ON QUANTIZATION OF A NILPOTENT ORBIT CLOSURE IN $G_2$

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Abstract. Let $G$ be the complex exceptional Lie group of type $G_2$. Among the five nilpotent orbits in its Lie algebra $\mathfrak{g}$, only the 8-dimensional orbit $O_8$ has non-normal orbit closure $\overline{O_8}$. In this manuscript, we will give a quantization model of $\overline{O_8}$, verifying a conjecture of Vogan made in 1984.

1. Introduction

Let $G$ be a complex simple Lie group. The $G$-conjugates of a nilpotent element $X \in \mathfrak{g}$ form a nilpotent orbit $O \subset \mathfrak{g}$. Following the ideas in [19] or [21], one would like to attach unitary representations to all such orbits along with their finite $G$-equivariant covers. More precisely, let $V$ be a finite $G$-equivariant cover of an affine Poisson $G$-variety containing a nilpotent orbit $O$ as an open set, with its ring of regular functions $R(V)$; then one would like to find a $(\mathfrak{g}_C, K_C)$-module $X_V$ such that we have the $G$-module isomorphism

$$X_V|_{K_C} \cong R(V)$$

(note that $K \leq G$ is the maximal compact subgroup of $G$, hence its complexification $K_C$ is isomorphic to $G$). Throughout this work, we will call $X_V$ a quantization of $V$.

As hinted in [19], one needs to pay special attention when the orbit closure $\overline{O}$ is not normal. One reason is due to the algebro-geometric fact that $R(\overline{O}) \cong R(O)$ if and only if $\overline{O}$ is normal. Following the spirit of the orbit method, one needs to give a quantization model for $V = O$ and $V = \overline{O}$ separately when $\overline{O}$ is not normal.

Here is a summary on the current progress of the above quantization scheme. In [3], Barbasch constructs such models for a large class of classical nilpotent orbits. Using a completely different method in [5], Ranee Brylinski constructs a Dixmier algebra for all classical nilpotent orbit closures. The reconciliation between the two models is the main theme of the Ph.D. thesis of the author [22].

Contrary to the classical setting, very little is known about the scheme for exceptional groups. We now focus on the case for $G = G_2$. Let $\{\alpha, \beta\}$ be the simple roots of $\mathfrak{g}$, with $\alpha$ being the short root. The fundamental weights of $\mathfrak{g}$ are therefore given by

$$\{\omega_1, \omega_2\} = \{2\alpha + \beta, 3\alpha + 2\beta\}.$$
By the Bala-Carter classification, we have five nilpotent orbits $O_0$, $O_6$, $O_8$, $O_{10}$ and $O_{12}$ in $\mathfrak{g}$. Following the study of completely prime primitive ideals of Joseph in [9], Vogan in [19] conjectured a quantization model for $O_8$ and $\overline{O}_8$ for $G_2$:

**Conjecture 1.1** ([19 Conjecture 5.6]). Let $\lambda \in \mathfrak{h}^*$ and $J(\lambda)$ be the maximal primitive ideal in $U(\mathfrak{g})$ with infinitesimal character $\lambda$. Then the $(\mathfrak{g}_C, K_C) \cong (\mathfrak{g} \times \mathfrak{g}, G)$-modules

$$U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2)), \quad U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2)),$$

are quantizations of $O_8$ and $\overline{O}_8$ respectively. In particular,

$$U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2))|_{K_C} \cong R(O_8), \quad U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2))|_{K_C} \cong R(\overline{O}_8).$$

As a consequence, $O_8$ has non-normal closure.

Interestingly, by the classification of spherical unitary dual of complex $G_2$ given by Duflo in [8], $U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2))$ is unitarizable while $U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2))$ is not (this fact is also observed by Vogan in p. 226 of [21]). Later, Levasseur and Smith in [10] proved that $U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2))|_{K_C} \cong R(O_8)$ and $\overline{O}_8$ are not normal, but were unable to prove the rest of the conjecture. The main result of this manuscript is the following:

**Theorem 1.2.** As $K_C \cong G$ modules,

$$U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2))|_{K_C} \cong R(\overline{O}_8).$$

**Remark 1.3.** This quantization model of nilpotent orbit closure is very different from the classical model given in [3]. Namely, the Brylinski model is not necessarily of the form $U(\mathfrak{g})/J(\lambda)$. In particular, when the classical nilpotent orbit closure $\overline{O}$ is not normal (the classification of all such orbit closures is given in [13]), the infinitesimal character of the Brylinski model $\lambda_{\overline{O}}$ always yields associated variety $AV(U(\mathfrak{g})/J(\lambda_{\overline{O}})) = \overline{O}'$, where $\overline{O}'$ is strictly smaller than $\overline{O}$.

In fact, it can be shown that the Brylinski model always contains the composition factor $U(\mathfrak{g})/J(\lambda_{\overline{O}})$. This is part of the on-going work of Barbasch and the author [4].

Before going to the proof of Theorem 1.2 it is worthwhile to mention the orbits other than $O_8$ in $\mathfrak{g}$. Indeed, Kraft in [12] confirmed that $O_8$ is the only nilpotent orbit with non-normal closure. So we just need to consider quantizations of the orbits (and their covers) only. For the zero orbit $O_0$ the quantization is trivial, and the quantization of the minimal orbit $O_6$ is $U(\mathfrak{g})/J(\frac{1}{3}(3\omega_1 + \omega_2))$, where $J(\frac{1}{2}(3\omega_1 + \omega_2))$ is the Joseph ideal. The 10-dimensional orbit $O_{10}$ is a special orbit with fundamental group $S_3$. It is a simple exercise to compare the formulas in [2] and [14] that the spherical unipotent representation attached to $O_{10}$ is a quantization of $\mathcal{O}$ (as a bonus, the other two unipotent representations attached to $O_{10}$ essentially give quantization of all covers of $O_{10}$ as well). Finally, the quantization of the principal orbit $O_{12}$ is well known to be the principal series representation with zero infinitesimal character. In conclusion, we completed the picture of quantization for all nilpotent orbits of $\mathfrak{g}$ and their closures.
2. Proof of the theorem

As mentioned in the Introduction, the non-normality of $\mathcal{O}_8$ implies that $R(\mathcal{O}_8) \not\subseteq R(\mathcal{O})$. In fact, Costantini in [7] gives the discrepancies in terms of $G$-modules:

**Theorem 2.1** ([7] Theorem 5.6). Let $V_{(a,b)}$ be the finite-dimensional irreducible representation of $G_2$ with highest weight $a\omega_1 + b\omega_2$, where $a$ and $b$ are non-negative integers. Then

$$R(\mathcal{O}_8) \cong R(\mathcal{O}) \oplus \bigoplus_{n \in \mathbb{N} \cup \{0\}} V_{(1,n)}.$$  

The following lemma gives another expression of the discrepancies between $R(\mathcal{O})$ and $R(\mathcal{O})$:

**Lemma 2.2.** As virtual $G$-modules,

$$\bigoplus_{n \in \mathbb{N} \cup \{0\}} V_{(1,n)} = \text{Ind}_{T}^{G}(1,0) - \text{Ind}_{T}^{G}(0,1) - \text{Ind}_{T}^{G}(2,0) + \text{Ind}_{T}^{G}(1,1) + \text{Ind}_{T}^{G}(0,2) - \text{Ind}_{T}^{G}(2,1),$$

where $\text{Ind}_{T}^{G}(a,b)$ is the shorthand for the induced module $\text{Ind}_{T}^{G}(e^{a\omega_1 + b\omega_2})$.

**Proof.** The lemma can be derived from the Weyl character formula. Namely, by the $W(G_2)$-symmetry of weights of $V_{(a,b)}$, we have

$$V_{(a,b)} = \sum_{w \in W(G_2)} \det(w) \text{Ind}_{T}^{G}(\lambda_w)$$

with $\lambda_w$ being the unique $W(G_2)$-conjugate of $w[(a,b) + (1,1)] - (1,1)$ lying in the dominant chamber. In fact, we have

$$V_{(1,n)} = \text{Ind}_{T}^{G}(1,n) - \text{Ind}_{T}^{G}(1,n+3) - \text{Ind}_{T}^{G}(2,n) + \text{Ind}_{T}^{G}(2,n+2)$$

$$- \text{Ind}_{T}^{G}(3,n-1) + \text{Ind}_{T}^{G}(3,n) - \text{Ind}_{T}^{G}(3,n+1) + \text{Ind}_{T}^{G}(3,n+2)$$

$$+ \text{Ind}_{T}^{G}(6,n-2) - \text{Ind}_{T}^{G}(6,n) - \text{Ind}_{T}^{G}(7,n-2) + \text{Ind}_{T}^{G}(7,n-1)$$

for $n > 1$, and

$$V_{(1,0)} = \text{Ind}_{T}^{G}(1,0) - \text{Ind}_{T}^{G}(1,3) - \text{Ind}_{T}^{G}(2,0) + \text{Ind}_{T}^{G}(2,2) - \text{Ind}_{T}^{G}(0,1) + \text{Ind}_{T}^{G}(3,0)$$

$$- \text{Ind}_{T}^{G}(3,1) + \text{Ind}_{T}^{G}(3,2) + \text{Ind}_{T}^{G}(0,2) - \text{Ind}_{T}^{G}(6,0) - \text{Ind}_{T}^{G}(1,2) + \text{Ind}_{T}^{G}(4,1);$$

$$V_{(1,1)} = \text{Ind}_{T}^{G}(1,1) - \text{Ind}_{T}^{G}(1,4) - \text{Ind}_{T}^{G}(2,1) + \text{Ind}_{T}^{G}(2,3) - \text{Ind}_{T}^{G}(3,0) + \text{Ind}_{T}^{G}(3,1)$$

$$- \text{Ind}_{T}^{G}(3,2) + \text{Ind}_{T}^{G}(3,3) + \text{Ind}_{T}^{G}(3,1) - \text{Ind}_{T}^{G}(6,1) - \text{Ind}_{T}^{G}(4,1) + \text{Ind}_{T}^{G}(7,0).$$

The lemma is proved by adding up the terms. \qed

We now study the two Harish-Chandra bi-modules $U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2))$ and $U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2))$:

**Proposition 2.3.** As $K_C \cong G$-modules

$$U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2))|_{K_C} = \text{Ind}_{T}^{G}(0,0) - \text{Ind}_{T}^{G}(0,1) - \text{Ind}_{T}^{G}(2,0) + \text{Ind}_{T}^{G}(1,1);$$

$$U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2))|_{K_C} = \text{Ind}_{T}^{G}(0,0) - \text{Ind}_{T}^{G}(1,0) - \text{Ind}_{T}^{G}(0,2) + \text{Ind}_{T}^{G}(2,1).$$
Proof. To cater for subsequent calculations, we let $\mathfrak{h}^* = \{(x, y, z) \in \mathbb{C}^3 \mid x + y + z = 0\}$, with short simple root $\alpha = (1, -1, 0)$ and long simple root $\beta = (-1, 2, -1)$. Then

$$\lambda_1 = \frac{1}{2}(\omega_1 + \omega_2) = (1, 1/2, -3/2); \quad \lambda_2 = \frac{1}{2}(5\omega_1 - \omega_2) = (2, -1/2, -3/2).$$

The character formulas of $U(\mathfrak{g})/J(\lambda)$ for regular $\lambda$ are well known by the work of Barbasch and Vogan [2]: Consider the subgroup $W_\lambda$ of $W(G_2)$ generated by roots $\alpha$ satisfying $2\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$. Then the formula is given by

$$U(\mathfrak{g})/J(\lambda) = \sum_{w \in W_\lambda} \det(w)X(\lambda, w\lambda),$$

where $X(\mu, \nu) = K$-finite part of $\text{Ind}^G_T(e^{(\mu, \nu)} \otimes 1)$ is the principal series representation with character $(\mu, \nu) \in \mathfrak{h}_\mathbb{C}$, the complexification of the maximal torus $\mathfrak{h}$ in $\mathfrak{g}$ (here we treat $G$ as a real Lie group). In particular, the $G \cong K_\mathbb{C}$-types of $X(\mu, \nu)$ are equal to $\text{Ind}^G_T(e^{\mu-\nu})$ (see Theorem 1.8 of [2] for more details on the principal series representations).

Now apply the above recipe for $\lambda_1 = (1, 1/2, -3/2)$: With the above notation, $W_{\lambda_1}$ is isomorphic to $W(A_1 \times \tilde{A}_1)$, generated by the roots $\{(0, 1, -1), (2, -1, -1)\}$. Hence the character formula of $U(\mathfrak{g})/J(\lambda_1)$ is given by

$$U(\mathfrak{g})/J(\lambda_1) = X((1, 1/2, -3/2), (1, 1/2, -3/2)) - X((1, 1/2, -3/2), (1, -3/2, 1/2)) - X((1, 1/2, -3/2), (-1, 3/2, -1/2)) + X((1, 1/2, -3/2), (-1, -1/2, 3/2)).$$

Upon restricting to $K_\mathbb{C}$, we have

$$U(\mathfrak{g})/J(\lambda_1)|_{K_\mathbb{C}} \cong \text{Ind}^G_T(e^{(0,0,0)}) - \text{Ind}^G_T(e^{(0,2,-2)}) - \text{Ind}^G_T(e^{(2,1,-1)}) + \text{Ind}^G_T(e^{(2,1,-3)}).$$

Again, by $W(G_2)$-symmetry of finite-dimensional irreducible $G$-modules, the above expression can be written in the form as in the proposition. The calculations for $U(\mathfrak{g})/J(\lambda_2)$ are identical to the one above. We omit the calculations here. $\square$

Proof of Theorem 1.2 By the result of Levasseur and Smith in [10],

$$U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2))|_{K_\mathbb{C}} \cong R(\mathcal{O}_8)$$

as $G$-modules. Therefore the first equation of Proposition 2.3 gives

$$(2.1) \quad R(\mathcal{O}_8) \cong \text{Ind}^G_T(0, 0) - \text{Ind}^G_T(0, 1) - \text{Ind}^G_T(2, 0) + \text{Ind}^G_T(1, 1)$$

as virtual $G$-modules. By Theorem 2.1 and Lemma 2.2 we need to show that

$$U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2))|_{K_\mathbb{C}} = R(\mathcal{O}_8) - \bigoplus_n V_{1,n}$$

$$= R(\mathcal{O}_8) - (\text{Ind}^G_T(1, 0) - \text{Ind}^G_T(0, 1) - \text{Ind}^G_T(2, 0) + \text{Ind}^G_T(1, 1) + \text{Ind}^G_T(0, 2) - \text{Ind}^G_T(2, 1)).$$

This is readily seen to be true from equation (2.1) and the second equation of Proposition 2.3 $\square$
3. Final remarks

In [18], Sommers gives some conjectures on the multiplicities of small representations of \(R(O)\) for the exceptional groups. In particular, given that his conjecture is true, one can show the non-normality of some orbit closures.

To describe more explicitly which orbits \(O\) are conjectured to have non-normal closures, recall that Lusztig in [11] partitioned all nilpotent orbits in \(g\) by special pieces, i.e., for all nilpotent orbit \(O'\), it must belong to exactly one of the special pieces

\[ S_O := \{O' \subseteq \overline{O} | O' \not\subseteq \overline{O}_{\text{spec}} \text{ for any other special orbit } O_{\text{spec}} \subseteq O\}, \]

where \(O\) runs through all special orbits in \(g\).

For each \(O' \in S_O\), Lusztig assigned a Levi subgroup \(H(O', O)\) of the Lusztig's quotient \(\overline{A}(O)\). For example, the largest orbit in the special piece \(O \in S_O\) has \(H(O, \mathcal{O}) = 1\), and the smallest orbit \(O'' \in S_O\) has \(H(O'', \mathcal{O}) = \overline{A}(\mathcal{O})\).

By the conjecture of Sommers, if \(O\) has a non-abelian Lusztig's quotient, i.e., \(\overline{A}(O) = S_3, S_4\) or \(S_6\), then all \(O' \in S_O\) with \(H(O', O)\) not equal to 1 or \(\overline{A}(O)\) (that is, not equal to \(O\) or \(O''\)) have non-normal closures.

For example, in the case of \(G_2\) we studied above, we have \(O_8 \in S_{O_10}\) and \(H(O_8, O_{10}) = S_2 \leq S_3 = \overline{A}(O_{10})\). So \(\overline{O}_8\) is conjectured to have non-normal closure, which has been shown to be true.

We would like to end our manuscript with the following:

**Conjecture 3.1.** Suppose \(O\) is a nilpotent orbit with \(\overline{A}(O) = S_3, S_4\) or \(S_5\), and \(O' \in S_O\) satisfies \(H(O', O) \neq 1, \overline{A}(O)\). Then there exists two distinct completely prime primitive ideals \(J(\lambda_1), J(\lambda_2)\) such that

\[ U(\mathfrak{g})/J(\lambda_1)|_{K_C} \cong R(O), \quad U(\mathfrak{g})/J(\lambda_2)|_{K_C} \cong R(\overline{O}). \]

REFERENCES


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