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ON QUANTIZATION OF A NILPOTENT ORBIT CLOSURE IN G_2

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Abstract. Let G be the complex exceptional Lie group of type G_2. Among the five nilpotent orbits in its Lie algebra $\mathfrak{g}$, only the 8-dimensional orbit $\mathcal{O}_8$ has non-normal orbit closure $\overline{\mathcal{O}_8}$. In this manuscript, we will give a quantization model of $\overline{\mathcal{O}_8}$, verifying a conjecture of Vogan made in 1984.

1. Introduction

Let G be a complex simple Lie group. The G-conjugates of a nilpotent element $X \in \mathfrak{g}$ form a **nilpotent orbit** $\mathcal{O} \subset \mathfrak{g}$. Following the ideas in [19] or [21], one would like to attach unitary representations to all such orbits along with their finite G-equivariant covers. More precisely, let V be a finite G-equivariant cover of an affine Poisson G-variety containing a nilpotent orbit $\mathcal{O}$ as an open set, with its ring of regular functions $R(V)$; then one would like to find a $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-module X_V such that we have the G-module isomorphism

$$X_V|_{K_{\mathbb{C}}} \cong R(V)$$

(note that $K \leq G$ is the maximal compact subgroup of G, hence its complexification $K_{\mathbb{C}}$ is isomorphic to G). Throughout this work, we will call X_V a **quantization** of V.

As hinted in [19], one needs to pay special attention when the orbit closure $\overline{\mathcal{O}}$ is not normal. One reason is due to the algebro-geometric fact that $R(\overline{\mathcal{O}}) \cong R(\mathcal{O})$ if and only if $\overline{\mathcal{O}}$ is normal. Following the spirit of the orbit method, one needs to give a quantization model for $V = \mathcal{O}$ and $V = \overline{\mathcal{O}}$ separately when $\overline{\mathcal{O}}$ is not normal.

Here is a summary on the current progress of the above quantization scheme. In [3], Barbasch constructs such models for a large class of classical nilpotent orbits. Using a completely different method in [5], Ranee Brylinski constructs a Dixmier algebra for all classical nilpotent orbit closures. The reconciliation between the two models is the main theme of the Ph.D. thesis of the author [22].

Contrary to the classical setting, very little is known about the scheme for exceptional groups. We now focus on the case for $G = G_2$. Let $\{\alpha, \beta\}$ be the simple roots of $\mathfrak{g}$, with α being the short root. The fundamental weights of $\mathfrak{g}$ are therefore given by

$$\{\omega_1, \omega_2\} = \{2\alpha + \beta, 3\alpha + 2\beta\}.$$

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By the Bala-Carter classification, we have five nilpotent orbits $\mathcal{O}_0$, $\mathcal{O}_6$, $\mathcal{O}_8$, $\mathcal{O}_{10}$ and $\mathcal{O}_{12}$ in $\mathfrak{g}$. Following the study of completely prime primitive ideals of Joseph in [9], Vogan in [19] conjectured a quantization model for $\mathcal{O}_8$ and $\overline{\mathcal{O}_8}$ for G_2:

Conjecture 1.1 ([19, Conjecture 5.6])**.** *Let $\lambda \in \mathfrak{h}^*$ and $J(\lambda)$ be the maximal primitive ideal in $U(\mathfrak{g})$ with infinitesimal character λ. Then the $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}}) \cong (\mathfrak{g} \times \mathfrak{g}, G)$-modules*

$$U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2)), \quad U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2)),$$

are quantizations of $\mathcal{O}_8$ and $\overline{\mathcal{O}_8}$ respectively. In particular,

$$U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2))|_{K_{\mathbb{C}}} \cong R(\mathcal{O}_8), \quad U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2))|_{K_{\mathbb{C}}} \cong R(\overline{\mathcal{O}_8}).$$

As a consequence, $\mathcal{O}_8$ has non-normal closure.

Interestingly, by the classification of spherical unitary dual of complex G_2 given by Duflo in [8], $U(\mathfrak{g})/J(\frac{1}{2}(\omega_1+\omega_2))$ is unitarizable while $U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1-\omega_2))$ is not (this fact is also observed by Vogan in p. 226 of [21]. Later, Levasseur and Smith in [10] proved that $U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2))|_{K_{\mathbb{C}}} \cong R(\mathcal{O}_8)$ and $\overline{\mathcal{O}_8}$ are not normal, but were unable to prove the rest of the conjecture. The main result of this manuscript is the following:

Theorem 1.2. *As $K_{\mathbb{C}} \cong G$ modules,*

$$U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2))|_{K_{\mathbb{C}}} \cong R(\overline{\mathcal{O}_8}).$$

Remark 1.3. This quantization model of nilpotent orbit closure is very different from the classical model given in [5]. Namely, the Brylinski model is not necessarily of the form $U(\mathfrak{g})/J(\lambda)$. In particular, when the classical nilpotent orbit closure $\overline{\mathcal{O}}$ is not normal (the classification of all such orbit closures is given in [13]), the infinitesimal character of the Brylinski model $\lambda_{\mathcal{O}}$ always yields associated variety $AV(U(\mathfrak{g})/J(\lambda_{\mathcal{O}})) = \overline{\mathcal{O}'}$, where $\mathcal{O}'$ is strictly smaller than $\mathcal{O}$.

In fact, it can be shown that the Brylinski model always contains the composition factor $U(\mathfrak{g})/\ J(\lambda_{\mathcal{O}})$. This is part of the on-going work of Barbasch and the author [4].

Before going to the proof of Theorem 1.2, it is worthwhile to mention the orbits other than $\mathcal{O}_8$ in $\mathfrak{g}$. Indeed, Kraft in [12] confirmed that $\mathcal{O}_8$ is the only nilpotent orbit with non-normal closure. So we just need to consider quantizations of the orbits (and their covers) only. For the zero orbit $\mathcal{O}_0$ the quantization is trivial, and the quantization of the minimal orbit $\mathcal{O}_6$ is $U(\mathfrak{g})/J(\frac{1}{3}(3\omega_1 + \omega_2))$, where $J(\frac{1}{3}(3\omega_1 + \omega_2))$ is the Joseph ideal. The 10-dimensional orbit $\mathcal{O}_{10}$ is a special orbit with fundamental group S_3. It is a simple exercise to compare the formulas in [2] and [14] that the spherical unipotent representation attached to $\mathcal{O}_{10}$ is a quantization of $\mathcal{O}$ (as a bonus, the other two unipotent representations attached to $\mathcal{O}_{10}$ essentially give quantization of all covers of $\mathcal{O}_{10}$ as well). Finally, the quantization of the principal orbit $\mathcal{O}_{12}$ is well known to be the principal series representation with zero infinitesimal character. In conclusion, we completed the picture of quantization for all nilpotent orbits of $\mathfrak{g}$ and their closures.

2. Proof of the theorem

As mentioned in the Introduction, the non-normality of $\overline{\mathcal{O}_8}$ implies that $R(\overline{\mathcal{O}_8}) \subsetneq R(\mathcal{O}_8)$. In fact, Costantini in [7] gives the discrepancies in terms of G-modules:

Theorem 2.1 ([7, Theorem 5.6])**.** *Let $V_{(a,b)}$ be the finite-dimensional irreducible representation of G_2 with highest weight $a\omega_1 + b\omega_2$, where a and b are non-negative integers. Then*

$$R(\mathcal{O}_8) \cong R(\overline{\mathcal{O}_8}) \oplus \bigoplus_{n\in\mathbb{N}\cup\{0\}} V_{(1,n)}.$$

The following lemma gives another expression of the discrepancies between $R(\mathcal{O})$ and $R(\overline{\mathcal{O}})$:

Lemma 2.2. *As virtual G-modules,*

$$\begin{aligned}\bigoplus_{n\in\mathbb{N}\cup\{0\}} V_{(1,n)} =& Ind_T^G(1,0) - Ind_T^G(0,1) - Ind_T^G(2,0)\\ &+ Ind_T^G(1,1) + Ind_T^G(0,2) - Ind_T^G(2,1),\end{aligned}$$

where $Ind_T^G(a,b)$ is the shorthand for the induced module $Ind_T^G(e^{a\omega_1+b\omega_2})$.

Proof. The lemma can be derived from the Weyl character formula. Namely, by the $W(G_2)$-symmetry of weights of $V_{(a,b)}$, we have

$$V_{(a,b)} = \sum_{w\in W(G_2)} \det(w) Ind_T^G(\lambda_w)$$

with λ_w being the unique $W(G_2)$-conjugate of $w[(a,b)+(1,1)]-(1,1)$ lying in the dominant chamber. In fact, we have

$$\begin{aligned}V_{(1,n)} =& Ind_T^G(1,n) - Ind_T^G(1,n+3) - Ind_T^G(2,n) + Ind_T^G(2,n+2)\\ &- Ind_T^G(3,n-1) + Ind_T^G(3,n) - Ind_T^G(3,n+1) + Ind_T^G(3,n+2)\\ &+ Ind_T^G(6,n-2) - Ind_T^G(6,n) - Ind_T^G(7,n-2) + Ind_T^G(7,n-1)\end{aligned}$$

for $n > 1$, and

$$\begin{aligned}V_{(1,0)} =& Ind_T^G(1,0) - Ind_T^G(1,3) - Ind_T^G(2,0) + Ind_T^G(2,2) - Ind_T^G(0,1) + Ind_T^G(3,0)\\ &- Ind_T^G(3,1) + Ind_T^G(3,2) + Ind_T^G(0,2) - Ind_T^G(6,0) - Ind_T^G(1,2) + Ind_T^G(4,1);\end{aligned}$$

$$\begin{aligned}V_{(1,1)} =& Ind_T^G(1,1) - Ind_T^G(1,4) - Ind_T^G(2,1) + Ind_T^G(2,3) - Ind_T^G(3,0) + Ind_T^G(3,1)\\ &- Ind_T^G(3,2) + Ind_T^G(3,3) + Ind_T^G(3,1) - Ind_T^G(6,1) - Ind_T^G(4,1) + Ind_T^G(7,0).\end{aligned}$$

The lemma is proved by adding up the terms. □

We now study the two Harish-Chandra bi-modules $U(\mathfrak{g})/J(\frac{1}{2}(\omega_1+\omega_2))$ and $U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1-\omega_2))$:

Proposition 2.3. *As $K_{\mathbb{C}} \cong G$-modules*

$$U(\mathfrak{g})/J(\frac{1}{2}(\omega_1+\omega_2))|_{K_{\mathbb{C}}} = Ind_T^G(0,0) - Ind_T^G(0,1) - Ind_T^G(2,0) + Ind_T^G(1,1);$$

$$U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1-\omega_2))|_{K_{\mathbb{C}}} = Ind_T^G(0,0) - Ind_T^G(1,0) - Ind_T^G(0,2) + Ind_T^G(2,1).$$

Proof. To cater for subsequent calculations, we let $\mathfrak{h}^* = \{(x, y, z) \in \mathbb{C}^3 | x + y + z = 0\}$, with short simple root $\alpha = (1, -1, 0)$ and long simple root $\beta = (-1, 2, -1)$. Then

$$\lambda_1 = \frac{1}{2}(\omega_1 + \omega_2) = (1, 1/2, -3/2); \quad \lambda_2 = \frac{1}{2}(5\omega_1 - \omega_2) = (2, -1/2, -3/2).$$

The character formulas of $U(\mathfrak{g})/J(\lambda)$ for regular λ are well known by the work of Barbasch and Vogan [2]: Consider the subgroup W_λ of $W(G_2)$ generated by roots α satisfying $2\frac{\langle\alpha,\lambda\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$. Then the formula is given by

$$U(\mathfrak{g})/J(\lambda) = \sum_{w \in W_\lambda} \det(w) X(\lambda, w\lambda),$$

where $X(\mu, \nu) = K$-finite part of $Ind_B^G(e^{(\mu,\nu)} \otimes 1)$ is the **principal series representation** with character $(\mu, \nu) \in \mathfrak{h}_\mathbb{C}$, the complexification of the maximal torus $\mathfrak{h}$ in $\mathfrak{g}$ (here we treat G as a real Lie group). In particular, the $G \cong K_\mathbb{C}$-types of $X(\mu, \nu)$ are equal to $Ind_T^G(e^{\mu-\nu})$ (see Theorem 1.8 of [2] for more details on the principal series representations).

Now apply the above recipe for $\lambda_1 = (1, 1/2, -3/2)$: With the above notation, W_{λ_1} is isomorphic to $W(A_1 \times \tilde{A}_1)$, generated by the roots $\{(0, 1, -1), (2, -1, -1)\}$. Hence the character formula of $U(\mathfrak{g})/J(\lambda_1)$ is given by

$$\begin{aligned} U(\mathfrak{g})/J(\lambda_1) = {} & X((1, 1/2, -3/2), (1, 1/2, -3/2)) - X((1, 1/2, -3/2), (1, -3/2, 1/2)) \\ & - X((1, 1/2, -3/2), (-1, 3/2, -1/2)) + X((1, 1/2, -3/2), (-1, -1/2, 3/2)). \end{aligned}$$

Upon restricting to $K_\mathbb{C}$, we have

$$\begin{aligned} U(\mathfrak{g})/J(\lambda_1)|_{K_\mathbb{C}} \cong {} & Ind_T^G(e^{(0,0,0)}) - Ind_T^G(e^{(0,2,-2)}) \\ & - Ind_T^G(e^{(2,-1,-1)}) + Ind_G^T(e^{(2,1,-3)}). \end{aligned}$$

Again, by $W(G_2)$-symmetry of finite-dimensional irreducible G-modules, the above expression can be written in the form as in the proposition. The calculations for $U(\mathfrak{g})/J(\lambda_2)$ are identical to the one above. We omit the calculations here. □

Proof of Theorem 1.2. By the result of Levasseur and Smith in [10],

$$U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2))|_{K_\mathbb{C}} \cong R(\mathcal{O}_8)$$

as G-modules. Therefore the first equation of Proposition 2.3 gives

$$R(\mathcal{O}_8) \cong Ind_T^G(0, 0) - Ind_T^G(0, 1) - Ind_T^G(2, 0) + Ind_T^G(1, 1) \tag{2.1}$$

as virtual G-modules. By Theorem 2.1 and Lemma 2.2, we need to show that

$$\begin{aligned} U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2))|_{K_\mathbb{C}} &= R(\mathcal{O}_8) - \bigoplus_n V_{(1,n)} \\ &= R(\mathcal{O}_8) - (Ind_T^G(1, 0) - Ind_T^G(0, 1) - Ind_T^G(2, 0) \\ &\quad + Ind_T^G(1, 1) + Ind_T^G(0, 2) - Ind_T^G(2, 1)). \end{aligned}$$

This is readily seen to be true from equation (2.1) and the second equation of Proposition 2.3. □

3. Final remarks

In [18], Sommers gives some conjectures on the multiplicities of small representations of $R(\mathcal{O})$ for the exceptional groups. In particular, given that his conjecture is true, one can show the non-normality of some orbit closures.

To describe more explicitly which orbits $\mathcal{O}$ are conjectured to have non-normal closures, recall that Lusztig in [11] partitioned all nilpotent orbits in $\mathfrak{g}$ by **special pieces**, i.e., for all nilpotent orbit $\mathcal{O}'$, it must belong to exactly one of the special pieces

$$S_{\mathcal{O}} := \{\mathcal{O}' \subseteq \overline{\mathcal{O}} | \mathcal{O}' \nsubseteq \overline{\mathcal{O}}_{spec} \text{ for any other special orbit } \mathcal{O}_{spec} \subsetneq \mathcal{O}\},$$

where $\mathcal{O}$ runs through all special orbits in $\mathfrak{g}$.

For each $\mathcal{O}' \in S_{\mathcal{O}}$, Lusztig assigned a Levi subgroup $H(\mathcal{O}', \mathcal{O})$ of the Lusztig's quotient $\overline{A}(\mathcal{O})$. For example, the largest orbit in the special piece $\mathcal{O} \in S_{\mathcal{O}}$ has $H(\mathcal{O}, \mathcal{O}) = 1$, and the smallest orbit $\mathcal{O}'' \in S_{\mathcal{O}}$ has $H(\mathcal{O}'', \mathcal{O}) = \overline{A}(\mathcal{O})$.

By the conjecture of Sommers, if $\mathcal{O}$ has a non-abelian Lusztig's quotient, i.e., $\overline{A}(\mathcal{O}) = S_3, S_4$ or S_5, then all $\mathcal{O}' \in S_{\mathcal{O}}$ with $H(\mathcal{O}', \mathcal{O})$ not equal to 1 or $\overline{A}(\mathcal{O})$ (that is, not equal to $\mathcal{O}$ or $\mathcal{O}''$) have non-normal closures.

For example, in the case of G_2 we studied above, we have $\mathcal{O}_8 \in S_{\mathcal{O}_{10}}$ and $H(\mathcal{O}_8, \mathcal{O}_{10}) = S_2 \leq S_3 = \overline{A}(\mathcal{O}_{10})$. So $\overline{\mathcal{O}_8}$ is conjectured to have non-normal closure, which has been shown to be true.

We would like to end our manuscript with the following:

Conjecture 3.1. *Suppose $\mathcal{O}$ is a nilpotent orbit with $\overline{A}(\mathcal{O}) = S_3$, S_4 or S_5, and $\mathcal{O}' \in S_{\mathcal{O}}$ satisfies $H(\mathcal{O}', \mathcal{O}) \neq 1$, $\overline{A}(\mathcal{O})$. Then there exists two distinct completely prime primitive ideals $J(\lambda_1)$, $J(\lambda_2)$ such that*

$$U(\mathfrak{g})/J(\lambda_1)|_{K_{\mathbb{C}}} \cong R(\mathcal{O}), \quad U(\mathfrak{g})/J(\lambda_2)|_{K_{\mathbb{C}}} \cong R(\overline{\mathcal{O}}).$$

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