TORIC MATRIX SCHUBERT VARIETIES
AND THEIR POLYTOPES

LAURA ESCOBAR AND KAROLA MÉSZÁROS

(Communicated by Patricia L. Hersh)

Abstract. Given a matrix Schubert variety $\overline{X}_\pi$, it can be written as $\overline{X}_\pi = Y_\pi \times \mathbb{C}^q$ (where $q$ is maximal possible). We characterize when $Y_\pi$ is toric (with respect to a $(\mathbb{C}^*)^{2n-1}$-action) and study the associated polytope $\Phi(\mathbb{P}(Y_\pi))$ of its projectivization. We construct regular triangulations of $\Phi(\mathbb{P}(Y_\pi))$ which we show are geometric realizations of a family of subword complexes. Subword complexes were introduced by Knutson and Miller in 2004, who also showed that they are homeomorphic to balls or spheres and raised the question of their polytopal realizations.

1. Introduction

In this paper we study the geometry of matrix Schubert varieties and use it to give geometric realizations of a family of subword complexes. Matrix Schubert varieties were introduced in [12] to study the degeneraci loci of flagged vector bundles. In [14] Knutson and Miller showed that Schubert polynomials are multidegrees of matrix Schubert varieties. On the other hand, in [13] Knutson and Miller introduced subword complexes to illustrate the combinatorics of Schubert polynomials and determinantal ideals. They proved that any subword complex is homeomorphic to a ball or a sphere and asked about their geometric realizations.

Given a matrix Schubert variety $\overline{X}_\pi$, it can be written as $\overline{X}_\pi = Y_\pi \times \mathbb{C}^q$ (where $q$ is maximal possible). Our main results are as follows. We characterize when $Y_\pi$ is toric (with respect to a $(\mathbb{C}^*)^{2n-1}$-action) and study the moment polytope $\Phi(\mathbb{P}(Y_\pi))$ of its projectivization. We construct regular triangulations of $\Phi(\mathbb{P}(Y_\pi))$ which we show are geometric realizations of a family of subword complexes. Since the appearance of Knutson’s and Miller’s work in [13,14] there has been a flurry of research into the geometric realization of subword complexes with progress in realizing families of subword complexes homeomorphic to spheres [2,7,8,22,23,26,27]. The first paper which tackled realizing a large family of subword complexes which are homeomorphic to balls is [10]. Our current paper is further progress in this direction. Our method relies on the discovery that the chute and ladder moves [1] which connect the pipe dreams of permutations are in some cases related to the relations of the subdivision algebra of root polytopes [15]. We show that this is the case for permutations $\pi$ when $Y_\pi$ is toric, and it appears that it is not possible to extend this relation beyond these permutations. This raises the question of whether there is a more general subdivision algebra which would bear a similar relation to pipe dreams of other permutations.

Received by the editors August 23, 2015 and, in revised form, February 9, 2016.
2010 Mathematics Subject Classification. Primary 14M25, 52B11, 05E45.

© 2016 Laura Escobar and Karola Mészáros

5081
The organization of this paper is as follows. In Section 2 we define matrix Schubert varieties $X_\pi = Y_\pi \times \mathbb{C}^q$ and calculate the moment polytope $\Phi(\mathbb{P}(Y_\pi))$ of the projectivization of $Y_\pi$. In Section 3 we characterize when $Y_\pi$ is toric and construct a regular triangulation of $\Phi(\mathbb{P}(Y_\pi))$. In Section 4 we define subword complexes and show that the aforementioned triangulations of $\Phi(\mathbb{P}(Y_\pi))$ are a geometric realization of a family of subword complexes homeomorphic to balls. Finally, in Section 5 we show how to view the results of [10] in terms of canonical triangulations of $\Phi(\mathbb{P}(Y_\pi))$.

2. Matrix Schubert varieties

Given a matrix Schubert variety $X_\pi$ we define a variety $Y_\pi \hookrightarrow X_\pi$ such that when $Y_\pi$ is a toric variety, we can construct a regular triangulation of its corresponding polytope, which we show is a geometric realization of a family of subword complexes; see Proposition 3.3 and Theorem 4.2. Furthermore, in Theorem 3.5 we use the diagram of $\pi$ (defined below) to characterize the permutations $\pi$ for which $Y_\pi$ is a toric variety.

Let $M_n$ denote $n \times n$ matrices over $\mathbb{C}$, $B_+$ denote upper triangular invertible $n \times n$ matrices and $B_-$ denote lower triangular invertible $n \times n$ matrices. We let $\pi \in S_n$ denote both a permutation and its corresponding permutation matrix, where its $(i,j)$-th entry is

$$(\pi)_{(i,j)} = \begin{cases} 1, & \text{if } \pi(j) = i, \\ 0, & \text{else}. \end{cases}$$

An $n \times n$-matrix can always be reduced into a partial permutation matrix, which is a matrix with only 0,1-entries having at most one nonzero entry on each row and column, by multiplying on the left by matrices in $B_-$ and on the right by matrices in $B_+$. The multiplication on the left corresponds to downward row operations and multiplication on the right corresponds to rightward column operations. This multiplication gives a left action of $B_- \times B_+$ on $M_n$ defined by

$$(X, Y) \cdot M := XMY^{-1}. \quad (2.1)$$

Given $1 \leq a \leq m$ and $1 \leq b \leq m$, let $M_{(a,b)}$ denote the upper left $a \times b$ submatrix of the matrix $M$. See Figure 1 for an illustration. Define a rank function of a matrix $M$ to be $r_M(a,b) := \text{rank}(M_{(a,b)})$. We then have that $M \in B_- \pi B_+$ if and only if $r_M(a,b) = r_\pi(a,b)$ for all $(a,b) \in [m] \times [m]$.

![Figure 1. The submatrix $M_{(a,b)}$ of $M$.](image)

**Definition 2.1.** The matrix Schubert variety of $\pi$ is $X_\pi := B_- \pi B_+$, i.e., the Zariski closure of its $(B_- \times B_+)$-orbit inside $M_n = \mathbb{C}^{n^2}$.

Fulton studied this affine variety in [12]. We summarize some of his results here.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Theorem 2.2 ([12, Proposition 3.3]). The matrix Schubert variety $X_\pi$ is an irreducible variety of dimension $n^2 - \ell(\pi)$ defined as a scheme by the equations $r_M(a,b) \leq r_\pi(a,b)$ for all $(a,b) \in [n] \times [n]$.

Some of these inequalities are implied by others, and Fulton described the minimal set of rank conditions.

Definition 2.3. The (Rothe) diagram of a permutation $\pi$ is the collection of boxes $D(\pi) = \{(\pi_j, i) : i < j, \pi_i > \pi_j\}$. It can be visualized by considering the boxes left in the $(n \times n)$-grid after we cross out the boxes appearing south and east of each 1 in the permutation matrix for $\pi$.

![Figure 2. The diagram for $\pi = [25413]$.](image)

Definition 2.4. Fulton’s essential set, denoted $\text{Ess}(\pi)$, is the set consisting of the south-east corners of $D(\pi)$.

Theorem 2.5 ([12, Lemma 3.10]). The ideal defining the variety $X_\pi$ is generated by the equations $r_M(a,b) \leq r_\pi(a,b)$ for all $(a,b) \in \text{Ess}(\pi)$.

We now define some regions inside the $(n \times n)$-grid and some varieties corresponding to these regions.

Definition 2.6. The dominant piece, denoted $\text{dom}(\pi)$, of a permutation $\pi$ is the connected component of the diagram of $\pi$ containing the box $(1,1)$, or empty if $\pi(1) = 1$.

We have that $r_\pi(a,b) = 0$ if and only if $(a,b) \in \text{dom}(\pi)$. Therefore the dominant piece of $\pi$ consists precisely of the coordinates in $M_n$ that are 0 on $X_\pi$.

Definition 2.7. Let $\text{NW}(\pi)$ denote the union over the entries north-west of some box in $D(\pi)$. Let $L(\pi) := \text{NW}(\pi) - \text{dom}(\pi)$ and let $L'(\pi) := L(\pi) - D(\pi)$.

See Figure 3 for an example.

![Figure 3. Given $\pi = [25413]$, $L(\pi)$ consists of all the gray boxes and $L'(\pi)$ consists of only the darker gray boxes.](image)
Definition 2.8. Given a permutation \( \pi \), let \( Y_\pi \) be the projection of \( \overline{ X_\pi } \) onto the entries inside \( L(\pi) \) and let \( V_\pi \) be the projection onto the entries not north-west of any box of \( D(\pi) \).

Theorem 2.5 implies that the entries in \( V_\pi \) are free in \( \overline{ X_\pi } \) and thus \( V_\pi \cong \mathbb{C}^q \), where \( q \) is the number of boxes in the region defining \( V_\pi \). We have that \( \overline{ X_\pi } = Y_\pi \times V_\pi \) and

\[
\dim(Y_\pi) = \dim(\overline{ X_\pi }) - \dim(V_\pi) = (n^2 - \ell(\pi)) - (n^2 - |NW(\pi)|)
\]

(2.2)

\[
= (n^2 - |D(\pi)|) - (n^2 - |NW(\pi)|) = |NW(\pi)| - |D(\pi)| = |L'(\pi)|.
\]

Consider permutations of the form \( \pi = 1\pi' \) where \( \pi' \) is a dominant permutation of \( \{2, \ldots, n\} \), i.e., the diagram of \( \pi \) is a partition with north-west most box at position \((2,2)\). For these types of permutations we have that \( L'(1\pi') \) is a hook.

The \((B_- \times B^+)-\)action on \( \overline{ X_\pi } \) defined in equation (2.1) restricts to a \((T^n \times T^n)-\)action on \( \overline{ X_\pi } \), where \( T^n \) consists of \( n \times n \) diagonal matrices. This action is not faithful because scaling acts the same way on both sides, so

\[
\text{Stab}(T^{2n}) = \{(a \cdot I, a \cdot I) : a \in \mathbb{C}^*\}.
\]

Notice that both \( Y_\pi \) and \( V_\pi \) are \( T^{2n} \)-invariant subspaces of \( \overline{ X_\pi } \). In Theorem 3.5 we will characterize the \( \pi \) for which \( Y_\pi \) is a toric variety with respect to \( T^{2n}/\text{Stab}(T^{2n}) \) in terms of \( L'(\pi) \). In other words, we will characterize the \( \pi \) such that \( Y_\pi \) has a dense \( T^{2n} \)-orbit. We will denote the quotient \( T^{2n}/\text{Stab}(T^{2n}) \) by \( T^{2n-1} \). We note that by Fulton’s realization of matrix Schubert varieties in [12] as subvarieties of Schubert varieties, which are normal [9][25], it follows that matrix Schubert varieties and \( Y_\pi \) are normal.

Let \( p \) be a general point of \( Y_\pi \); then \( \overline{T^{2n}} \cdot p \subset Y_\pi \) is the affine toric variety associated to the \( T^{2n} \)-moment cone \( \Phi(Y_\pi) \) of which we denote by \( \Phi(\overline{ X_\pi }) \) and \( \dim(T^{2n} \cdot p) = \dim(\Phi(Y_\pi)) \). Since both \( Y_\pi \) and \( T^{2n} \cdot p \) are irreducible, if they have the same dimension they must be equal. In Theorem 3.5 we classify when \( Y_\pi \) is a toric variety by classifying the \( \pi \) for which \( \dim(\Phi(Y_\pi)) = \dim(Y_\pi) \).

In order to describe the cone \( \Phi(Y_\pi) \), we start by describing the cone \( \Phi(\overline{ X_\pi }) \) corresponding to a \( T^{2n} \)-orbit of a general point \( q \) in \( \overline{ X_\pi } \), which we can assume without loss of generality to be \( q = (1, \ldots, 1) \). The cone \( \phi(\overline{ X_\pi }) \) is spanned by the weights by which \( T^{2n} \) acts on \( T^{2n} \cdot q \). More explicitly, the orbit \( T^{2n} \cdot q \) is the Zariski closure of the image of a map \( \varphi : T^{2n} \to \mathbb{C}^{n^2} \) where \( \varphi(t) = (t^{a(1,1)}q_{(1,1)}, \ldots, t^{a(n,n)}q_{(n,n)}) \) and \( \Phi(\overline{ X_\pi }) \) is the cone spanned by the exponents \( a_{(i,j)} \) of the monomials. Notice that if \( A \) and \( B \) are the diagonal matrices with diagonal entries \( (a_1, \ldots, a_n) \) and \( (b_1, \ldots, b_n) \), respectively, then for any matrix \( M \) the \((i,j)\)-th entry of \( AMB^{-1} \) is \( a_i b_j^{-1} M_{(i,j)} \). We therefore have that the exponents are \( x_i - y_j \), where the \( x_i \) are the standard basis for \( \mathbb{R}^n \times 0 \), and the \( y_j \) are the standard basis for \( 0 \times \mathbb{R}^n \). In other words, the moment cone \( \Phi(\overline{ X_\pi }) \) is the cone spanned by the vectors in the set

---

1The reason we use the word moment for these convex objects is because they arise in the context of symplectic and pre-symplectic geometry. For readers interested in the connection, we refer them to [8][9].
The same is true for $Y_\pi$. In this paper we study the \textbf{moment polytope} $\Phi(\mathbb{P}(Y_\pi))$ of the projectivization of $Y_\pi$. This polytope is the convex hull of $(x_i - y_j)$ for $(i,j)$ inside $L(\pi)$. The next section studies the properties of the moment polytopes $\Phi(\mathbb{P}(Y_\pi))$.

3. Understanding the polytope $\Phi(\mathbb{P}(Y_\pi))$

This section is devoted to the study of $\Phi(\mathbb{P}(Y_\pi)) = \text{ConvHull}(x_i - y_j \mid (i,j) \in L(\pi))$ for $\pi \in S_n$, the moment polytope of the projectivization of $Y_\pi$. The polytope $\Phi(\mathbb{P}(Y_\pi))$ is a root polytope, since its vertices are positive roots of type $A_{n-1}$. We set our notation for root polytopes now.

3.1. Root polytopes and their triangulations. A root polytope (of type $A_{n-1}$) in this section will be the convex hull of some of the points $e_i - e_j$ for $1 \leq i < j \leq n$. Given a graph $G$ on the vertex set $[n]$ we associate to it the root polytope

\begin{equation}
Q_G = \text{ConvHull}(e_i - e_j \mid (i,j) \in E(G), i < j).
\end{equation}

We will also need a different root polytope, defined as:

\begin{equation}
\tilde{Q}_G = \text{ConvHull}(0, e_i - e_j \mid (i,j) \in E(G), i < j).
\end{equation}

Note that for every $\pi \in S_n$ we have that $L(\pi)$ is a \textit{skew Ferrers diagram}. Given a skew Ferrers diagram $D$ with $r$ rows and $c$ columns, label its rows by $1, 2, \ldots, r$ from top to bottom and its columns by $1, 2, \ldots, c$ from left to right. Define

\begin{equation}
G_D = \{(x_1, \ldots, x_r, y_1, \ldots, y_c), (x_i, y_j) \mid (i,j) \in D\}.
\end{equation}

Then

\begin{equation}
\Phi(\mathbb{P}(Y_\pi)) = Q_{G_{L(\pi)}}.
\end{equation}

Note that an edge $(x_i, y_j) \in G_D$ yields the vertex $e_i - e_{r+j}$ of the root polytope $Q_{G_D}$.

Given a drawing of a graph $G$ so that its vertices $v_1, \ldots, v_n$ are arranged in this order on a horizontal line, and its edges are drawn above this line, we say that $G$ is \textbf{noncrossing} if it has no edges $(v_i, v_k)$ and $(v_j, v_l)$ with $i < j < k < l$. A vertex $v_i$ of $G$ is said to be \textbf{nonalternating} if it has both an incoming and an outgoing edge; it is called \textbf{alternating} otherwise. The graph $G$ is alternating if all its vertices are alternating.

Since being noncrossing depends on the drawing of the graph it is essential that we set a way to draw $G_D$. For the purposes of this paper the vertices of $G_D$ are drawn from left to right in the following order: $x_r, \ldots, x_1, y_c, \ldots, y_1$.

\footnote{See footnote\textsuperscript{1}}
Lemma 3.1. Given a skew diagram $D$ for which $G_D$ has $k$ components, the root polytope $Q_{G_D} = \bigcup_F Q_F$, where the union runs over all noncrossing alternating spanning forests of $G_D$ with $|V(G_D)| - k$ edges and the simplices $Q_F$ are interior disjoint and of the same dimension as $Q_{G_D}$.

Before proving Lemma 3.1 we prove Lemma 3.2, which gives another way to view the triangulation described in Lemma 3.1. Special cases of these lemmas were used in [16], and the triangulation of the polytope $Q_{G_D}$ into simplices labeled by the lattice paths appearing in Lemma 3.2 was proven by different methods in [4].

Lemma 3.2. Given a skew diagram $D$ with $k$ components $D_1, \ldots, D_k$, the noncrossing alternating spanning trees of $G_D$ with $|V(G_D)| - k$ edges are in bijection with $k$-tuples of lattice paths $(p_1, \ldots, p_k)$, where $p_i$ is a lattice path inside $D_i$ starting at $D_i$'s south-westmost box and ending at $D_i$'s north-eastmost box taking steps either one unit south or west.

Proof. The bijection is given by the map that takes a noncrossing alternating spanning forest $F = \{(x_1, \ldots, x_r, y_c, \ldots, y_1), \{(x_i, y_j) \mid (i, j) \in S(F)\}\}$ of $G_D$ to the $k$-tuple of paths $(p_1, \ldots, p_k)$, where the set of edges of $p_1 \cup \cdots \cup p_k$ is given by the set $S(F)$. See Figure 4 for an example.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{The correspondence between noncrossing alternating spanning trees of $G_D$ and lattice paths from $(1, c)$ to $(r, 1)$ inside $D$ that take steps either one unit south or west.}
\end{figure}

Proof of Lemma 3.1 If $D$ is connected, then $G_D$ is a connected bipartite graph on the parts $\{x_1, \ldots, x_r\}$ and $\{y_1, \ldots, y_c\}$. Thus, $G_D \subset K_{r,c}$, the bipartite graph with parts of sizes $r$ and $c$. By [24, Proposition 13.3] the polytopes $Q_{K_{r,c}}$ are the facets of $\tilde{Q}_{K_{r+c}}$, where $K_{r+c}$ is the complete graph on $r+c$ vertices, and since $G_D$ is connected we have that $Q_{G_D}$ and $Q_{K_{r,c}}$ are of the same dimension. Thus, it suffices to prove that $\tilde{Q}_{G_D} = \bigcup_T \tilde{Q}_T$, where the union runs over all noncrossing alternating trees of $G_D$ and the simplices $\tilde{Q}_T$ are interior disjoint and of the same dimension as $\tilde{Q}_{G_D}$. The latter statement follows by application of [24, Lemma 12.6].

If $D$ is not connected, there exist partitions $\lambda, \mu, \omega$ such that $D = \lambda \setminus \mu$ and $D' := \lambda \setminus \omega$ is the unique smallest such that $G_{D'}$ is connected; see Figure 5. The number of rows and columns of $D$ and $D'$ are the same, $r$ and $c$, respectively, and $G_D \subset G_{D'}$. Since $G_{D'}$ is connected the first paragraph of this proof applies to it. Denoting by $B$ the set of boxes in $D'$ that are not in $D$, it follows that every noncrossing alternating spanning tree of $G_{D'}$ contains the set of edges $E_B := \{(x_i, y_j) \mid (i, j) \in B\}$. An easy way to see this is to notice that every lattice path described in Lemma 3.2 must contain the boxes in $B$. Note that if we delete the edges in $E_B$ from the noncrossing alternating spanning trees of $G_{D'}$ we exactly
obtain the forests described in the lemma. We get that in the triangulation of \( Q_{G_{D'}} \), proven in the first paragraph the vertices corresponding to the edges in \( E_B \) are all cone points, and a triangulation of \( Q_{G_{D'}} \) can be obtained from a triangulation of \( Q_{G_D} \) by adding the vertices corresponding to the edges in \( E_B \) (and thereby raising the dimension by \(|E_B|\)). It is clear that \( \bigcup F \cap Q_{G_D} \subset Q_{G_D} \) and the simplices \( Q_F \) are interior disjoint, intersect on common faces and are of the same dimension as \( Q_{G_D} \), since they are the restrictions of the noncrossing alternating trees of \( G_{D'} \) to \( G_D \). It also follows that \( Q_{G_D} \subset \bigcup F \cap Q_F \), since if a triangulation of \( Q_{G_D} \) were to include simplices other than the \( Q_F \)’s, then the triangulation of \( Q_{G_D} \) would include simplices other than the \( Q_T \)’s, which would contradict the statement proven in the first paragraph.

Figure 5. For \( D \) the unshaded region, \( G_D \) is disconnected. In this case, \( D' \) equals \( D \) together with the shaded square.

We call the triangulation of \( Q_{G_D} \) given in Lemma 3.1 the noncrossing alternating triangulation, or NAT for short. Recall that a triangulation of a polytope \( P \) is regular if there exists a concave piecewise linear function \( f : P \to \mathbb{R} \) such that the regions of linearity of \( f \) are the maximal simplices in the triangulation.

Proposition 3.3. For a skew diagram \( D \), the NAT triangulation of \( Q_{G_D} \) described in Lemma 3.1 is a regular triangulation.

Proof. We start by describing how to extend a function defined on a set \( A \) of points into a piecewise linear function defined on their convex hull. Let \( f : A \to \mathbb{R} \) be a function on the set \( A \) and consider the polytope \( P = \text{ConvHull}(A) \). Let \( \overline{P} = \text{ConvHull}(\{(a, f(a)) \mid a \in A\}) \) and define then \( f(p) = \max\{x \mid (p, x) \in \overline{P}\} \), \( p \in P \). The function \( f : P \to \mathbb{R} \) is concave by definition.

Consider the root polytope \( Q_{G_D} \) with vertices \( e_i - e_{r+j} \), for \((i,j) \in D \). Let \( f(e_i - e_{r+j}) = (i - (r+j))^2 \) for \((i,j) \in D \). Extend this to a concave piecewise linear function on \( Q_{G_D} \) as described above. It can be shown that this function yields the NAT triangulation of \( Q_{G_D} \), thus showing its regularity.

3.2. Characterizing when \( Y_\pi \) is a toric variety. Now we are ready to use the above lemmas in order to characterize when \( Y_\pi \) is a toric variety.

Lemma 3.4. Given a skew diagram \( D \) with \( r \) rows and \( c \) columns, for which \( G_D \) has \( k \) components, the dimension of \( Q_{G_D} \) is \( r + c - k - 1 \).

Proof. The dimension of \( Q_F \) for some noncrossing alternating spanning forest \( F \) of \( G_D \) is \( r + c - k - 1 \), since \( F \) has \( r + c - k \) edges, which correspond to the vertices of an \((r + c - k - 1)\)-dimensional simplex. Since by Lemma 3.1 the dimension of \( Q_{G_D} \) equals the dimension of \( Q_F \) for some noncrossing alternating spanning forest \( F \) of \( G_D \) with \( |V(G_D)| - k \) edges, the statement follows.
Theorem 3.5. \(Y_\pi\) is a toric variety with respect to the \(T^{2n-1}\)-action if and only if \(L'(\pi)\) consists of disjoint hooks that do not share a row or a column with each other.

Proof. In equation (2.2) we computed the dimension of \(Y_\pi\). Lemma 3.4 yields that the dimension of \(\Phi(P(Y_\pi))\) equals \(|L'(\pi)| - 1\) if and only if \(L'(\pi)\) consists of disjoint hooks that do not share a row or a column with each other. This suffices to prove the theorem.

An immediate corollary of Theorem 3.5 is the following.

Corollary 3.6. If \(\pi'\) is a dominant permutation on \(2, 3, \ldots, n\), then \(Y_{1_{\pi'}}\) is a toric variety.

4. ON GEOMETRIC REALIZATIONS OF SUBWORD COMPLEXES

In this section we show that the NAT triangulations studied in the previous section geometrically realize certain subword complexes. Thus, we extend the family of subword complexes homeomorphic to balls that have known geometric realizations as triangulations of polytopes.

The symmetric group \(S_n\) is generated by the adjacent transpositions \(s_1, \ldots, s_{n-1}\), where \(s_i\) transposes \(i \leftrightarrow i + 1\). Let \(Q = (q_1, \ldots, q_m)\) be a word in \(\{s_1, \ldots, s_{n-1}\}\), i.e., \(Q\) is an ordered sequence. A subword \(J = (r_1, \ldots, r_m)\) of \(Q\) is a word obtained from \(Q\) by replacing some of its letters by \(\pi\). There are a total of \(2^{|Q|}\) subwords of \(Q\). Given a subword \(J\), we denote by \(Q \setminus J\) the subword with \(k\)-th entry equal to \(-\) if \(r_k \neq -\) and equal to \(q_k\) otherwise for, \(k = 1, \ldots, m\). For example, \(J = (s_1, -, s_3, -, s_2)\) is a subword of \(Q = (s_1, s_2, s_3, s_1, s_2)\) and \(Q \setminus J = (-, s_2, -, s_1, -)\). Given a subword \(J\) we denote by \(\prod J\) the product of the letters in \(J\), from left to right, with \(-\) behaving as the identity.

Definition 4.1 (13, 14). Let \(Q = (q_1, \ldots, q_m)\) be a word in \(\{s_1, \ldots, s_{n-1}\}\) and \(\pi \in S_n\). The subword complex \(\Delta(Q, \pi)\) is the simplicial complex on the vertex set \(Q\) whose facets are the subwords \(F\) of \(Q\) such that the product \(\prod (Q \setminus F)\) is a reduced expression for \(\pi\).

Given a permutation \(\pi \in S_n\), let \(\overline{L(\pi)}\) be the mirror image of the skew shape \(L(\pi)\) obtained by reflecting the skew shape \(L(\pi)\) vertically. We then fill in the boxes of \(\overline{L(\pi)}\) with transpositions starting with \(s_1, s_2, \ldots\) on the first column, \(s_2, s_3, \ldots\) on the second column, and so on; see Figure 6. Let \(Q(\overline{L(\pi)})\) be the word given by reading the transpositions in the boxes of \(\overline{L(\pi)}\) from left to right and from bottom to top. Let \(P(\pi) = \overline{L(\pi)} - B(\pi)\) where \(B(\pi)\) is as follows. In each connected part of \(\overline{L(\pi)}\) draw the lowestmost path from its top left box to its bottommost rightmost box. These boxes constitute \(B(\pi)\). Let \(p(\pi)\) be the permutation obtained from reading the transpositions in the boxes of \(P(\pi)\) from left to right and from bottom to top. See Figure 6.

Theorem 4.2. For \(\pi \in S_n\), the noncrossing alternating triangulation of \(Q_{G_L(\pi)}\) is a geometric realization of the subword complex \(\Delta(Q(\overline{L(\pi)}), p(\pi))\).

Proof. The dimension of \(\Delta(Q(\overline{L(\pi)}), p(\pi))\) is \(|B(\pi)| - 1\) by definition. On the other hand the dimension of \(Q_{G_L(\pi)}\) equals this same thing by Lemma 3.1 (this is clear through Lemma 3.2). Viewing the noncrossing alternating spanning forests labeling
of the directed transitive closure of the acyclic graph get a bijection to reduced pipe dreams of
the top dimensional simplices in NAT as a union of lattice paths via Lemma 3.2, we
been utilized in solving various problems in [16–21].

Figure 6. On the left we have $L(\pi)$ and on the right $\overline{L(\pi)}$ for
$\pi = [14523]$. Note that $p(\pi) = s4s2s3 = [13524]$.

the top dimensional simplices in NAT as a union of lattice paths via Lemma 3.2 we
get a bijection to reduced pipe dreams of $p(\pi)$, a special type of subword complexes
defined in Section 5 by filling by elbow the boxes that are in the paths labeling the
simplices of NAT (or rather in their mirror images in $\overline{L(\pi)}$) and putting crosses in
the rest of $\overline{L(\pi)}$. Since $\Delta(Q(L(\pi)), p(\pi))$ and the NAT of $Q_{G\cup(\pi)}$ are pure simplicial
complexes, the above proves the theorem.

5. Degeneration of moment polytopes into acyclic root polytopes

In [10] Theorem 1 we have shown that the pipe dream complex $PD(\pi)$ of a
permutations $\pi = 1\pi'$, with $\pi'$ dominant, can be geometrically realized as canonical
triangulations of acyclic root polytopes $P(T(\pi))$. The pipe dream complex
$PD(\pi)$ is the subword complex $\Delta(Q, \pi)$ corresponding to the triangular word $Q =
(s_{n-1}, s_{n-2}, s_{n-1}, \ldots, s_1, s_2, \ldots, s_{n-1})$ and $\pi$. In this section we review [10] Theorem 1 and relate the moment polytopes $\Phi(P(\mathbb{P}(Y_n)))$ to the acyclic root polytopes
$P(T(\pi))$. In turn, we give another way to see $PD(\pi)$ as a triangulation of $\Phi(P(\mathbb{P}(Y_n)))$.

5.1. Acyclic root polytopes. Let $G$ be an acyclic graph on the vertex set $[n + 1]$. Define

$$V_G = \{e_i - e_j \mid (i, j) \in E(G), i < j\},$$

a set of vectors associated to $G$;

$$\text{cone}(G) := \{ \sum_{e_i - e_j \in V_G} c_{ij} (e_i - e_j) \mid c_{ij} \geq 0\},$$

the cone associated to $G$; and

$$\overline{V_G} = \Phi^+ \cap \text{cone}(G),$$

the set of positive roots of type $A_n$ contained in $\text{cone}(G)$, where $\Phi^+ = \{e_i - e_j \mid 1 \leq i < j \leq n + 1\}$ is the set of positive roots of type $A_n$.

The acyclic root polytope $\mathcal{P}(G)$ associated to the acyclic graph $G$ is

$$\mathcal{P}(G) = \text{ConvHull}(0, e_i - e_j \mid e_i - e_j \in \overline{V_G}).$$

Theorem 5.1 ([15]). Let $T_1, \ldots, T_k$ be the noncrossing alternating spanning trees
of the directed transitive closure of the acyclic graph $G$. Then $\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)$ are
top dimensional simplices in a regular triangulation of $\mathcal{P}(G)$ called the canonical
triangulation.

The main tool developed in [15] which is used to construct the canonical triangulation
of Theorem 5.1 is the subdivision algebra. Subdivision algebras have since
been utilized in solving various problems in [16–21].

Theorem 5.2 ([10] Theorem 1]). Let $\pi = 1\pi' \in S_n$, where $\pi'$ is dominant. Let
$C^2(\pi)$ be the core of $PD(\pi)$ coned over twice. The canonical triangulation of the
root polytope $\mathcal{P}(T(\pi))$ is a geometric realization of $C^2(\pi)$. 
5.2. Definition of $T(\pi)$ (appearing in Theorem 5.2). We now review the original description from [10] of $T(\pi)$ used in Theorem 5.2. For a definition of a reduced pipe dream, also called RC-graphs, as well as further background on it see [1,11,13] or [10].

Since we are only considering permutations of the form $1\pi'$ with $\pi'$ a dominant permutation, $\text{core}(1\pi')$ is easy to describe. Given a diagram of a permutation there are two natural reduced pipe dreams for $\pi$, referred to as the bottom reduced pipe dream of $\pi$ and the top reduced pipe dream of $\pi$, one obtained by aligning the diagram to the left and replacing the boxes with crosses and the other one by aligning the diagram up. See Figure 7. The core of $1\pi'$ is the simplicial complex obtained by restricting $PD(\pi)$ to the vertices corresponding to the positions of the crosses in the superimposition of these two pipe dreams. We refer to the region itself as the core region, and denote it by $\text{cr}(\pi)$. See Figure 8 for an example.

Let $\pi = 1\pi'$, where $\pi'$ is dominant. Denote by $\mathcal{S}(\pi)$ the subword complex which is the core($\pi$) coned over the vertex of $PD(\pi)$ corresponding to the entry $(1, 1)$. Denote the region which is the union of $(1, 1)$ and $\text{cr}(\pi)$ by $R(\pi)$. In order to determine the tree $T(\pi)$, we will label the south-east boundary with numbers and we will place some dots in $R(\pi)$; see Figure 9. The boundary of the core region starting from the south-west (SW) corner of it to the north-east (NE) corner can be described as a series of east (E) and north (N) steps. Let $A$ be the set consisting of all the $N$ steps together with some $E$ steps. The step $E \in A$ if the bottom reduced pipe dream is bounded by $E$ but not by the $N$ step directly preceding $E$. As we traverse this lower boundary from the SW corner we write the numbers $1, \ldots, m$ in increasing fashion below the $E$ steps and to the right of the $N$ steps that belong

![Figure 7](image1.png)

**Figure 7.** Two reduced pipe dreams for [164235] obtained by aligning the diagram to the left and to the top.

![Figure 8](image2.png)

**Figure 8.** The core region of [164235]
to $A$. For the $E$ steps not assigned a number, we consider their number to be the number assigned to the $N$ step directly preceding them. Consider the bottom reduced pipe dream drawn inside $R(\pi)$ with elbows replaced by dots. Drop these dots south. Define $T(\pi)$ to be the tree on $m$ vertices such that there is an edge between vertices $i < j$ if there is a dot in the entry in the column of the $E$ step labeled $i$ and in the row of the $N$ step labeled $j$.

\[ \begin{array}{cccccc}
\bullet & \bullet & \bullet & 6 \\
\bullet & 1 & 2 & 3 & 5 \\
\bullet & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & 5 & 6
\end{array} \quad \longrightarrow \quad \begin{array}{cccccc}
\bullet & \bullet & \bullet & 6 \\
\bullet & 1 & 2 & 3 & 5 \\
\bullet & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & 5 & 6
\end{array} \]

Figure 9. Let $\pi = [15342]$. On the left we see the bottom reduced pipe dream for $\pi$ drawn inside $R(\pi)$ with dots instead of elbows. This gives the labeling of the boundary. We then drop the dots to the south to get the dots encoding $T(\pi)$, which is depicted on the right.

5.3. Alternative description of $T(\pi)$ and degeneration of $\Phi(P(Y_\pi))$ to $P(T(\pi))$. Here we give an alternative way to describe the above through the following propositions.

**Proposition 5.3.** Let $\pi = 1\pi'$ with $\pi'$ dominant. Then the union of $(1,1)$ and $\text{cr}(\pi)$ consists of all entries north-west of some box in $D(\pi)$ that are not inside Fulton’s essential set. In equations, this says that $R(\pi) = \text{NW}(\pi) - \text{Ess}(\pi)$.

**Proof.** It is clear that $R(\pi) \subset \text{NW}(\pi)$. Given $(i, j) \in \text{Ess}(\pi)$, we have that neither the bottom or the top reduced pipe dreams of $\pi$ have a cross on this position since $(i, j)$ is a south-east corner of $D(\pi)$ and so $(i, j) \notin R(\pi)$. Since $D(\pi)$ is a partition, then $D(\pi)$ restricted to a row or to a column is connected. Therefore, any entry strictly west of an entry in $\text{Ess}(\pi)$ must be in the bottom reduced pipe dream of $\pi$ and thus inside $R(\pi)$. Similarly, any entry strictly north of $\text{Ess}(\pi)$ must be in the top reduced pipe dream. \hfill $\square$

Recall that the graph associated to $\text{cr}(\pi)$ for $\pi = 1\pi'$ with $\pi'$ dominant has as vertices the elements of the set $A$ which consists of some $E$ and $N$ steps that bound the SW boundary of $\text{cr}(\pi)$.

**Proposition 5.4.** Let $\pi = 1\pi'$ with $\pi'$ dominant. An $E$ step is in $A$ if and only if it does not bound a box in $\text{Ess}(\pi)$.

**Proof.** Proposition 5.3 tells us that given an $E$ step the $N$ step directly preceding $E$ bounds $R(\pi)$ if and only if $E$ bounds a box in $\text{Ess}(\pi)$.

These propositions allow us to rephrase the construction of $T(\pi)$, for permutations $\pi = 1\pi'$, $\pi'$ dominant. We construct the graph $T(\pi)$ by looking at the $E$ and $N$ steps bounding the SE boundary of $\text{NW}(\pi) - \text{Ess}(\pi)$. Let $A$ be the set consisting of all the $N$ steps together with the $E$ steps that do not bound a box in $\text{Ess}(\pi)$. Suppose $|A| = m$, as we transverse this lower boundary from the SW corner, we write $\alpha_1, \ldots, \alpha_m$ in increasing fashion below the $E$ steps and to the right.
of the $N$ steps that belong to $A$. For the $E$ steps that we did not assign an $\alpha_i$, we consider their label to be the $\alpha_i$ assigned to the $N$ step directly preceding them. Let $\mathcal{T}(\pi)$ be the tree with vertices $V = \{\alpha_1, \ldots, \alpha_m\}$ and edges corresponding to the entries in $NW(\pi) - \text{Ess}(\pi)$. More precisely $(\alpha_i, \alpha_j)$ is an edge of $\mathcal{T}(\pi)$ if the entry $(i, j) \in NW(\pi) - \text{Ess}(\pi)$ is north of the $E$ step labeled $\alpha_j$ and east of the $N$ step labeled $\alpha_i$. See Figures 10 and 11 for an example.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig10.png}
\caption{The diagram for $\pi = [14523]$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig11.png}
\caption{On the left we have $NW(\pi) - \text{Ess}(\pi)$ for $\pi = [14523]$ with its SW boundary labeled by $A$. On the right we have the corresponding tree $\mathcal{T}(\pi)$.}
\end{figure}

Note that Propositions 5.3 and 5.4 tell us that for $\pi = 1\pi'$ with $\pi'$ dominant this construction coincides with the one we had before.

**Theorem 5.5.** Given $\pi = 1\pi'$, with $\pi'$ dominant, the moment polytope $\Phi(\mathcal{P}(Y_\pi))$ can be degenerated into the root polytope $\mathcal{P}(\mathcal{T}(\pi))$.

**Proof.** We will give a linear map from $\Phi(\mathcal{P}(Y_\pi)) \to \mathcal{P}(\mathcal{T}(\pi))$ as a composition of two maps $K$ and $L$. Let $L$ be the map given by

$$L(x_i) = -e_j, \text{ where } \alpha_j \text{ is the label of step } N \text{ on row } i, \text{ and}$$

$$L(y_i) = \begin{cases} 
0 & \text{if } (a, i) \in \text{Ess}(\pi) \text{ for some } a, \\
-e_j & \text{where } \alpha_j \text{ is the label of step } E \text{ on column } i.
\end{cases}$$

Let $K$ be the map given by

$$K(y_j) = \begin{cases} 
x_i & \text{if } (i, j) \in \text{Ess}(\pi), \\
y_j & \text{if there is no } a \text{ such that } (a, j) \in \text{Ess}(\pi).
\end{cases}$$

$$K(x_i) = x_i.$$ 

See Figure 12 for an example of these maps. Now notice that if $(x_i - y_j) \in \Phi((\mathcal{P}Y_\pi)^{2n-1})$, i.e., it is the image of a fixed point, then we have that $L \circ K((x_i - y_j)) = e_{k_j} - e_{k_i}$, where $\alpha_{k_i}$ is the label of step $N$ in $A$ on row $i$ and $\alpha_{k_j}$ is the label of step $E$ in $A$ on column $j$. Notice that $K$ corresponds
to subtracting the vector $x_a - y_b$ from each vector $x_i - y_i$ in column $b$ whenever $(a, b) \in \text{Ess}(\pi)$. Notice also that $L$ is the map induced by the relabeling of the entries in $\text{NW}(\pi) = \text{Ess}(\pi)$ into the tree labeling. In the language of $\phi$, we then have that $(x_i - y_j) \mapsto e_{\phi^{-1}(j)} - e_{\phi^{-1}(i)}$. The root polytope of $T(\pi)$ is

$$\text{ConvHull}(0, e_{\phi^{-1}(j)} - e_{\phi^{-1}(i)} \mid (i, j) \in \text{NW}(\pi) - \text{Ess}(\pi))$$

so the theorem follows.

$$\begin{array}{ccc}
\begin{bmatrix} x_1 - y_1, x_1 - y_2, x_1 - y_3 \\ x_2 - y_1, x_2 - y_2, x_2 - y_3 \\ x_3 - y_1, x_3 - y_2, x_3 - y_3 \end{bmatrix} & \mapsto & \begin{bmatrix} x_1 - y_1, x_1 - y_2, x_1 - x_3 \\ x_2 - y_1, x_2 - y_2, x_2 - x_3 \\ x_3 - y_1, x_3 - y_2, 0 \end{bmatrix} \\
\begin{bmatrix} e_1 - e_5, e_2 - e_5, e_3 - e_5 \\ e_1 - e_4, e_2 - e_4, e_3 - e_4 \\ e_1 - e_3, e_2 - e_3, 0 \end{bmatrix} \alpha_5 & \mapsto & \begin{bmatrix} e_1 - 3, e_2 - 3, \alpha_3 \\ e_1 - 3, e_2 - 3, \alpha_3 \end{bmatrix}
\end{array}$$

**Figure 12.** Degeneration for $\pi = [1243]$. By Propositions 5.3 and 5.4 we have that for $\pi = 1\pi'$ with $\pi'$ dominant the image $L \circ K(\Phi(\mathbb{P}(Y_{\pi})))$ is precisely the root polytope $\mathcal{P}(T(\pi))$. The following proposition allows us to describe the degeneration in Theorem 5.5 in words.

**Proposition 5.6.** The vertices of $\Phi(\mathbb{P}(Y_{\pi}))$ corresponding to the entries in $\text{Ess}(\pi)$ form a face of $\Phi(\mathbb{P}(Y_{\pi}))$.

**Proof.** Let $\text{Ess}(\pi) = \{(a_1, b_1), \ldots, (a_k, b_k)\}$ and consider the linear functional $\ell$ defined by

$$\ell \left( \sum_i s_i x_i + \sum_j t_j y_j \right) := \sum_{i=1}^k is_{a_i} + \sum_{j=1}^{k-j} (j-k+1)t_{b_j}.$$ 

Note that since $(a_i, b_j) \notin L(\pi)$ if $i < j$, then the maximum value $\ell$ can attain on $\Phi(\mathbb{P}(Y_{\pi}))$ is $k + 1$. Moreover, this value is only attained for the vectors $x_a - y_b$ with $(a, b) \in \text{Ess}(\pi)$.

The degeneration $L \circ K : \Phi(\mathbb{P}(Y_{\pi})) \to \mathcal{P}(T(\pi))$ consists of contracting the face of $\Phi(\mathbb{P}(Y_{\pi}))$ corresponding to $\text{Ess}(\pi)$ to a point and moving this point to the origin while tweaking the vertices of $\Phi(\mathbb{P}(Y_{\pi}))$ that are of the form $\frac{1}{2}(x_i - y_j)$ where $(i, j)$ is north of an entry of $\text{Ess}(\pi)$.

5.4. Triangulating $\Phi(\mathbb{P}(Y_{\pi}))$ and geometric realization of subword complexes. Next we show that the image of the canonical triangulation of $\mathcal{P}(T(\pi))$ for $\pi = 1\pi'$, with $\pi'$ dominant, under the linear map $L \circ K$ is a triangulation of $\Phi(\mathbb{P}(Y_{\pi}))$, which is yet another way to geometrically realize the pipe dream complex $PD(\pi)$ for these permutations.

**Theorem 5.7.** Let $\Delta_1, \ldots, \Delta_k$ be the top dimensional simplices in the canonical triangulation of $\mathcal{P}(T(\pi))$ for $\pi = 1\pi'$, where $\pi'$ is dominant. Then $P_i := (L \circ K)^{-1}(\Delta_i)$, $i \in [k]$, are the top dimensional simplices in a triangulation of $\Phi(\mathbb{P}(Y_{\pi}))$ which we call its canonical triangulation.
Proof. Since $L \circ K(\Phi(\mathbb{P}(Y_\pi))) = \mathcal{P}(T(\pi))$, it follows that $\bigcup_{i \in [k]} P_i = \Phi(\mathbb{P}(Y_\pi))$. We will show that the $P_i$ are interior disjoint simplices that induce a triangulation of $\Phi(\mathbb{P}(Y_\pi))$.

Note that each vertex of $\mathcal{P}(T(\pi))$ except 0 has a unique vertex of $\Phi(\mathbb{P}(Y_\pi))$ that maps to it under $L \circ K$ and there are $|\text{Ess}(\pi)|$ many vertices of $\Phi(\mathbb{P}(Y_\pi))$ mapping to 0. For $v \neq 0$ a vertex of $\mathcal{P}(T(\pi))$, let $w(v)$ be the vertex of $\Phi(\mathbb{P}(Y_\pi))$ that maps to it under $L \circ K$. Let $q_1, \ldots, q_{|\text{Ess}(\pi)|}$ be the vertices of $\Phi(\mathbb{P}(Y_\pi))$ mapping to 0 under $L \circ K$. We have that

$$
(5.2) \quad P_i = \text{ConvHull}(q_1, \ldots, q_{|\text{Ess}(\pi)|}, w(v) \mid v \text{ is a nonzero vertex of } \Delta_i),
$$

since $(L \circ K)^{-1}(0) = \text{ConvHull}(q_1, \ldots, q_{|\text{Ess}(\pi)|})$ and $(L \circ K)^{-1}(v) = w(v)$ for $v$ a nonzero vertex of $\mathcal{P}(T(\pi))$. Notice that $\dim(\Delta_i) = \dim(\mathcal{P}(T(\pi)))$ which equals to the number of edges of $T(\pi)$ which is $r + c - 1 - |\text{Ess}(\pi)|$ and so it follows that $P_i$ is the convex hull of $r + c - 1$ points. It remains to show that the dimension of each $P_i$ equals the dimension of $\Phi(\mathbb{P}(Y_\pi))$. This can be done by fixing a vertex $u$ of $\Delta_i$ and showing that the set $\{w(u) - q_1, \ldots, w(u) - q_{|\text{Ess}(\pi)|}, w(v) - w(u) \mid v \text{ is a nonzero vertex of } \Delta_i\}$ is linearly independent. We leave the details of this argument to the interested reader. \hfill $\Box$

Denote by $\mathcal{C}(\pi)$ the core of the pipe dream complex $PD(\pi)$. Let $\mathcal{C}^i(\pi)$ be the core $\mathcal{C}(\pi)$ coned over $i$ times.

**Theorem 5.8.** The canonical triangulation of $\Phi(\mathbb{P}(Y_\pi))$, for $\pi = 1\pi'$, with $\pi'$ dominant, is a geometric realization of $\mathcal{C}^{|\text{Ess}(\pi)|+1}(\pi)$.

**Proof.** By [10] Theorem 1] the canonical triangulation of $\mathcal{P}(T(\pi))$ is a geometric realization of $\mathcal{C}^2(\pi)$. For $\pi$ as in the theorem, we have that $\text{dom}(\pi)$ is empty and $L(\pi)$ is a partition with $r$ rows and $c$ columns. Then by equation (3.7) and Lemma 3.1 we get that $\dim(\Phi(\mathbb{P}(Y_\pi))) = r + c - 2$. On the other hand, the dimension of $\mathcal{P}(T(\pi))$ equals the number of edges of $T(\pi)$, which is $r + c - 1 - |\text{Ess}(\pi)|$. Thus, $\dim(\mathcal{P}) - \dim(\mathcal{P}(T(\pi))) = |\text{Ess}(\pi)| - 1$. Together with Theorem 5.7 we obtain the theorem. \hfill $\Box$

Using the characterization of toric $Y_\pi$ of Theorem 5.5 one can extend this geometric realization to realizations of pipe dream complexes for all $\pi$ such that $Y_\pi$ is toric.

**Acknowledgements**

We are grateful to Allen Knutson for many valuable discussions. We thank Maksim Maydanskiy for his interest and in-depth comments on our research. We also thank Vic Reiner for several interesting conversations.

**References**


Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801

E-mail address: lescobar@illinois.edu

Department of Mathematics, Cornell University, Ithaca, New York 14853

E-mail address: karola@math.cornell.edu