

ON TENSOR THIRD L -FUNCTIONS OF AUTOMORPHIC REPRESENTATIONS OF $\mathrm{GL}_n(\mathbb{A}_F)$

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(Communicated by Ken Ono)

ABSTRACT. Langlands' beyond endoscopy proposal for establishing functoriality motivates interesting and concrete problems in the representation theory of algebraic groups. We study these problems in a setting related to the Langlands L -functions $L(s, \pi, \otimes^3)$, where π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ and F is a global field.

1. INTRODUCTION

Let F be a number field and let \mathbb{A}_F be the adèles of F . Let G be a reductive group over F . Given a representation

$${}^L G \longrightarrow \mathrm{GL}_n(\mathbb{C}),$$

Langlands' functoriality conjectures [L1] predict that there should be a corresponding transfer of automorphic representations of $G(\mathbb{A}_F)$ to automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$.

One can ask for a characterization of those automorphic representations in the image. By the conjectural Langlands correspondence, these should correspond to L -parameters

$$\varphi : \mathcal{L}_F \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

such that a $\mathrm{GL}_n(\mathbb{C})$ -conjugate factors through ${}^L G$, where \mathcal{L}_F is the conjectural Langlands group. From this optic, one is led to ask how one can detect such parameters.

Let ${}^\lambda G$ be the Zariski closure of $\mathrm{Im}({}^L G)$ viewed as a reductive group over \mathbb{C} . Then a theorem of Chevalley states that there exists a representation $\mathrm{GL}_n \longrightarrow \mathrm{GL}(V)$ such that ${}^\lambda G$ is the stabilizer of a line in V ; see [M]. Moreover, one knows via a result of Larsen and Pink [LP] that if ${}^\lambda G$ is connected semisimple, then the dimensions $\dim(V^{{}^\lambda G})$ for all representations V of GL_n characterize ${}^\lambda G$ up to conjugation. Thus the following definition is natural:

Definition 1.1. Let G be an irreducible reductive subgroup of GL_n . We say a representation $r : \mathrm{GL}_n \longrightarrow \mathrm{GL}(V)$ *detects* G if G stabilizes a line in V .

Remark. If G is connected, then r detects G if and only if it detects G^{der} .

The following conjecture is the crux of Langlands' beyond endoscopy proposal [L2], which aims to prove Langlands functoriality in general.

Received by the editors September 8, 2015 and, in revised form, February 1, 2016.
2010 *Mathematics Subject Classification.* Primary 11F70; Secondary 11F66, 11E57.

Conjecture 1.2. *Let π be a unitary cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. If π is a functorial transfer from G , then $L(s, \pi, r \otimes \chi)$ has a pole at $s = 1$ for some character $\chi \in F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$ whenever r detects ${}^\lambda G$.*

Thus implicit in Langlands' proposal is a very concrete question in algebraic group theory.

Question 1.3. Given a representation

$$r : \mathrm{GL}_n \longrightarrow \mathrm{GL}(V),$$

which algebraic subgroups of GL_n are detected by r ?

If $r = \mathrm{Sym}^2$, one knows that every irreducible reductive subgroup of GL_n detected by r is conjugate to a subgroup of the orthogonal similitude group GO_n . Moreover, in this case Conjecture 1.2 is proven by work of Arthur [A], work of Cogdell, Kim, Piatetski-Shapiro and Shahidi [CKPSS] and work of Ginzburg, Rallis and Soundry [GRS01]. There is a similar statement for $r = \Lambda^2$.

Apart from these special cases, explicit results are hard to come by. We mention one case that was discussed in a recent paper of Getz and Klassen [GK]. Let

$$RS : \mathrm{GL}_m \times \mathrm{GL}_m \hookrightarrow \mathrm{GL}_{m^2}$$

be the representation induced by the usual tensor product. Then it is known that $RS(\mathrm{GL}_m \times \mathrm{GL}_m)$ is detected by Sym^m (see [GK, §3] for instance). Moreover, the analytic properties of the relevant Langlands L -function are well understood via the Rankin-Selberg theory, although we do not know its automorphicity. It is important to note that even in this situation Getz and Klassen do not discuss whether there are any other maximal subgroups of GL_{m^2} that are detected by Sym^m .

Remark. Let π_1 and π_2 be cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A}_F)$. Then one knows that

$$L(s, \pi_1 \times \pi_2, RS) = L(s, \pi_1 \times \pi_2).$$

Here the function on the left is the Langlands L -function and the function on the right is the usual Rankin-Selberg L -function.

In this paper, we examine subrepresentations of the representation

$$\bigotimes^3 : \mathrm{GL}_n \longrightarrow \mathrm{GL}_{n^3}.$$

In some sense this is the easiest case to consider after $r = \mathrm{Sym}^2$ and $r = \Lambda^2$. This coincides with the setting of [GK] when we take $m = 3$ and $n = 9$. In this case we can obtain much more information than what is proven in [GK] (see Theorem 1.8 below).

Our first result is a lower bound on $\mathbb{S}_\lambda(\mathrm{SL}_m)$ not detected by \bigotimes^3 with respect to the number of nonzero parts of a partition λ . Here \mathbb{S}_λ is the usual Schur functor associated to a partition λ .

Theorem 1.4. *Let λ be a partition with $\ell \leq m$ nonzero parts. If $m > 3\ell$, then the representation \bigotimes^3 does not detect $\mathbb{S}_\lambda(\mathrm{SL}_m)$.*

Theorem 1.4 leads us to ask whether $\mathbb{S}_\lambda(\mathrm{SL}_m)$ is detected by \bigotimes^3 if $\ell \leq m \leq 3\ell$. We consider only the simplest case where $\lambda = (k)$ (see §2 for notation). So $\mathbb{S}_\lambda = \mathrm{Sym}^k$. In this case, $\ell = 1$, so we ought to study when $m = 2$ and $m = 3$.

Theorem 1.5. *Let*

$$G := \text{Sym}^{n-1}(\text{SL}_2) \hookrightarrow \text{GL}_n.$$

The representation Sym^3 detects G if and only if $n \equiv 1 \pmod{4}$.

Theorem 1.5 admits a pleasant partition-theoretic interpretation. Recall the following partition function: For fixed $j, k \in \mathbb{N}$, let $p(k, j, n)$ be the number of partitions of n into at most j parts, with the largest part at most k . Then we have the following:

Corollary 1.6. *For any integer $\ell \geq 1$, one has*

$$(1.0.1) \quad p(4\ell - 2, 3, 6\ell - 3) = p(4\ell - 2, 3, 6\ell - 4)$$

and

$$(1.0.2) \quad p(4\ell, 3, 6\ell) - p(4\ell, 3, 6\ell - 1) = 1.$$

The proof of Corollary 1.6 relies on representation theory. It would be interesting to find combinatorial proofs of (1.0.1) and (1.0.2). A. J. Yee pointed out to the author that one might deduce the identities by employing symmetries of the coefficients of Gaussian polynomials, but the details are not obvious.

For the case $m = 3$, we set

$$(1.0.3) \quad n = \frac{(k + 2)(k + 1)}{2},$$

for $k \in \mathbb{N}$. Then $\text{Sym}^k(\text{SL}_3)$ is naturally an irreducible reductive subgroup of GL_n . One has the following result:

Theorem 1.7. *Let n be given as in (1.0.3) and let*

$$G := \text{Sym}^k(\text{SL}_3) \hookrightarrow \text{GL}_n.$$

Then for any $k > 0$, the representation \otimes^3 detects G .

Finally, we examine closely irreducible connected reductive subgroups of GL_9 that are detected by Sym^3 . They are completely classified via the following theorem:

Theorem 1.8. *Let $G \leq \text{GL}_9$ be an irreducible connected reductive subgroup of GL_9 . Then if the representation Sym^3 detects G , then either $\text{Lie}(G^{\text{der}}) = \mathfrak{sl}_2$ or a GL_9 -conjugate of G is contained in $RS(\text{GL}_3 \times \text{GL}_3)$.*

Remark. It would be very interesting to investigate nonconnected reductive subgroups $G \leq \text{GL}_9$.

We close the introduction by outlining the paper. In §2, we provide an upper bound on m such that $\mathbb{S}_\lambda(\text{SL}_m)$ is detected by the representation \otimes^3 in terms of the number of nonzero parts of λ . In §3, we discuss when Sym^3 detects $\text{Sym}^k(\text{SL}_2)$. As an application, we present a partition theoretic interpretation of Theorem 1.5. In §4, we prove Theorem 1.7, and finally in §5, we prove Theorem 1.8.

2. SCHUR FUNCTORS

Let λ be a partition with at most m parts written as

$$(2.0.1) \quad \lambda = (\lambda_1, \lambda_2, \dots, \lambda_m), \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 0,$$

and let $|\lambda|$ be the number partitioned by λ .

For any m -dimensional vector space V over \mathbb{C} and any partition λ of $|\lambda|$ with at most m parts as in (2.0.1), we can apply the Schur functor \mathbb{S}_λ to V to obtain a representation $\mathbb{S}_\lambda(V)$ of GL_m . It remains irreducible when restricted to SL_m .

By the Littlewood-Richardson formula (compare with [FH, Exercise 15.23]), one knows the decomposition of a tensor product of any two irreducible representations of SL_m , namely

$$(2.0.2) \quad \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V) = \bigoplus_{\nu} C_{\lambda\mu\nu} \mathbb{S}_\nu(V).$$

Here the coefficient $C_{\lambda\mu\nu}$ is given by the Littlewood-Richardson rule and the sum is over partition ν of $|\lambda| + |\mu|$.

Then one has the following result:

Theorem 2.1. *Let λ be a partition with $\ell \leq m$ nonzero parts. If $m > 3\ell$, then the representation \otimes^3 does not detect $\mathbb{S}_\lambda(\mathrm{SL}_m)$.*

Proof. Let λ be a partition of $|\lambda|$ whose number of nonzero parts is $\ell \leq m$. In other words, λ can be written as

$$\lambda = (\lambda_1, \dots, \lambda_\ell, \underbrace{0, \dots, 0}_{m-\ell \text{ times}}), \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0.$$

Let V be the standard representation of SL_m . Consider

$$(2.0.3) \quad \begin{aligned} \mathbb{S}_\lambda(V)^{\otimes 3} &\cong \mathrm{Hom}(\mathbb{S}_\lambda(V)^\vee, \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\lambda(V)) \\ &\cong \bigoplus_{\nu} C_{\lambda\lambda\nu} \mathrm{Hom}(\mathbb{S}_\lambda(V)^\vee, \mathbb{S}_\nu(V)), \end{aligned}$$

where we employ the Littlewood-Richardson formula (2.0.2) and denote by $\mathbb{S}_\lambda(V)^\vee$ the dual space of $\mathbb{S}_\lambda(V)$. Therefore the partitions ν indexing the direct sum of (2.0.3) are partitions of $2|\lambda|$ with at most m parts.

Let λ^\vee be the partition such that $\mathbb{S}_\lambda(V)^\vee = \mathbb{S}_{\lambda^\vee}(V)$. Then it is easy to check that λ^\vee is

$$(2.0.4) \quad \lambda^\vee = (\underbrace{\lambda_1, \dots, \lambda_1}_{m-\ell \text{ times}}, \lambda_1 - \lambda_\ell, \lambda_1 - \lambda_{\ell-1}, \dots, \lambda_1 - \lambda_2, 0)$$

(compare with [FH, Exercise 15.50]). Therefore λ^\vee must have at least $m - \ell$ nonzero parts.

Notice that two partitions μ and μ' with at most m parts determine the same representation of SL_m if and only if there is an integer b such that

$$\mu = \mu' + b,$$

that is, $\mu_i = \mu'_i + b$ for all $1 \leq i \leq m$. Thus it suffices to prove that if $m > 3\ell$, then

$$\mathrm{Hom}_{\mathrm{SL}_m}(\mathbb{S}_{\lambda^\vee+b}(V), \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\lambda(V)) = 0$$

for all $b \in \mathbb{Z}$ such that $\lambda^\vee + b$ is a partition (i.e. all of its entries are nonnegative). In fact, such a b is necessarily nonnegative by (2.0.4). Thus partitions of the form $\lambda^\vee + b$ always have at least $m - \ell$ nonzero parts.

On the other hand, the Littlewood-Richardson rule forces the partition ν indexing the decomposition in (2.0.2) to have at most 2ℓ nonzero parts. This is a well-known fact, but we will provide a quick proof. To prove this, one first recalls that the number of boxes in the first column of the Young diagram of a given partition denotes the number of nonzero parts of it. Therefore λ will have ℓ boxes in the

first column in its Young diagram. Now we have to fill this Young diagram of λ by the Littlewood-Richardson rule to obtain the Young diagram for ν . Starting with ℓ boxes in the first column corresponding to λ , one is allowed to add only as many boxes in the first column as the number of parts of λ , which is in turn ℓ . Therefore the maximum number of boxes in the first column of the Young diagram of ν is 2ℓ (see [FH, Appendix A] for example).

Assuming $m - \ell > 2\ell$, we conclude that the partition $\lambda^\vee + b$ can never appear in the decomposition of tensor product; hence we prove that

$$\text{Hom}_{\text{SL}_m}(\mathbb{S}_\lambda(V)^\vee, \mathbb{S}_\nu(V)) = 0.$$

This completes the proof. □

3. THE SUBGROUP $\text{Sym}^k(\text{SL}_2)$

In this section, we only consider a special type of irreducible reductive subgroup of GL_n , namely $G := \text{Sym}^k(\text{SL}_2)$. In fact, we provide a precise condition on k for G to be detected by the representation Sym^3 , which implies that it is detected by \otimes^3 .

Recall the Gaussian polynomial

$$(3.0.1) \quad \begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{(1 - q^a)(1 - q^{a-1}) \cdots (1 - q^{a-b+1})}{(1 - q)(1 - q^2) \cdots (1 - q^b)}$$

and the plethysm decomposition for GL_2 , namely:

Theorem 3.1 (Theorem 5.5, [Do] (p. 81)). *Let $\dim V = 2$. Then there is an isomorphism of GL_2 -representations*

$$(3.0.2) \quad \text{Sym}^j(\text{Sym}^k V) \cong \bigoplus_{w=0}^{\lfloor jk/2 \rfloor} (\text{Sym}^{jk-2w} V \otimes \det^{jk-w})^{\oplus N(j,k,w)},$$

where $N(j, k, w)$ is the coefficient of q^w in the polynomial $(1 - q) \begin{bmatrix} j + k \\ k \end{bmatrix}_q$.

In particular if $j = 2$ in Theorem 3.1 and V is a vector space of characteristic zero such that $\dim V = 2$, then one has

$$(3.0.3) \quad \text{Sym}^2(\text{Sym}^k V) \cong \bigoplus_{w \geq 0}^{\lfloor k/2 \rfloor} \text{Sym}^{2k-4w} V$$

as SL_2 -representations (see [Do, Exercise 5.16 (p. 90)] for instance). One should notice that some terms in (3.0.2) vanish in the decomposition in (3.0.3). In the following theorem, we give a complete answer to the question of when the representation Sym^3 detects the subgroup $\text{Sym}^{n-1}(\text{SL}_2)$ of GL_n :

Theorem 3.2. *Let*

$$G := \text{Sym}^{n-1}(\text{SL}_2) \hookrightarrow GL_n.$$

The representation Sym^3 detects G if and only if $n \equiv 1 \pmod{4}$.

Proof. Let W be a representation of SL_2 . Then it is well known that one has an isomorphism of SL_2 representations

$$W^{\otimes 3} \cong \text{Hom}(W^\vee, W \otimes W) \cong \text{Hom}(W^\vee, \text{Sym}^2 W) \oplus \text{Hom}(W^\vee, \Lambda^2 W),$$

where W^\vee denotes the dual space of W . Considering the highest weights, it is easy to see that

$$\text{Sym}^3 W \hookrightarrow \text{Hom}(W^\vee, \text{Sym}^2 W)$$

as SL_2 -representations. Let $W = \text{Sym}^k V$, where V is a complex vector space with $\dim V = 2$. Then

$$\begin{aligned} \text{Hom}(W^\vee, \text{Sym}^2 W) &= \text{Hom}((\text{Sym}^k V)^\vee, \text{Sym}^2(\text{Sym}^k V)) \\ &= \text{Hom}(\text{Sym}^k(V^\vee), \text{Sym}^2(\text{Sym}^k V)) \\ &= \text{Hom}(\text{Sym}^k V, \text{Sym}^2(\text{Sym}^k V)) \\ &\cong \bigoplus_{i=0}^{\lfloor k/2 \rfloor} \text{Hom}(\text{Sym}^k V, \text{Sym}^{2k-4i} V), \end{aligned}$$

where in the last equality we use the plethysm decomposition formula (3.0.3) and the fact that $V^\vee \cong V$ for $\dim V = 2$. By Shur’s lemma,

$$(3.0.4) \quad \text{Hom}_{\text{SL}_2}(\text{Sym}^k V, \text{Sym}^{2k-4i} V) \leq 1,$$

and $\text{Hom}_{\text{SL}_2}(\text{Sym}^k V, \text{Sym}^{2k-4i} V) = 1$ if and only if $k = 2k - 4i$. Therefore $k \equiv 0 \pmod{4}$. By taking $k = n - 1$, we obtain the result. \square

One then obtains an immediate partition theoretic interpretation of Theorem 3.2. It is well known that the generating function of $p(k, j, n)$ is the Gaussian polynomial

$$(3.0.5) \quad \sum_{n \geq 0} p(k, j, n) q^n = \left[\begin{matrix} j+k \\ k \end{matrix} \right]_q$$

(see [St, Proposition 1.7.3], for instance). We deduce the following corollary:

Corollary 3.3. *For any integer $\ell \geq 1$, one has*

$$p(4\ell, 3, 6\ell) - p(4\ell, 3, 6\ell - 1) = 1$$

and

$$p(4\ell - 2, 3, 6\ell - 3) = p(4\ell - 2, 3, 6\ell - 4).$$

Proof. Note that when $3k - 2w = 0$ in the decomposition (3.0.2), one has only one 1-dimensional summand. We then use the fact (3.0.4) in the proof of Theorem 3.2 to deduce that

$$N(3, k, 3k/2) = \begin{cases} 1 & \text{if } k \equiv 0 \pmod{4}, \\ 0 & \text{if } k \equiv 2 \pmod{4}. \end{cases}$$

On the other hand, by Theorem 3.1, since $N(3, k, 3k/2)$ denotes the coefficient of $q^{3k/2}$ in the polynomial $(1 - q) \left[\begin{matrix} 3+k \\ k \end{matrix} \right]_q$, then $N(3, k, 3k/2)$ is equal to the coefficient of $q^{3k/2}$ in $\left[\begin{matrix} 3+k \\ k \end{matrix} \right]_q$ minus the coefficient of $q^{3k/2-1}$ in $\left[\begin{matrix} 3+k \\ k \end{matrix} \right]_q$. Using the partition interpretation of the Gaussian polynomial as in (3.0.5), we derive that

$$p(k, 3, 3k/2) - p(k, 3, 3k/2 - 1) = \begin{cases} 1 & \text{if } k \equiv 0 \pmod{4}, \\ 0 & \text{if } k \equiv 2 \pmod{4}, \end{cases}$$

which in turn completes the proof. \square

4. THE SUBGROUP $\text{Sym}^k(\text{SL}_3)$

In this section, we closely study the reductive subgroups $\text{Sym}^k(\text{SL}_3)$ for $k > 0$.

Theorem 4.1. *Let $n = \frac{(k+2)(k+1)}{2}$ and let*

$$G := \text{Sym}^k(\text{SL}_3) \hookrightarrow \text{GL}_n$$

for $k > 0$. Then the representation \otimes^3 detects G .

Proof. Let V be the standard representation of SL_3 . For $k > 0$, we consider

$$(4.0.1) \quad (\text{Sym}^k V)^{\otimes 3} \cong \text{Hom}((\text{Sym}^k V)^\vee, \text{Sym}^k V \otimes \text{Sym}^k V),$$

where $(\text{Sym}^k V)^\vee$ denotes the dual space of $\text{Sym}^k V$. We wish to investigate to see if

$$\text{Hom}((\text{Sym}^k V)^\vee, \text{Sym}^k V \otimes \text{Sym}^k V) \neq 0.$$

As a special case of Pieri’s formula (compare with [FH, Exercise 15.33]), one has that

$$(4.0.2) \quad \text{Sym}^k V \otimes \text{Sym}^k V = \bigoplus_{i=0}^k \mathbb{S}_{(k+i, k-i)}(V).$$

Let λ (resp. λ^\vee) be the partition corresponding to $\text{Sym}^k V$ (resp. $(\text{Sym}^k V)^\vee$). It is easy to check that if $m = 3$, we have $\lambda^\vee = (k, k, 0)$ (see [FH, Exercise 15.50] for example). In particular, λ^\vee appears in the decomposition of (4.0.2) if one takes $i = 0$. □

5. CONNECTED REDUCTIVE SUBGROUPS OF GL_9 DETECTED BY Sym^3

Throughout this section, we assume that $G \leq \text{GL}_9$ is an irreducible connected reductive subgroup. We characterize those G detected by Sym^3 :

Theorem 5.1. *Let $G \leq \text{GL}_9$ be an irreducible connected reductive subgroup. If the representation Sym^3 detects G , then either $\text{Lie}(G^{\text{der}}) = \mathfrak{sl}_2$ or a GL_9 -conjugate of G is contained in $RS(\text{GL}_3 \times \text{GL}_3)$.*

Proof. Let

$$\rho : G \hookrightarrow \text{GL}_9$$

be the natural representation. Recall that $G = Z_G G^{\text{der}}$, where Z_G is the center of G .

Notice that the degree of a tensor product of two representations is the product of the degrees of the two representations. Therefore if $\text{Lie}(G^{\text{der}})$ is not simple, then it is clear that $\text{Lie}(G^{\text{der}})$ is contained in a GL_9 -conjugate of $\text{Lie}(RS(\text{GL}_3 \times \text{GL}_3))$. It follows that $G^{\text{der}} \leq RS(\text{GL}_3 \times \text{GL}_3)$. On the other hand, since the representation ρ is irreducible, $\rho(Z_G)$ must be contained in the subgroup of scalar matrices, and hence $G \leq RS(\text{GL}_3 \times \text{GL}_3)$.

If, on the other hand, $\text{Lie}(G^{\text{der}})$ is simple, then its rank is less than or equal to 9. Thus after eliminating the set of possibilities in which the minimal dimension of a faithful representation of $\text{Lie}(G^{\text{der}})$ is bigger than 9 (see [BM, §2.2] for example), $\text{Lie}(G^{\text{der}})$ must be isomorphic to \mathfrak{sl}_i for $2 \leq i \leq 9$, or \mathfrak{so}_i for $2 \leq i \leq 9$, or \mathfrak{sp}_{2i} for $2 \leq i \leq 4$.

With these considerations in mind if we consult [FK, Tables A.1-A.26], one finds that if $\text{Lie}(G^{\text{der}})$ is simple, then it must be either \mathfrak{sl}_2 , \mathfrak{so}_9 or \mathfrak{sl}_9 since none of the

other Lie algebras mentioned above admit irreducible representations of degree 9. In consulting the tables one must bear in mind that the irreducible representations of the compact group $SU(n)$ are in bijective correspondence with the irreducible representations of \mathfrak{sl}_n by the unitary trick.

The case $\text{Lie}(G^{\text{der}}) = \mathfrak{sl}_2$ coincides with Theorem 3.2 taking $n = 9$. If $\text{Lie}(G^{\text{der}}) = \mathfrak{sl}_9$, clearly G cannot be detected by Sym^3 because $\text{Sym}^3 \circ \rho$ is irreducible. Finally, for the case where $\text{Lie}(G^{\text{der}}) = \mathfrak{so}_9$, we let V be the standard representation of \mathfrak{so}_9 . Then from the decomposition of $\text{Sym}^3 V$ into the direct sum of Schur functors [FH, Exercise 19.21(ii)], one knows that there is no trivial representation occurring in the decomposition. This completes our proof. \square

Lastly we want to note that it would be interesting to investigate irreducible non-connected reductive subgroups of GL_9 , since the rank is not so big. This would require finite group theory and Clifford theory.

ACKNOWLEDGEMENTS

The author is grateful to J. R. Getz for suggesting this project and to Leslie Saper for answering various questions on representation theory. The author also thanks A. J. Yee for a useful conversation on the Gaussian polynomials and an anonymous referee for a careful reading that improved the exposition of the paper.

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