$$

where the supremum is taken over all balls $B := B(x_B, r_B) \subset \mathbb{R}^n$, with $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, and $B^{\complement} := \mathbb{R}^n \backslash B$. Here and hereafter, x_B denotes the center of the ball B and r_B its radius,

$$w(B) := \int_B w(z)\,dz \quad \text{and} \quad b_B := \frac{1}{|B|}\int_B b(z)\,dz.$$

It should be pointed out that the space $\mathcal{BMO}_w(\mathbb{R}^n)$ has been considered first by Bloom [2] when studying the pointwise multipliers of weighted BMO spaces (see also [15]).

Recall that a locally integrable function b is said to be in $\mathrm{BMO}(\mathbb{R}^n)$ if

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |b(x)-b_B|\,dx < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

Remark 1.2. (i) $\mathcal{BMO}_w(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$ and the inclusion is continuous (see Proposition 2.1 of Section 2).

(ii) For any $w \in A_{1+\delta/n}(\mathbb{R}^n)$, with $\delta \in (0,1]$, satisfying $\int_{\mathbb{R}^n} \frac{w(x)}{1+|x|^n} dx < \infty$, the space $\mathcal{BMO}_w(\mathbb{R}^n)$ is not a trivial function space, moreover, any Lipschitz function b with compact support belongs to $\mathcal{BMO}_w(\mathbb{R}^n)$.

Indeed, let b be a Lipschitz function with compact support. For the simplicity of the presentation, without loss of generality, we may assume that $\operatorname{supp} b \subset B(\vec{0},1)$. In what follows, let

$$\|b\|_{\mathrm{Lip}(\mathbb{R}^n)} := \sup_{\substack{x,\,y\in\mathbb{R}^n\\ x\neq y}} \frac{|b(x)-b(y)|}{|x-y|}. \tag{1.7}$$

Then, for any ball $B := B(x_B, r_B) \subset B(\vec{0},1)$ with some $x_B \in \mathbb{R}^n$ and $r_B \in (0,1)$, we have

$$\begin{aligned}(1.8)\quad & \frac{1}{w(B)}\int_{B^{\complement}} \frac{w(x)}{|x-x_B|^n}\,dx \int_B |b(y)-b_B|\,dy \\ &= \frac{1}{w(B)}\left[\int_{r_B\le|x-x_B|<1} \frac{w(x)}{|x-x_B|^n}\,dx + \int_{|x-x_B|\ge 1}\cdots\right]\int_B |b(y)-b_B|\,dy \\ &=: \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

Let $M \in \mathbb{Z}_+$ be such that $2^{M-1}r_B \le 1 < 2^M r_B$. Since $w \in A_{1+\delta/n}(\mathbb{R}^n)$, it follows that there exists $p \in [1, 1+\delta/n)$ such that $w \in A_p(\mathbb{R}^n)$. By this and (1.7), we obtain

$$\begin{aligned}\mathrm{I}_1 &\le \sum_{i=1}^{M} \frac{w(2^iB)}{|2^iB|} \frac{1}{w(B)|B|} \int_B \int_B |b(x)-b(y)|\,dy\,dx \\ &\lesssim \sum_{i=1}^{M} 2^{in(p-1)} \frac{\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}}{|B|^2} \int_B \int_B |x-y|\,dy\,dx \lesssim r_B^{1-\delta}\|b\|_{\mathrm{Lip}(\mathbb{R}^n)} < \infty.\end{aligned} \tag{1.9}$$

For I_2, by the assumption that $\int_{\mathbb{R}^n} \frac{w(x)}{1+|x|^n}dx < \infty$, $w \in A_p(\mathbb{R}^n)$ and (1.7), we conclude that

$$\begin{aligned}\mathrm{I}_2 &\le \frac{1}{w(B)|B|} \int_B \int_B |b(x)-b(y)|\,dy\,dx \lesssim \frac{|B|^{1+\frac{1}{n}}}{w(B)} \|b\|_{\mathrm{Lip}(\mathbb{R}^n)} \\ &\lesssim \frac{|B|^{1+\frac{1}{n}-p}\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}}{w(B(\vec{0},1))} \lesssim \frac{\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}}{w(B(\vec{0},1))} < \infty,\end{aligned}$$

which, together with (1.8) and (1.9), further implies that $\|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)} < \infty$ and hence $b \in \mathcal{BMO}_w(\mathbb{R}^n)$. This shows the above claim.

To state our main results, we first recall the definition of Calderón-Zygmund operators. For $\delta \in (0,1]$, a linear operator T is called a *δ-Calderón-Zygmund operator* if T is a linear bounded operator on $L^2(\mathbb{R}^n)$ and there exist a kernel K on $(\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(x,x): x \in \mathbb{R}^n\}$ and a positive constant C such that, for all x, y, $z \in \mathbb{R}^n$,

$$|K(x,y)| \le C\frac{1}{|x-y|^n} \quad \text{if} \quad x \ne y,$$

$$|K(x,y)-K(x,z)| + |K(y,x)-K(z,x)| \le C\frac{|y-z|^\delta}{|x-y|^{n+\delta}} \quad \text{if} \quad |x-y| > 2|y-z|$$

and, for any $f \in L^2(\mathbb{R}^n)$ with compact support and $x \notin \mathrm{supp}\,(f)$,

$$Tf(x) = \int_{\mathrm{supp}\,(f)} K(x,y)f(y)\,dy.$$

The main result of this paper is the following theorem.

Theorem 1.3. *Let $\delta \in (0,1]$, $w \in A_{1+\delta/n}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \frac{w(x)}{1+|x|^n}\,dx < \infty$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then the following two statements are equivalent:*

(i) *for every δ-Calderón-Zygmund operator T, the commutator $[b,T]$ is bounded from $H^1_w(\mathbb{R}^n)$ into $L^1_w(\mathbb{R}^n)$;*

(ii) *$b \in \mathcal{BMO}_w(\mathbb{R}^n)$.*

Remark 1.4. When $w(x) \equiv 1$ for all $x \in \mathbb{R}^n$, we see that $\int_{\mathbb{R}^n} \frac{1}{1+|x|^n}\,dx = \infty$ and hence, in this case, for any ball $B := B(x_B, r_B) \subset \mathbb{R}^n$ with some $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, $\int_{B^\complement} \frac{1}{|x-x_B|^n}\,dx = \infty$, which, together with (1.6), implies that $b \in \mathcal{BMO}_w(\mathbb{R}^n)$ if and only if b is a constant. In this sense, $\mathcal{BMO}_w(\mathbb{R}^n)$ when $w \equiv 1$ can be seen as a zero space in $\mathrm{BMO}(\mathbb{R}^n)$ and, in this case, Theorem 1.3 coincides with the result in [9].

The next theorem gives a sufficient condition for the boundedness of $[b,T]$ on $H^1_w(\mathbb{R}^n)$. Recall that, for $w \in A_p(\mathbb{R}^n)$ with $p \in (1,\infty)$ and $q \in [p,\infty]$, a measurable function a is called an $(H^1_w(\mathbb{R}^n), q)$-*atom* related to a ball $B \subset \mathbb{R}^n$ if

(i) $\operatorname{supp} a \subset B$,
(ii) $\int_{\mathbb{R}^n} a(x)\, dx = 0$ and
(iii) $\|a\|_{L^q_w(\mathbb{R}^n)} \le [w(B)]^{1/q-1}$,

and also that $T^*1 = 0$ means $\int_{\mathbb{R}^n} Ta(x)\, dx = 0$ holds true for all $(H^1_w(\mathbb{R}^n), q)$-atoms a.

Theorem 1.5. *Let $\delta \in (0, 1]$, T be a δ-Calderón-Zygmund operator, $w \in A_{1+\delta/n}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \frac{w(x)}{1+|x|^n}\, dx < \infty$ and $b \in \mathcal{BMO}_w(\mathbb{R}^n)$. If $T^*1 = 0$, then the commutator $[b, T]$ is bounded on $H^1_w(\mathbb{R}^n)$, namely, there exists a positive constant C such that, for all $f \in H^1_w(\mathbb{R}^n)$,*

$$\|[b, T](f)\|_{H^1_w(\mathbb{R}^n)} \le C\|f\|_{H^1_w(\mathbb{R}^n)}.$$

Finally we make some conventions on notation. Throughout the whole article, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. The *symbol* $A \lesssim B$ means that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. For any measurable subset E of $\mathbb{R}^n$, we denote by $E^{\complement}$ the *set* $\mathbb{R}^n \setminus E$ and its *characteristic function* by χ_E. We also let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$.

2. Proofs of Theorems 1.3 and 1.5

We begin with pointing out that, if $w \in A_\infty(\mathbb{R}^n)$, then there exist $p \in [1, \infty)$ and $r \in (1, \infty)$ such that $w \in A_p(\mathbb{R}^n) \cap RH_r(\mathbb{R}^n)$ (see, for example, [8, Chapter IV, Lemma 2.5]), where $RH_r(\mathbb{R}^n)$ denotes the *reverse Hölder class* of weights w satisfying that there exists a positive constant C, depending on $[w]_{A_p(\mathbb{R}^n)}$, such that, for any ball $B \subset \mathbb{R}^n$,

$$\left\{\frac{1}{|B|}\int_B [w(x)]^r\, dx\right\}^{1/r} \le C\frac{1}{|B|}\int_B w(x)\, dx;$$

moreover, there exist positive constants $C_1 \le C_2$, depending on $[w]_{A_p(\mathbb{R}^n)}$, such that, for any measurable sets $E \subset B$,

$$C_1\left(\frac{|E|}{|B|}\right)^p \le \frac{w(E)}{w(B)} \le C_2\left(\frac{|E|}{|B|}\right)^{(r-1)/r} \tag{2.1}$$

(see, for example, [8, Chapter IV, (1.6) and Theorem 2.9]).

In order to prove Theorems 1.3 and 1.5, we need the following proposition and several technical lemmas.

Proposition 2.1. *Let $w \in A_\infty(\mathbb{R}^n)$. Then there exists a positive constant C such that, for any $f \in \mathcal{BMO}_w(\mathbb{R}^n)$,*

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C\|f\|_{\mathcal{BMO}_w(\mathbb{R}^n)}.$$

Proof. By (2.1), for any ball $B := B(x_B, r_B) \subset \mathbb{R}^n$ with some $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$, we have

$$\begin{aligned}\int_{B^{\complement}} \frac{w(x)}{|x - x_B|^n}\, dx \frac{1}{w(B)} &\ge \int_{2B\setminus B} \frac{w(x)}{|x - x_B|^n}\, dx \frac{1}{w(B)}\\ &\ge \frac{w(2B\setminus B)}{|2B|}\frac{1}{w(B)} \gtrsim \frac{1}{|B|}.\end{aligned}$$

This proves that $\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{BMO}_w(\mathbb{R}^n)}$ and hence finishes the proof of Proposition 2.1. $\square$

Lemma 2.2. *Let f be a measurable function such that* supp $f \subset B := B(x_0, r)$ *with some $x_0 \in \mathbb{R}^n$ and $r \in (0,\infty)$. Then there exists a positive constant $C := C_{(\phi,n)}$, depending only on ϕ and n, such that, for all $x \notin B$,*

$$\frac{1}{|x-x_0|^n}\left|\int_{B(x_0,r)} f(y)\,dy\right| \le C\mathcal{M}_\phi f(x).$$

Proof. For $x \notin B(x_0,r)$ and any $y \in B(x_0,r)$, it follows that

$$\frac{|x-y|}{2|x-x_0|} < \frac{|x-x_0|+r}{2|x-x_0|} \le 1,$$

which, together with $\phi \equiv 1$ on $B(\vec{0},1)$, further implies that $\phi(\frac{x-y}{2|x-x_0|}) = 1$. Thus, we know that

$$\begin{aligned}\mathcal{M}_\phi f(x) = \sup_{t\in(0,\infty)} |f*\phi_t(x)| &\ge |f*\phi_{2|x-x_0|}(x)|\\ &= \frac{1}{2^n|x-x_0|^n}\left|\int_{B(x_0,r)} f(y)\phi\left(\frac{x-y}{2|x-x_0|}\right)dy\right|\\ &\gtrsim \frac{1}{|x-x_0|^n}\left|\int_{B(x_0,r)} f(y)\,dy\right|,\end{aligned}$$

which completes the proof of Lemma 2.2. $\square$

Lemma 2.3. *Let $w \in A_\infty(\mathbb{R}^n)$ and $q \in [1,\infty)$. Then there exists a positive constant C such that, for any $f \in \mathrm{BMO}(\mathbb{R}^n)$ and any ball $B := B(x_B, r_B) \subset \mathbb{R}^n$ with some $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$,*

$$\left[\frac{1}{w(B)}\int_B |f(x)-f_B|^q w(x)\,dx\right]^{1/q} \le C\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Proof. It follows, from the John-Nirenberg inequality (see [10, Lemma 1]), that there exist two positive constants c_1 and c_2, depending only on n, such that, for all $\lambda \in (0,\infty)$,

$$|\{x\in B : |f(x)-f_B| > \lambda\}| \le c_1 e^{-c_2\frac{\lambda}{\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}}}|B|.$$

Since $w \in A_\infty(\mathbb{R}^n)$, it follows that there exists $r \in (1,\infty)$ such that (2.1) holds true, which implies that

$$\begin{aligned}\frac{1}{w(B)}\int_B |f(x)-f_B|^q w(x)\,dx &= q\int_0^\infty \lambda^{q-1}\frac{w(\{x\in B : |f(x)-f_B|>\lambda\})}{w(B)}\,d\lambda\\ &\lesssim \int_0^\infty \lambda^{q-1}\left[\frac{|\{x\in B : |f(x)-f_B|>\lambda\}|}{|B|}\right]^{(r-1)/r} d\lambda\\ &\lesssim \int_0^\infty \lambda^{q-1}e^{-c_2\frac{r-1}{r}\frac{\lambda}{\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}}}\,d\lambda \lesssim \|f\|^q_{\mathrm{BMO}(\mathbb{R}^n)},\end{aligned}$$

which completes the proof of Lemma 2.3. $\square$

Lemma 2.4. *Let $\delta \in (0,1]$, $q \in (1, 1+\delta/n)$ and $w \in A_q(\mathbb{R}^n)$. Assume that T is a δ-Calderón-Zygmund operator. Then there exists a positive constant C such that, for any $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $(H^1_w(\mathbb{R}^n), q)$-atom a related to a ball $B \subset \mathbb{R}^n$,*

$$\|(b - b_B)Ta\|_{L^1_w(\mathbb{R}^n)} \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Proof. It suffices to show that

$$\mathrm{I}_1 := \int_{2B} |[b(x) - b_B]Ta(x)|w(x)\,dx \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$$

and

$$\mathrm{I}_2 := \int_{(2B)^\complement} |[b(x) - b_B]Ta(x)|w(x)\,dx \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

For I_1, by the boundedness of T from $H^1_w(\mathbb{R}^n)$ to $L^1_w(\mathbb{R}^n)$ and from $L^q_w(\mathbb{R}^n)$ to itself with $q \in (1, 1+\delta/n)$ (see [7, Theorem 2.8]), the Hölder inequality and Lemma 2.3, we conclude that

(2.2)
$$\begin{aligned}
\mathrm{I}_1 &= \int_{2B} |[b(x) - b_B]Ta(x)|w(x)\,dx \\
&\le |b_{2B} - b_B|\|Ta\|_{L^1_w(\mathbb{R}^n)} + \int_{2B} |[b(x) - b_{2B}]Ta(x)|w(x)\,dx \\
&\lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} + \left[\int_{2B} |b(x) - b_{2B}|^{q'} w(x)\,dx\right]^{1/q'} \left[\int_{2B} |Ta(x)|^q w(x)\,dx\right]^{1/q} \\
&\lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} + [w(2B)]^{1/q'} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|a\|_{L^q_w(\mathbb{R}^n)} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)},
\end{aligned}$$

here and hereafter, $1/q' + 1/q = 1$.

On the other hand, by the Hölder inequality, (1.3), Lemma 2.3, (2.1),

$$q \in (1, 1+\delta/n)$$

and the fact that, for all $k \in \mathbb{N}$,

$$\begin{aligned}
|b_{2^{k+1}B} - b_B| &\le \sum_{j=0}^{k} |b_{2^{j+1}B} - b_{2^j B}| \\
&\le \sum_{j=0}^{k} \frac{1}{|2^j B|} \int_{2^j B} |b(x) - b_{2^{j+1}B}|\,dx \lesssim (k+1)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)},
\end{aligned}$$

we know that

$$\begin{aligned}
\mathrm{I}_2 &= \int_{(2B)^\complement} |[b(x) - b_B]Ta(x)|w(x)\,dx \\
&= \int_{(2B)^\complement} |b(x) - b_B| \left|\int_B a(y)[K(x,y) - K(x,x_0)]\,dy\right| w(x)\,dx \\
&\le \int_B |a(y)| \int_{(2B)^\complement} |b(x) - b_B|\,|K(x,y) - K(x,x_0)|\,w(x)\,dx\,dy \\
&= \int_B |a(y)| \sum_{k=1}^{\infty} \int_{2^{k+1}B\setminus 2^k B} |b(x) - b_B|\,|K(x,y) - K(x,x_0)|\,w(x)\,dx\,dy
\end{aligned}$$

$$\begin{aligned}
&\lesssim \int_B |a(y)|\,dy \sum_{k=1}^{\infty} \int_{2^{k+1}B\setminus 2^kB} \frac{r^\delta}{(2^k r)^{n+\delta}} |b(x)-b_B| w(x)\,dx \\
&\lesssim \left[\int_B |a(y)|^q w(y)\,dy\right]^{1/q} \left[\int_B [w(y)]^{-q'/q}\,dy\right]^{1/q'} \\
&\quad \times \sum_{k=1}^{\infty} 2^{-k\delta} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} \left[|b(x)-b_{2^{k+1}B}| + |b_{2^{k+1}B}-b_B|\right] w(x)\,dx \\
&\lesssim \frac{|B|}{w(B)} \sum_{k=1}^{\infty} 2^{-k\delta} k \frac{w(2^{k+1}B)}{|2^{k+1}B|} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sum_{k=1}^{\infty} k 2^{-k(\delta+n-nq)} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)},
\end{aligned}$$

which, together with (2.2), further completes the proof of Lemma 2.4. □

The following lemma is due to Bownik et al. [3, Theorem 7.2].

Lemma 2.5. *Let $w \in A_{1+\delta/n}(\mathbb{R}^n)$ and $\mathcal{X}$ be a Banach space. Assume that T is a linear operator defined on the space of finite linear combinations of continuous $(H^1_w(\mathbb{R}^n),\infty)$-atoms with the property that*

$$\sup\left\{\|T(a)\|_{\mathcal{X}} : a \text{ is a continuous } (H^1_w(\mathbb{R}^n),\infty)\text{-atom}\right\} < \infty.$$

Then T admits a unique continuous extension to a bounded linear operator from $H^1_w(\mathbb{R}^n)$ into $\mathcal{X}$.

Let $w \in A_{1+\delta/n}(\mathbb{R}^n)$ and $\varepsilon \in (0,\infty)$. Recall that m is called an $(H^1_w(\mathbb{R}^n),\infty,\varepsilon)$-*molecule* related to a ball $B \subset \mathbb{R}^n$ if

(i) $\int_{\mathbb{R}^n} m(x)dx = 0$,

(ii) $\|m\|_{L^\infty(S_j)} \le 2^{-j\varepsilon}[w(S_j)]^{-1}$, $j \in \mathbb{Z}_+$, here and hereafter $S_0 := B$ and, for any $j \in \mathbb{N}$, $S_j := 2^{j+1}B \setminus 2^jB$ with 2^jB being the ball having the center the same as the center of B and the radius 2^j times the radius of B.

Lemma 2.6. *Let $w \in A_{1+\delta/n}(\mathbb{R}^n)$ and $\varepsilon \in (0,\infty)$. Then there exists a positive constant C such that, for any $(H^1_w(\mathbb{R}^n),\infty,\varepsilon)$-molecule m related to a ball B, it holds true that*

$$m = \sum_{j=0}^{\infty} \lambda_j a_j,$$

where $\{a_j\}_{j=0}^{\infty}$ are $(H^1_w(\mathbb{R}^n),\infty)$-atoms related to the balls $\{2^{j+1}B\}_{j\in\mathbb{Z}_+}$ and there exists a positive constant C such that $|\lambda_j| \le C2^{-j\varepsilon}$ for all $j \in \mathbb{Z}_+$.

Proof. The proof of this lemma is standard (see, for example, [14, Theorem 4.7]), the details being omitted. □

Now we are ready to give the proofs of Theorems 1.3 and 1.5.

Proof of Theorem 1.3. First, we prove that (ii) implies (i). Since $w \in A_{1+\delta/n}(\mathbb{R}^n)$, it follows that there exists $q \in (1, 1+\delta/n)$ such that $w \in A_q(\mathbb{R}^n)$. By Lemma 2.5, it suffices to prove that, for any continuous $(H^1_w(\mathbb{R}^n),\infty)$-atom a related to some ball $B := B(x_0, r)$ with $x_0 \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$\|[b,T](a)\|_{L^1_w(\mathbb{R}^n)} \lesssim \|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)}. \tag{2.3}$$

By Lemma 2.4 and the boundedness of T from $H^1_w(\mathbb{R}^n)$ to $L^1_w(\mathbb{R}^n)$, (2.3) is reduced to showing that

$$\|(b-b_B)a\|_{H^1_w(\mathbb{R}^n)} \lesssim \|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)}. \tag{2.4}$$

To do this, for every $x\in(2B)^{\complement}$ and $y\in B$, we see that $|x-y|\sim|x-x_0|$ and

$$\begin{aligned}\mathcal{M}_\phi([b-b_B]a)(x) &\lesssim \sup_{t\in(0,\infty)}\frac{1}{t^n}\int_B\int_B |b(y)-b_B||a(y)|\left|\phi\left(\frac{x-y}{t}\right)\right|dy\\ &\lesssim \frac{1}{|x-x_0|^n}\int_B |b(y)-b_B||a(y)|\,dy.\end{aligned}$$

Hence,

$$\int_{(2B)^{\complement}} \mathcal{M}_\phi([b-b_B]a)(x)w(x)\,dx \lesssim \|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)}.$$

In addition, by the boundedness of $\mathcal{M}_\phi$ on $L^q_w(\mathbb{R}^n)$ with $q\in(1,1+\delta/n)$ (see, for example, [8, p. 400, Theorem 2.8]), Lemma 2.3 and Proposition 2.1, we know that

$$\begin{aligned}\int_{2B}\mathcal{M}_\phi([b-b_B]a)(x)w(x)\,dx &\lesssim [w(2B)]^{1/q'}\|(b-b_B)a\|_{L^q_w(\mathbb{R}^n)}\\ &\lesssim \left[\frac{1}{w(B)}\int_B |b(x)-b_B|^q w(x)\,dx\right]^{1/q}\\ &\lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \lesssim \|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)},\end{aligned}$$

which concludes the proof of (ii) implying (i).

We now prove that (i) implies (ii). Let $\{R_j\}_{j=1}^n$ be the classical Riesz transforms. Then, by Lemma 2.4, we find that, for any $(H^1_w(\mathbb{R}^n),\infty)$-atom a related to some ball B and $j\in\{1,\dots,n\}$,

$$\begin{aligned}\|R_j([b-b_B]a)\|_{L^1_w(\mathbb{R}^n)} &\le \|[b,R_j](a)\|_{L^1_w(\mathbb{R}^n)} + \|(b-b_B)R_ja\|_{L^1_w(\mathbb{R}^n)}\\ &\lesssim \|[b,R_j]\|_{H^1_w(\mathbb{R}^n)\to L^1_w(\mathbb{R}^n)} + \|b\|_{\mathrm{BMO}(\mathbb{R}^n)},\end{aligned}$$

here and hereafter,

$$\|[b,R_j]\|_{H^1_w(\mathbb{R}^n)\to L^1_w(\mathbb{R}^n)} := \sup_{\|f\|_{H^1_w(\mathbb{R}^n)}\le 1}\|[b,R_j]f\|_{L^1_w(\mathbb{R}^n)}.$$

By the Riesz transform characterization of $H^1_w(\mathbb{R}^n)$ (see [4, Theorem 1.5]), we see that $(b-b_B)a\in H^1_w(\mathbb{R}^n)$ and, moreover,

$$\|(b-b_B)a\|_{H^1_w(\mathbb{R}^n)} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} + \sum_{j=1}^n \|[b,R_j]\|_{H^1_w(\mathbb{R}^n)\to L^1_w(\mathbb{R}^n)}. \tag{2.5}$$

For any ball $B:=B(x_0,r)\subset\mathbb{R}^n$ with some $x_0\in\mathbb{R}^n$ and $r\in(0,\infty)$, let

$$a := \frac{1}{2w(B)}(f-f_B)\chi_B,$$

where $f:=\operatorname{sign}(b-b_B)$. It is easy to see that a is an $(H^1_w(\mathbb{R}^n),\infty)$-atom related to the ball B. Moreover, for every $x\notin B$, Lemma 2.2 gives us that

$$\begin{aligned}\frac{1}{|x-x_0|^n}\frac{1}{2w(B)}\int_B |b(x)-b_B|\,dx &= \frac{1}{|x-x_0|^n}\int_B [b(x)-b_B]a(x)\,dx\\ &\lesssim \mathcal{M}_\phi([b-b_B]a)(x).\end{aligned}$$

This, together with (2.5), allows one to conclude that $b \in \mathcal{BMO}_w(\mathbb{R}^n)$ and, moreover,

$$\|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} + \sum_{j=1}^{n} \|[b, R_j]\|_{H^1_w(\mathbb{R}^n) \to L^1_w(\mathbb{R}^n)},$$

which completes the proof of Theorem 1.3. $\square$

Proof of Theorem 1.5. By Lemma 2.5, it suffices to prove that, for any continuous $(H^1_w(\mathbb{R}^n), \infty)$-atom a related to some ball B,

$$\|[b, T](a)\|_{H^1_w(\mathbb{R}^n)} \lesssim \|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)}. \tag{2.6}$$

By (2.4) and the boundedness of T on $H^1_w(\mathbb{R}^n)$ (see [11, Theorem 1.2]), (2.6) is reduced to proving that

$$\|(b - b_B)Ta\|_{H^1_w(\mathbb{R}^n)} \lesssim \|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)}.$$

Since $w \in A_{1+\delta/n}(\mathbb{R}^n)$, it follows that there exists $q \in (1, 1 + \delta/n)$ such that $w \in A_q(\mathbb{R}^n)$. By this and the fact that T is a δ-Calderón-Zygmund operator, together with a standard argument, we find that Ta is an $(H^1_w(\mathbb{R}^n), \infty, \varepsilon)$-molecule related to the same ball B with $\varepsilon := n + \delta - nq > 0$. Therefore, by Lemma 2.6, we have

$$Ta = \sum_{j=0}^{\infty} \lambda_j a_j,$$

where $\{a_j\}_{j=0}^{\infty}$ are $(H^1_w(\mathbb{R}^n), \infty)$-atoms related to the balls $\{2^{j+1}B\}_{j=0}^{\infty}$ and $|\lambda_j| \lesssim 2^{-j\varepsilon}$ for all $j \in \mathbb{Z}_+$. Thus, by (2.4) and Proposition 2.1, we obtain

$$\begin{aligned}
\|(b - b_B)Ta\|_{H^1_w(\mathbb{R}^n)} &\le \sum_{j=0}^{\infty} |\lambda_j| \left[\|(b - b_{2^{j+1}B})a_j\|_{H^1_w(\mathbb{R}^n)} + \|(b_{2^{j+1}B} - b_B)a_j\|_{H^1_w(\mathbb{R}^n)}\right] \\
&\lesssim \|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)} \sum_{j=0}^{\infty} 2^{-j\varepsilon} + \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sum_{j=0}^{\infty} (j+1) 2^{-j\varepsilon} \\
&\lesssim \|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)},
\end{aligned}$$

which completes the proof of (i) implying (ii) and hence Theorem 1.5. $\square$

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